Cubic forms on adjoint representations of exceptional groups

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Abstract

We construct cubic forms on the adjoint representation of the Chevalley group of type E_7, whose partial derivatives are linear combinations of equations on the orbit of the highest weight vector. In order to describe the forms we introduce new combinatorial notions related to maximal squares in root systems of exceptional types.

1 Introduction

Algebraic groups are often given as transformations preserving some multilinear forms. For example, an orthogonal group (say, over a field of characteristic not 2) is by definition the group of linear transformations preserving a non-degenerate quadratic form, or, equivalently, preserving the corresponding bilinear form. Similarly, a symplectic group is a group of linear transformations preserving a symplectic bilinear form.

In 1905 Leonard Dickson constructed an invariant cubic form for the group of type E_6. Later this form in 27 variables was studied in the works of Claude Chevalley, Hans Freudenthal, and many others.

Michael Aschbacher proved (see [8]) that the group of linear transformations of the 27-dimensional space preserving this form coincides with the simply-connected Chevalley group of type E_6 over an arbitrary field (even in the cases of characteristics 2 and 3).

Even before that, Leonard Dickson described an invariant form of degree 4 for the group of type E_7. This form acts on the 56-dimensional space of the minimal representation of the simply-connected Chevalley group of

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type E_7. It is also known that this space possesses an invariant symplectic form. Bruce Cooperstein ([10], see also [9]) showed that the group of linear transformations preserving both these forms coincides with the Chevalley group of type E_7 for the case of a field of characteristic not 2. In [7] the restriction on the characteristic was removed by replacing a form of degree 4 with a *non-symmetric* four-linear one.

The study of minimal representations for the groups of types E_6 and E_7 is simplified by the fact that these representations are *microweight*. At the same time, the group of type E_8 has no microweight representations. Its minimal representation is the adjoint one. Therefore it is instructive to study adjoint representations of exceptional groups and their invariant multilinear forms.

The multilinear forms described above are intimately connected to the equations on the orbit of the highest weight vector. It is well known (see [11]) that the orbit of the highest weight vector in any representation is cut out by quadratic equations. Let us differentiate an invariant trilinear form for the group of type E_6 with respect to every variable. We get a set of 27 quadratic polynomials. It turns out that these polynomials cut out precisely the orbit of the highest weight vector (over an algebraically closed field). Similarly, the second order partial derivatives of a four-linear invariant form for the group of type E_7 are polarizations of quadratic polynomials describing the orbit of the highest weight vector of the minimal representation of this group (however, there are subtleties due to the existence of an invariant symplectic form on this 56-dimensional representation).

The equations on the orbit of the highest weight vector in adjoint representations for the groups of type E_6, E_7, E_8 are described in [12]. They are distributed among three types, identified [12] as "$\pi/2$-equations", "$2\pi/3$-equations", and "π-equations". In the present work, which is a development of the bachelor thesis of the first author under supervision of the second author, we construct cubic forms on the space of the adjoint representation for the Chevalley group of type E_7. The partial derivatives of these forms with respect to any variable are linear combinations of the equations of the first two types. Thus various $\pi/2$- and $2\pi/3$-equations are compactly presented in a comparatively small number of cubic forms. It is easy to show that in fact various partial derivatives of these forms provide *all* the $\pi/2$- and $2\pi/3$-equations on the orbit of the highest weight vector. In the process of constructing these forms we unravel additional combinatorial structures on root systems, which complement the *maximal squares* defined by Vavilov (see [3], [4]); we are convinced that these structures (below we call them *batches*) will be important in a future development of geometric methods of computations in exceptional groups in their adjoint representations.

2 Main notation

One can find many details relating to Chevalley groups over rings and further references in [13], [14], [15], [16], [17]. Here we only fix the main notation.

Let Φ be a reduced irreducible root system of rank l. Let $\Pi = \alpha_1, \dots, \alpha_l$ be a fundamental system in Φ (its elements will be called *fundamental* roots). Our numbering of the fundamental roots always follows Bourbaki [1]. For $\alpha \in \Phi$ we have $\alpha = \sum_{s=1}^{l} m_s(\alpha)\alpha_s$

Let $G = G(\Phi, R)$ be a simply-connected Chevalley group of type Φ over a commutative ring R with 1. We are going to work with the adjoint representation $G(\Phi, R)$, which gives us an irreducible action of $G(\Phi, R)$ on a free R-module V of rank $|\Phi| + l$. Let Λ denote the set of roots of our representation *with multiplicities*. Thus $\Lambda = \Lambda^* \sqcup \Delta$, where $\Lambda^* = \Phi$ is the set of non-zero roots, and $\Delta = \{0_1, \dots, 0_l\}$ is the set of zero roots. We fix an admissible base $e^\lambda, \lambda \in \Lambda$ in V. Then any vector $v \in V$ can be uniquely expressed as

$$v = \sum_{\lambda \in \Lambda} v_\lambda e^\lambda = \sum_{\alpha \in \Phi} v_\alpha e^\alpha + \sum_{i=1}^{l} \hat{v}_i \hat{e}^i.$$

We often write simply $v = (v_\lambda)$.

The set of roots of Φ is a subset of the euclidean space E equipped with the scalar product $(\cdot, \cdot)$. We shall also make use of the product defined by $\langle \alpha, \beta \rangle = 2(\alpha, \beta)/(\beta, \beta)$ for $\alpha, \beta \in E$ (for $\alpha, \beta \in \Phi$, we get the Cartan number). We only consider *simply-laced* root systems, which means that all roots have length 1. Therefore we have $\langle \alpha, \beta \rangle = 2(\alpha, \beta)$ for any $\alpha, \beta \in \Phi$. The angle between $\alpha, \beta \in E$ will be denoted by $\angle(\alpha, \beta)$.

In the present work we treat the cases $\Phi = \mathsf{E}_6, \mathsf{E}_7, \mathsf{E}_8$; the last two sections contain a construction of cubic forms for the case $\Phi = \mathsf{E}_7$. Nevertheless, many of the preliminary lemmas hold true even for $\Phi = \mathsf{D}_l$; sometimes we give remarks concerning this case.

The structure constants $N_{\alpha,\beta}, \alpha, \beta \in \Phi$ of the simple complex Lie algebra of type Φ are described in [2], and we freely use the identities listed there without explicit reference. Note that in our case $N_{\alpha,\beta} = 0$ or ± 1.

Let $k = l - 1, 4, 5, 7$ for $\Phi = \mathsf{D}_l, \mathsf{E}_6, \mathsf{E}_7, \mathsf{E}_8$, respectively.

Definition 1. A set of roots $\Omega = \{\beta_i\},\ i = 1, \dots, k, -k, \dots, -1$ such that $\angle(\beta_i, \beta_{-i}) = \pi/2$ for $i = 1, \dots, k$, and $\angle(\beta_i, \beta_j) = \pi/3$ for $i \neq \pm j$, is called a *maximal square*.

In general, a set of roots $\{\beta_i\}$ that satisfies the aforementioned conditions on the angles is called a *square*; a maximal set with this property contains exactly $2k$ roots. In what follows we only need maximal squares, so the adjective "maximal" will often be omitted.

Since the sum $\beta_i + \beta_{-i}$ does not depend on i, it is the same for the whole square Ω. We denote this vector by $\sigma(\Omega)$.

Conversely, for every pair $\alpha, \beta \in \Phi$ of orthogonal roots there exists a unique square containing this pair: we can just take all the pairs of roots with the same sum. This square will be denoted by $\Omega(\alpha, \beta)$. Let us number its elements as in Definition 1: put $\Omega(\alpha, \beta) = \{\beta_1, \ldots, \beta_k, \beta_{-k}, \ldots, \beta_{-1}\}$, where $\beta_1 = \alpha$, $\beta_{-1} = \beta$. Consider the following polynomials in $\mathbb{Z}[\{x_\alpha\}_{\alpha\in\Phi}, \{\widehat{x}_s\}_{s=1}^l]$:

$$f^{\pi/2}_{\alpha,\beta} = x_\alpha x_\beta - \sum_{i\geq 2} N_{\alpha,-\beta_i} N_{\beta,-\beta_{-i}} x_{\beta_i} x_{\beta_{-i}};$$

$$f^{2\pi/3}_{\alpha,\beta} = \sum_{i\neq\pm 1} N_{\alpha,-\beta_i} x_{\alpha-\beta_i} x_{\beta_i} - x_\alpha \sum_{s=1}^{l} \langle \beta, \alpha_s\rangle \widehat{x}_s;$$

$$f^{\pi}_{\alpha,\beta} = \sum_{i\neq\pm 1} (x_{\alpha-\beta_i} x_{\beta_i-\alpha} - x_{-\beta_i} x_{\beta_i}) - \sum_{s=1}^{l} \langle \alpha, \alpha_s\rangle \widehat{x}_s \cdot \sum_{s=1}^{l} \langle \beta, \alpha_s\rangle \widehat{x}_s.$$

Suppose that $v = (v_\lambda)_{\lambda\in\Lambda} \in V$ is a vector from the *highest weight vector orbit*, i. .e., $v \in G\cdot e^\rho$, where ρ is the maximal root of Φ (which is exactly the highest weight of the adjoint representation). As shown in [12], the coordinates of v satisfy the equations $f^{\pi/2}_{\alpha,\beta}(v) = 0$, $f^{2\pi/3}_{\alpha,\beta}(v) = 0$, and $f^{\pi}_{\alpha,\beta}(v) = 0$.

3 Combinatorial lemmas

Recall that the Weyl group $W = W(\Phi)$ acts on the root system by reflections:

$$s_\alpha(\beta) = \beta - \langle \alpha, \beta\rangle \alpha.$$

Since we only consider reduced crystallographic simply-laced root systems, the value of $\langle \alpha, \beta\rangle$ can only be 2, -2, 1, -1, or 0. The angle $\angle(\alpha, \beta)$ equals 0, π, $\pi/3$, $2\pi/3$, or $\pi/2$, respectively. Hence $\angle(\alpha, \beta) = \pi/3$ is equivalent to $\alpha - \beta \in \Phi$, while $\angle(\alpha, \beta) = 2\pi/3$ is equivalent to $\alpha + \beta \in \Phi$.

Consider a square Ω. Let us reflect all its roots with respect to an arbitrary root $\alpha \in \Phi$.

Definition 2. A set $S_\alpha(\Omega) = \{s_\alpha(\gamma) \mid \gamma \in \Omega\}$ is called the *reflection of the square* Ω *with respect to* α.

Lemma 1. *The set $S_\alpha(\Omega)$ is itself a square. Moreover, if $\alpha \in \Omega$, then $\sigma(S_\alpha(\Omega)) = \sigma(\Omega) - 2\alpha$.*

Proof. Suppose that $\Omega = \{\beta_1, \ldots, \beta_k, \beta_{-k}, \ldots, \beta_{-1}\}$. Without loss of generatlity we may assume that $\alpha = \beta_1$. By the definition of the square, β_1 is orthogonal to β_{-1}, while having angle $\pi/3$ with the rest of β_i's. Therefore, our reflection acts as follows:

$$(\beta_1,\ \beta_{-1}) \mapsto (-\beta_1, \beta_{-1})$$
$$(\beta_i, \beta_{-i}) \mapsto (\beta_i - \beta_1, \beta_{-i} - \beta_1) \text{ for any } i \neq \pm 1$$

Note that the sum of all the pairs we get is the same; it equals $\sigma(\Omega) - 2\alpha$, and the pair $(-\beta_1, \beta_{-1})$ is orthogonal. $\square$

Note also that (in the notation of the proof of Lemma 1)

$$\beta_i + \beta_{-i} - 2\beta_1 = \beta_1 + \beta_{-1} - 2\beta_1 = \beta_{-1} - \beta_1.$$

Consider an arbitrary square

$$\Omega = \{\beta_1, \ldots, \beta_k, \beta_{-k}, \ldots, \beta_{-1}\}.$$

A square formed by the opposite roots will be denoted by $-\Omega$.

Definition 3. The set of squares obtained by reflection of the square Ω with respect to all of its roots, together with $\pm\Omega$, is called a *batch*. It will be denoted by $s(\Omega) = \{S_\alpha(\Omega) | \alpha \in \Omega\} \cup \{\pm\Omega\}$.

Below we show that a batch (as a set of roots) is closed under reflections. Consider the following binary relation $\sim$ on the set of all squares: $\Omega \sim \Omega'$ if and only if $\Omega' \in s(\Omega)$. It turns out that $\sim$ is an equivalence relation, so the set of all squares is partitioned into a disjoint union of batches. Before proving this, let us draw a picture of a batch.

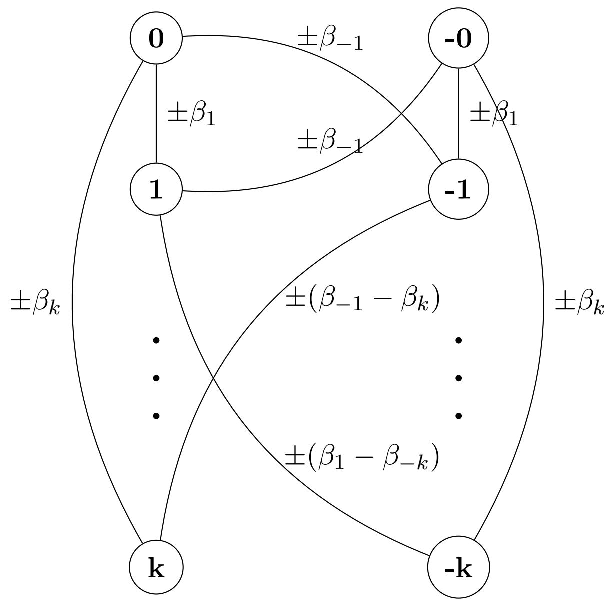
For any batch we consider a graph with labelled vertices and edges. Its vertices are squares of the batch, while its edges are reflections. A square can be transformed into another square by a reflection which labels an edge joining these two squares. In order to preserve the symmetry, we actually label an edge not by a root, but by a *pair* of opposite roots (because the opposite roots define the same reflection).

By the definition of a batch, our graph has $2(k+1)$ vertices. Let us number them by $-\mathbf{k}, \ldots, -\mathbf{1}, -\mathbf{0}, \mathbf{0}, \mathbf{1}, \ldots, \mathbf{k}$, which corresponds to the squares

$$S_{\beta_{-k}}(\Omega), \ldots, S_{\beta_{-1}}(\Omega),\ -\Omega,\ \Omega,\ S_{\beta_1}(\Omega), \ldots, S_{\beta_k}(\Omega)$$

Note that labels $\mathbf{i}, -\mathbf{i}$ correspond to the opposite squares, since $S_{\beta_{-i}}(\Omega) = -S_{\beta_i}(\Omega)$.

Now we join two non-zero vertices $\mathbf{i}, \mathbf{j}$ with $\mathbf{i} \neq \pm\mathbf{j}$, by an edge labelled by $\pm(\beta_i - \beta_j)$. Also, we label the edge $(\mathbf{0}, \mathbf{i})$ by $\pm\beta_i$, and the edge $(-\mathbf{0}, \mathbf{i})$ by $\pm\beta_{-i}$.



The only pairs of vertices not joined by an edge correspond to opposite squares.

Lemma 2. *Every edge joins two squares obtained from one another by the reflection that labels this edge.*

Proof. Since any reflection is idempotent, it suffices to prove the assertion for any one direction of each edge.

Consider an arbitrary edge. If it goes from $\mathbf{0}$, the assertion is true by the definition of a batch.

Now take an arbitrary edge going out from $-\mathbf{0}$. Without loss of generality we may consider the edge $(-\mathbf{0}, \mathbf{1})$. The square labelled by $\mathbf{1}$ has an orthogonal pair $(-\beta_1, \beta_{-1})$ that goes to $(-\beta_1, -\beta_{-1})$ after the reflection with respect to β_{-1}. The pair obtained lies in the square labelled by $-\mathbf{0}$.

It remains to consider an edge joining two non-zero vertices, e. g., $(\mathbf{1}, \mathbf{2})$. The square $\mathbf{1}$ contains an orthogonal pair $(\beta_2 - \beta_1, \beta_{-2} - \beta_1)$, while the square $\mathbf{2}$ contains an orthogonal pair $(\beta_1 - \beta_2, \beta_{-1} - \beta_2)$. Note that the second roots in these pairs are the same: $\beta_{-2} - \beta_1 = \beta_{-1} - \beta_2$, since $\beta_1 + \beta_{-1} =$

$\beta_2 + \beta_{-2}$. It follows that the first pair goes to the second pair when we consider the reflection with respect to $\beta_1 - \beta_2$. Since a square is determined by an orthogonal pair, the whole square **1** goes to the square **2**. $\square$

Corollary 1. *1. An intersection of any two non-opposite squares from a batch is a single root.*

2. Either two batches have no common squares, or they coincide.

3. The edges going out from a vertex with non-zero sign are labelled by the roots of the corresponding square.

4. Any root appears in a batch either two times, or does not appear at all.

Proof. Consider an edge joining two non-opposite squares. Suppose it is labelled by $\pm\alpha$. The proof of Lemma 2 shows that the root corresponding to a "+" sign on the edge's label is contained in a square corresponding to one of this edge's vertex, while the root corresponding to a "$-$" sign appears in the other square. The roots orthogonal to these two roots are the same. All the other roots from one of the squares have the angle $\pi/3$ with α, and all the other roots from the second square have the angle $2\pi/3$ with α; hence they can not coincide. We proved the first assertion. The other assertions easily follow from this and from the proof of Lemma 2. $\square$

Lemma 3. *A batch is closed under reflections of its squares with respect to the roots of this batch.*

Proof. Looking at the graph we constructed, we see that it remains to consider the reflections of a square with respect to the roots that are not in this square. Consider the square labelled by **1** and its reflection with respect to β_2. This square contains the orthogonal pair $(-\beta_1, \beta_{-1})$. After the reflection with respect to β_2 we get $(-\beta_1 + \beta_2, \beta_{-1} - \beta_2)$. Note that the sum of roots in this pair is the same. This means that the whole square is mapped to itself after this reflection. By symmetry, the same is true for any square. $\square$

Since a square is determined by a pair of orthogonal roots α, β, we sometimes denote a batch that contains a square Ω with $\sigma(\Omega) = \alpha + \beta$ by $s(\alpha, \beta)$ instead of $s(\Omega)$. Also, we sometimes consider a batch as a set of squares, and sometimes as a set of roots contained in these squares. The precise meaning should be clear from the context.

Now let us consider a batch as a set of roots (the union of all the squares in this batch). Note that (by definition) a batch is closed under reflections with respect to its elements. Also, it is finite. It follows that a batch is a root system. An easy calculation (using the transitivity of the Weyl group action on the batches) can be performed to determine its type. We obtain the following statement.

Lemma 4. *If* $\Phi = \mathsf{E}_6, \mathsf{E}_7, \mathsf{E}_8$*, then the set of roots in any batch is a root system of type* $\mathsf{D}_5, \mathsf{D}_6, \mathsf{D}_8$*, respectively.*

Note that $\mathsf{D}_5, \mathsf{D}_6, \mathsf{D}_8$ are maximal subsystems of type D in $\mathsf{E}_6, \mathsf{E}_7, \mathsf{E}_8$.

In the case $\Phi = \mathsf{D}_l$ the situation becomes more complicated: the pairs of orthogonal roots in D_l form *two orbits* under the Weyl group action. These orbits give us two types of squares: the "long" ones (they contain $2l-2$ roots) and the "short" ones (they contain 6 roots). The equations on the highest weight vector orbit, described in [12], are parametrized by "long" squares. If $l = 4$, the situation is even more complicated due to the fact that these squares have the same size $(2 \cdot 4 - 2 = 6)$ — this is explained by triality. The construction of the batches for $\Phi = \mathsf{D}_l$ (and its properties) still hold only for the "long" squares.

In what follows we denote by ρ the maximal root of E_8. The root system E_7 consists of the roots E_8 that have zero coefficient at the fundamental root α_8. Similarly, E_6 is the set of roots that have zero coefficients at α_8 and α_7.

The *type* of a vector in E is its coordinate at α_8. For example, the maximal root has type 2. Note that the type of a root $\lambda \in \Phi$ is equal to $0, \pm 1, \pm 2$ when $\lambda \in \mathsf{E}_7$, $\lambda \notin \mathsf{E}_7 \cup \{\pm\rho\}$, $\lambda = \pm\rho$, respectively. The set of all roots in E_8 of type i will be denoted by $\mathsf{E}_8^{(i)}$. The *type* of a square Ω is the type of $\sigma(\Omega)$.

Lemma 5. *A square in* E_8 *can only have type* $0, \pm 1$*, or* ± 2.

Proof. Since σ is a sum of two roots, the discussion above implies that its type is an integer between -4 and 4. The values ± 3, ± 4 can by obtained only by using $\pm\rho$. But such a square would countain no more than two roots. □

Lemma 1 implies that the types of the squares in one batch have the same parity. Therefore every batch is either *even* or *odd*.

We say that a batch s in E_8 *contains an* E_7*-batch* if after intersecting every square in s with E_7 (and omitting empty intersections) we get a batch in E_7. It is easy to see that such a batch is even (it contains squares of type 0). The batch obtained from s by restriction to E_7 will be denoted by $s|_{\mathsf{E}_7}$. Similarly, we say that a square *lies in* E_7, if its intersection with E_7 is itself a square in E_7. Note that, by construction, a batch that contains a square in E_7 automatically contains an E_7-batch.

Lemma 6. *Let s be an arbitrary batch in E_8. The batch s contains an E_7-batch if and only if $\rho \in s$.*

Proof. Suppose that $\rho \in s$. Then the batch s is even. Consider a square $\Omega \in s$ that contains ρ. This square has type $2 + 0 = 2$. Therefore all the other orthogonal pairs in this square have type $1+1$. Consider a reflection of Ω with respect to one of its roots of type 1. Then one pair of the orthogonal roots goes to $1 + (-1)$, while the other go to $0 + 0$. The pairs of roots of type $0 + 0$ form a square in E_7, therefore s contains an E_7-batch.

Conversely, suppose that s contains an E_7-batch. Consider a square Ω in s that lies in E_7. It has type 0, and contains $2(k-1)$ pairs of type $0+0$. Consider the remaining two pairs. The sums of types in each of these pairs should equal 0, hence they are either $2 + (-2)$ or $1 + (-1)$. But types 2, -2 occur only when we have roots ρ, $-\rho$, which are not orthogonal. Therefore, the remaining two pairs have type $1 + (-1)$. After reflecting these two pairs with respect to a root of type -1 we obtain sums $2 + 0$ and $1 + 1$. Hence we found a root of type 2 in s — this root is ρ. $\square$

Let s be a batch in E_8 that contains an E_7-batch. Note that the pair of ρ in these batch lies in E_7. Conversely, for any root in E_7 we may consider the set of the roots in E_7 orthogonal to it; this set turns out to be a batch.

Lemma 7. *Let α be an arbitrary root in E_7. Then*

$$\{\gamma \in \mathsf{E}_7 \mid \gamma \perp \alpha\} = \{\gamma \in \mathsf{E}_7 \mid \gamma \in s(\rho, \alpha)|_{\mathsf{E}_7}\}.$$

Proof. Suppose that $\beta \in s(\rho, \alpha)|_{\mathsf{E}_7}$ is not orthogonal to α. Note that $\rho \perp \beta$, since $\rho \perp \mathsf{E}_7$. Consider the reflection of $\Omega(\rho, \alpha)$ with respect to β. We get a square that contains ρ and is distinct from $\Omega(\rho, \pm\alpha)$. But each root in a batch appears twice, and we get a contradiction. The number of roots in our batch is equal to the number of roots in D_6, i. e., 60. The number of roots in E_7 that are orthogonal to a given one is the same. Hence these sets coincide. $\square$

Therefore, the batches in E_7 are parametrized by the pairs of opposite roots in E_7: for a pair $\pm\alpha \in \mathsf{E}_7$ we have a batch $s(\rho, \alpha)|_{\mathsf{E}_7}$, and for a batch s we have a unique (up to a sign) root in E_7 that is orthogonal to all roots in s. This is exactly the correspondence between the roots in E_7 (up to a sign) and (orthogonal to them) subsystems of type D_6 in E_7.

4 $\pi/2$-forms

From now on we set $\Phi = \mathsf{E}_7$. Let A denote the set of all (unordered) triples of pairwise orthogonal roots $\alpha, \beta, \gamma \in \mathsf{E}_7$ such that the roots $\alpha, \beta, \gamma, \rho$ generate a subsystem of type D_4 in E_8. Note that we have $\alpha + \beta + \gamma + \rho = 2\delta$, where $\delta \in \mathsf{E}_8$ is a root of type 1. Conversely, for a root $\delta \in \mathsf{E}_8^{(1)}$ let A_δ denote the set of triples of pairwise orthogonal roots $\alpha, \beta, \gamma \in \mathsf{E}_7$ such that $\alpha + \beta + \gamma + \rho = 2\delta$. Then

$$A = \bigcup_{\delta \in \mathsf{E}_8^{(1)}} A_\delta.$$

Since the subgroup $W(\mathsf{E}_7)$ in $W(\mathsf{E}_8)$ acts transitively on $\mathsf{E}_8^{(1)}$, all the sets A_δ have the same cardinality; it is easy to see that each of them contains 45 elements. For example, one can look at $\delta = \alpha_8$. In this case A_δ is the set of triples of pairwise orthogonal roots $\alpha, \beta, \gamma \in \mathsf{E}_7$ such that $\alpha + \beta + \gamma = -\substack{2465430\\3}$. The coefficient at α_7 of a root in E_7 can be equal to 0 or ± 1. Therefore α, β, γ has coefficient -1 at α_7. This means that α, β, γ can be considered as a triple of pairwise orthogonal roots of the minimal representation of E_6 that arises from the standard embedding $\mathsf{E}_6 \leq \mathsf{E}_7$ and its action on the set of roots of type $-\substack{*****1\\*}$ in E_7. These triples are well-known; usually they are called *triads*. It is known that there are 45 of them (see, for example, [6]). On the minimal representation of E_6 there exists a unique (up to a scalar multiple) invariant cubic form; it is the sum of monomials $\pm x_\alpha x_\beta x_\gamma$, where $\{\alpha, \beta, \gamma\}$ runs over all triads. Let $c_{\alpha\beta\gamma}$ be a sign at $x_\alpha x_\beta x_\gamma$ (for some choice of signs in this cubic form).

Since everything is symmetric with respect to the Weyl group action, we can repeat these construction for any root $\delta \in \mathsf{E}_8^{(1)}$: A_δ is the set of pairwise orthogonal triples of weights for a minimal representation of E_6 arising from *some* embedding of the root systems $\mathsf{E}_6 \to \mathsf{E}_7$. Therefore $|A_\delta| = 45$ and we can choose the signs $c_{\alpha\beta\gamma}$ in some manner for all the triples $\{\alpha, \beta, \gamma\} \in A_\delta$.

Therefore, for each of the fifty-six roots $\delta \in \mathsf{E}_8^{(1)}$ we constructed some cubic form

$$F_\delta^{\pi/2} = \sum_{\{\alpha,\beta,\gamma\} \in A_\delta} c_{\alpha\beta\gamma} x_\alpha x_\beta x_\gamma \in \mathbb{Z}[\{x_\alpha\}_{\alpha \in \mathsf{E}_7}].$$

Naturally, this form coincides with the invariant cubic form on a minimal representation of E_6 for some embedding of root systems $\mathsf{E}_6 \to \mathsf{E}_7$ and the corresponding embedding of the weights of this representation into E_7.

Theorem 1. *Let F be a linear combination of cubic forms $F_\delta^{\pi/2}$ with arbitrary scalar coefficients. Then for any $\alpha \in \mathsf{E}_7$ the formal derivative $\partial F / \partial x_\alpha$*

is a linear combination of $\pi/2$-equations corresponding to the squares in the batch $s(\rho,\alpha)|_{\mathsf{E}_7}$.

Proof. It is well known (see, for example, [6] or [5]), that the derivatives of the invariant cubic form on the minimal representation of the group of type E_6 are exactly the square equations for this representation. Namely, each derivative $\partial F_\delta^{\pi/2}/\partial x_\alpha$ is a sum of five square monomials corresponding to the pairs of orthogonal weights with the same sum $2\delta-\rho-\alpha$. Choose such a pair (β,γ). The sum obtained is comprised of exactly five terms in the polynomial $f_{\beta,\gamma}^{\pi/2}$. It remains to note that the signs of these terms in the $\pi/2$-polynomial agree with the signs in the derivative, since we know that the orbit of the highest weight vector satisfies both of these polynomials (see [12] and [5]). □

5 $2\pi/3$-forms

Definition 4. A triple of roots $\alpha,\beta,\gamma\in\mathsf{E}_7$ is called a $2\pi/3$-*triple* if after renumbering we have $\angle(\alpha,\beta)=\angle(\alpha,\gamma)=\pi/2$ and $\angle(\beta,\gamma)=2\pi/3$. The set of all $2\pi/3$-triples in E_7 will be denoted by Υ.

Note that if $\{\alpha,\beta,\gamma\}$ is a $2\pi/3$-triple such that $\angle(\beta,\gamma)=2\pi/3$, then $\beta+\gamma\in\mathsf{E}_7$ and $\alpha\perp\beta+\gamma$. Given a pair of orthogonal roots $\alpha,\beta+\gamma$, we can construct a unique maximal square Ω that contains this pair, and $\sigma(\Omega)=\alpha+\beta+\gamma$. Conversely, for an arbitrary maximal square Ω in E_7 one can consider the set $\Upsilon(\Omega)$ of all $2\pi/3$-triples α,β,γ such that $\alpha+\beta+\gamma=\sigma(\Omega)$. Therefore
$$\Upsilon=\bigcup_{\Omega\in\mathrm{M}(\mathsf{E}_7)}\Upsilon(\Omega),$$
where $\mathrm{M}(\mathsf{E}_7)$ denotes the set of all maximal squares in E_7. Note that $|\mathrm{M}(\mathsf{E}_7)|=756$. Symmetry considerations show that all maximal squares (as well as the sets $\Upsilon(\Omega)$) are conjugate under the action of the Weyl group $W(\mathsf{E}_7)$.

For every triple $\{\alpha,\beta,\gamma\}\in\Upsilon(\Omega)$ we may assume (after renaming the roots) that $\alpha\perp\beta$, $\alpha\perp\gamma$, and $\angle(\beta,\gamma)=2\pi/3$. By Lemma 7 there exists a unique (up to a sign) root τ that is orthogonal to all α's from the triples $\{\alpha,\beta,\gamma\}\in\Upsilon(\Omega)$. Choose one of these two opposite roots and denote it by τ. Then $\tau\perp(\beta+\gamma)$; one of the expressions $\langle\tau,\beta\rangle$, $\langle\tau,\gamma\rangle$ equals 1, while the other equals -1. Switching, if necessary, β with γ, we may assume that $\langle\tau,\beta\rangle=1$ and $\langle\tau,\gamma\rangle=-1$ for all triples α,β,γ for $\Upsilon(\Omega)$. From now on we may assume that $\Upsilon(\Omega)$ consists of *ordered* $2\pi/3$-triples (α, β, γ) such that $\angle(\beta, \gamma) = 2\pi/3$ and $\langle \tau, \beta \rangle = 1$.

Let us fix an arbitrary triple $(\alpha_0, \beta_0, \gamma_0) \in \Upsilon(\Omega)$. For $\alpha, \delta \in \Omega$ with $\alpha \perp \delta$ define

$$c_{\alpha,\delta} = \begin{cases} N_{\alpha_0,-\alpha} N_{\beta_0+\gamma_0,-\delta}, & \text{if } \alpha \neq \alpha_0 \text{ and } \alpha \neq \beta_0 + \gamma_0; \\ -1, & \text{if } \alpha = \alpha_0 \text{ or } \alpha = \beta_0 + \gamma_0. \end{cases}$$

Now define a cubic form

$$F_\Omega^{2\pi/3} = \sum_{(\alpha,\beta,\gamma)\in\Upsilon(\Omega)} c_{\alpha,\beta+\gamma} N_{\beta,\gamma} x_\alpha x_\beta x_\gamma - \sum_{\substack{\{\alpha,\delta\}\subseteq\Omega, \\ \alpha\perp\delta}} c_{\alpha,\delta} x_\alpha x_\delta \sum_{s=1}^{l} \langle \tau, \alpha_s \rangle \widehat{x}_s.$$

Theorem 2. *Let F be a linear combination of cubic forms $F_\Omega^{2\pi/3}$ with arbitrary scalar coefficients. Then for any $\alpha \in \mathsf{E}_7$ the formal derivative of F with respect to any of the variables x_α, $\widehat{x}_s$ is a linear combination of $\pi/2$-polynomials and $2\pi/3$-polynomials.*

Proof. It suffices to prove this for $F = F_\Omega^{2\pi/3}$ for some fixed maximal square Ω. Note that the derivatives with respect to variables corresponding to roots not in any of the triples in $\Upsilon(\Omega)$ are zero. Consider a $2\pi/3$-triple $(\alpha_1, \beta_1, \gamma_1)$ in $\Upsilon(\Omega)$ (recall that $\angle(\beta_1, \gamma_1) = 2\pi/3$ by our conventions).

Consider first the derivative $\partial F/\partial x_{\alpha_1}$. For sake of simplicity we shall assume at first that $\alpha_1 \neq \alpha_0$ and $\alpha_1 \neq \beta_0 + \gamma_0$. It is easy to see that our derivative is equal to $N_{\alpha_0,-\alpha_1} N_{\beta_0+\gamma_0,-\beta_1-\gamma_1} f^{2\pi/3}_{\beta_1+\gamma_1,\tau}$. Indeed, it suffices to note that there exists exactly one root $\delta_1 \in \Omega$ orthogonal to α_1 (namely, $\delta_1 = \beta_1 + \gamma_1$), so the terms in $\partial F/\partial x_{\alpha_1}$ containing the variables $\widehat{x}_s$ are

$$N_{\alpha_0,-\alpha_1} N_{\beta_0+\gamma_0,-\beta_1+\gamma_1} x_{\beta_1+\gamma_1} \sum_{s=1}^{l} \langle \tau, \alpha_s \rangle \widehat{x}_s.$$

The cases $\alpha_1 = \alpha_0$ and $\alpha_1 = \beta_0 + \gamma_0$ can be treated similarly.

Now consider the derivative of $F_\Omega^{2\pi/3}$ with respect to x_{β_1}. Note that for every $2\pi/3$-triple (α, β, γ) in $\Upsilon(\Omega)$ we have

$$\begin{aligned} N_{\alpha_0,-\alpha} N_{\beta_0+\gamma_0,-\beta-\gamma} N_{\beta,\gamma} &= -N_{\alpha_0,-\alpha} N_{-\beta_0,-\gamma_0,\beta+\gamma} N_{\beta,\gamma} \\ &= -N_{\alpha_0,-\alpha} N_{-\beta_0-\gamma_0+\beta,\gamma} N_{-\beta_0-\gamma_0,\beta} \\ &= -N_{\alpha_0,-\alpha} N_{\beta_0+\gamma_0-\beta,-\gamma} N_{-\beta_0-\gamma_0,\beta} \end{aligned}$$

Suppose that $\beta = \beta_1$ in such a triple. Then the third factor in our product is constant, so after differentiating we obtain a sum of terms $N_{\alpha_0,-\alpha}N_{\beta_0+\gamma_0-\beta,-\gamma}x_\alpha x_\gamma$ that runs over all orthogonal pairs α, γ with a fixed sum $\alpha + \gamma = \sigma(\Omega) - \beta_1$, and one more term $-N_{\beta_1,\gamma}x_{\alpha_0}x_{\beta_0+\gamma_0-\beta_1}$. It remains to note that $N_{\beta_1,\gamma} = N_{-\beta_0-\gamma_0,\beta_1}$. Summing everything up, we see that $\partial F/\partial x_{\beta_1}$ equals $N_{-\beta_0-\gamma_0,\beta_1}f^{\pi/2}_{\alpha_0,\beta_0+\gamma_0-\beta_1}$.

The only remaining case is the derivative $\partial F/\partial \widehat{x}_s$ for $s = 1, \dots, l$; it is easy to see that this is equal to $\langle \tau, \alpha_s \rangle f^{\pi/2}_{\alpha_0,\delta_0}$ (since the coefficient $c_{\alpha,\delta}$ are exactly the signs used in the definition of $f^{\pi/2}$; see. [12, p. 3]) $\square$

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