ISSN 1063-4541, Vestnik St. Petersburg University. Mathematics, 2013, Vol. 46, No. 1, pp. 29–34. © Allerton Press, Inc., 2013.
Original Russian Text © A.Yu. Luzgarev, 2013, published in Vestnik Sankt-Peterburgskogo Universiteta. Seriya 1. Matematika, Mekhanika, Astronomiya, 2013, No. 1, pp. 44–51.

Fourth-Degree Invariants for $G(\mathrm{E}_7, R)$ not Depending on the Characteristic

A. Yu. Luzgarev
St. Petersburg State University, Universitetskaya nab. 7/9, St. Petersburg, 199034 Russia
e-mail: mahalex@yandex.ru

Received September 20, 2012

Abstract—The Chevalley group of type E_7 over a field of characteristic different from 2 coincides with the stabilizer of a fourth-degree form on a 56-dimensional vector space. Removing the constraint on the characteristic requires considering asymmetric forms. The space of four-linear forms stabilized by the Chevalley group of type E_7 in the minimal representation over an arbitrary commutative ring is described.

DOI: 10.3103/S106345411301007X

To my teacher Nikolai Vavilov

One of the most important tools in studying the Chevalley group of type E_7 in the 56-dimensional representation is an invariant biquadratic form and its partial polarizations. This form was first constructed by ‘Elie Cartan (for a field of characteristic 0); subsequently, it was studied by Hans Freudenthal, Jacques Tits, Robert Brown, Michael Aschbacher, Bruce Cooperstein, Tony Springer, and other authors; see, in particular, [4–7, 12–14] and the references therein. It was usually assumed that $2 \in R^*$ or, sometimes, even $6 \in R^*$.

In this paper, we describe all four-linear forms stabilized by the Chevalley group $G(\mathrm{E}_7, R)$ of type E_7 over any commutative ring R. For this purpose, we construct an *asymmetric* four-linear form on the module $V(\varpi_7)$ without constraints on the characteristic of the base ring R. The biquadratic form associated with its symmetrization coincides (up to a multiplier) with a form constructed by Cartan in the case of a field of characteristic different from 2 (it was discussed in [4, 8, 9, 15]).

We do not recall definitions related to Chevalley groups, Weyl modules, the choice of a basis, etc. They can be found in, e.g., [2]. Let $\mathfrak{g}$ be a Lie algebra of type E_8 with simple roots $\alpha_1, \ldots, \alpha_8$, and let ρ be the highest root of E_8. The coefficient of the simple root α_8 in the decomposition of a root from E_8 in the simple root basis (the α_8-altitude root) can take values -2, -1, 0, 1, and 2. This fact determines a grading of length 5 on $\mathfrak{g}$:

$$\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2.$$

Namely, the subspace of $\mathfrak{g}$ spanned by e_α is contained in $\mathfrak{g}_i$ if the coefficient α of α_8 equals i. Moreover, $\mathfrak{g}_0$ contains a Cartan subalgebra $\mathfrak{h}$.

Note that $\mathfrak{g}_0$ is the direct sum of the Lie algebra of type E_7 and a one-dimensional Abelian space contained in a Cartan subalgebra of the algebra $\mathfrak{g}$. Thus, the adjoint action of the Lie algebra of type E_7 on the subspace $\mathfrak{g}_1$ is defined, which determines an action of the group $G(\mathrm{E}_7, R)$ on $\mathfrak{g}_1$. This action coincides with the action of $G(\mathrm{E}_7, R)$ on the internal Chevalley module $V(\varpi_7)$ considered in [2]. In particular, the 56-dimensional space $\mathfrak{g}_1$ has the basis consisting of the elementary root elements e_α, where α ranges over the roots of α_8-height 1, that is, the weights of the representation of $V(\varpi_7)$. In what follows, we identify $\mathfrak{g}_1$ with $V(\varpi_7)$ and treat the root system of E_7 as the subset of the root system of E_8 which consists of the roots whose decompositions contain the simple root α_8 with zero coefficient. The spaces $\mathfrak{g}_{-2}$ and $\mathfrak{g}_2$ are one-dimensional and spanned by $e_{-\rho}$ and e_ρ, respectively, where ρ is the maximal root of E_8.

Let Λ denote the set of weights of the representation of $V(\varpi_7)$. In what follows, we extensively use the fact that the elements of Λ can be considered as roots of E_8. Many of our calculations are based on the structure of the weight diagram of the representation under consideration. We fix an enumeration of the weights (see the figure); in [2], this enumeration was called the *natural enumeration*. Note that the weight

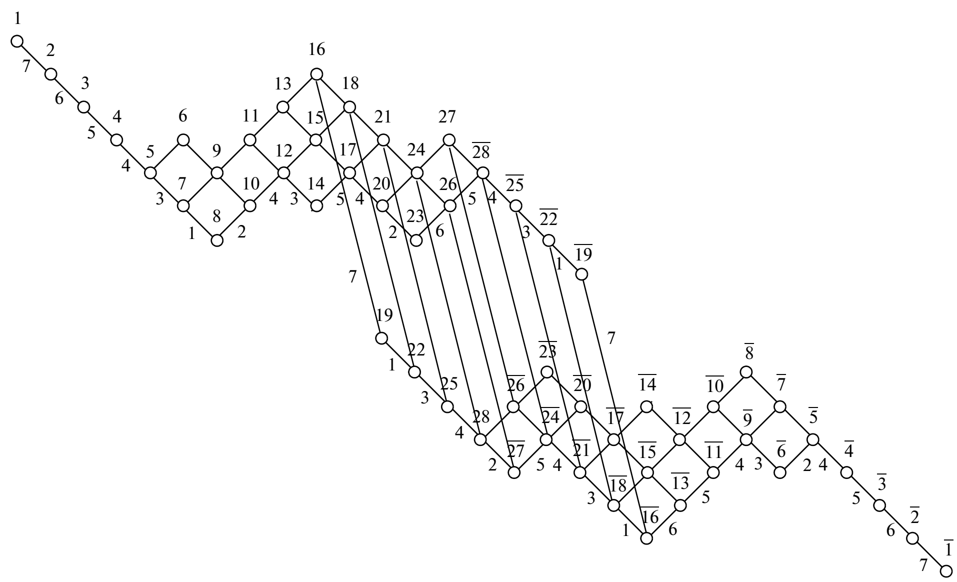

diagram of $V(\varpi_7)$ is *symmetric* (we say that weights are symmetric if their sum equals ρ). This symmetry is reflected in the enumeration: the numbers of symmetric weights are opposite. It is more convenient for us to write $-n$ instead of $\bar{n}$.

Let $\alpha, \beta, \gamma, \delta \in \Lambda$. Consider the element $[[[[e_{-\rho}, e_\alpha], e_\beta], e_\gamma], e_\delta]$. The root $-\rho$ has α_8-height -2, and the roots α, β, γ, and δ have α_8-height 1. Therefore, the above element belongs to $\mathfrak{g}_2$ and is proportional to e_ρ. We define $c(\alpha, \beta, \gamma, \delta)$ by

$$[[[[e_{-\rho}, e_\alpha], e_\beta], e_\gamma], e_\delta] = c(\alpha, \beta, \gamma, \delta)e_\rho.$$

It is easy to see that $c(\alpha, \beta, \gamma, \delta) \in \mathbb{Z}$. For all other quadruples of weights α, β, γ, and δ, we set $c(\alpha, \beta, \gamma, \delta) = 0$. Each set of coefficients $c(\alpha, \beta, \gamma, \delta)$ determines the four-linear form q on $V(\varpi_7)$ defined by

$$q(u, v, w, z) = \sum_{\alpha, \beta, \gamma, \delta \in \Lambda} c(\alpha, \beta, \gamma, \delta)u^\alpha v^\beta w^\gamma z^\delta.$$

Considering the decompositions of the vectors u, v, w, and z in the basis of e_α, we obtain

$$[[[[e_{-\rho}, u], v], w], z] = q(u, v, w, z)e_\rho. \tag{1}$$

Note that if a quadruple of weights $(\alpha, \beta, \gamma, \delta)$ is such that $c(\alpha, \beta, \gamma, \delta) = 0$, then $\alpha + \beta + \delta + \delta = 2\rho$. We say that a quadruple of weights is *degenerate* if it contains symmetric weights. Otherwise, a quadruple of weights whose sum is 2ρ is said to be *nondegenerate*. Degenerate and nondegenerate quadruples exhaust all quadruples of weight whose sum is 2ρ; we refer to such quadruples as *significant*. Finally, quadruples of weights whose sum differs from 2ρ are said to be *insignificant*.

Let us introduce yet another invariant on V, a bilinear symplectic form. Given two weights $\alpha, \beta \in \Lambda$, consider the commutator $[e_\alpha, e_\beta]$. Since there are no opposite weights in Λ, this commutator equals $N(\alpha, \beta)$ if $\alpha + \beta$ is a root and vanishes otherwise (see [16]). But α and β are roots of α_8-height 1; therefore, their sum is a root only if $\beta = \rho - \alpha$, that is, if the weights α and β are opposite in the weight diagram of $V(\varpi_7)$. In this case, $N(\alpha, \rho - \alpha) = N(-\rho, \alpha)$. We set $c'(\alpha, \beta) = N(\alpha, \beta)$ if $\alpha + \beta = \rho$ and $c'(\alpha, \beta) = 0$ for all other pairs of weights (α, β). Each set of coefficients $c'(\alpha, \beta)$ determines the bilinear form h on V defined by

$$h(u, v) = \sum_{\alpha, \beta \in \Lambda} c'(\alpha, \beta)u^\alpha v^\beta.$$

A coordinate-free expression for this form is

$$[u, v] = h(u, v)e_\rho.$$

This can easily be shown by decomposing u and v in the basis of e_α. Note that if $\alpha + \beta = \rho$, then $c'(\alpha, \beta) = N(\alpha, \beta) = -N(\beta, \alpha) = -c'(\beta, \alpha)$; therefore, the form h is symplectic.

Theorem 1. *The forms q and h are invariant with respect to the action of $G(\mathrm{E}_7, R)$ on the module $V = V(\varpi_7)$. In other words,*

$$q(u, v, w, z) = q(gu, gv, gw, gz) \quad \text{and} \quad h(u, v) = h(gu, gv)$$

for all $u, v, w, z \in V$ and $g \in G(\mathrm{E}_7, R)$.

Proof. Choosing arbitrary $g \in G(\mathrm{E}_7, R)$ and $u, v, w, z \in V$ and applying g to both sides of relation (1), we obtain

$$[[[[ge_{-\rho}, gu], gv], gw], gz] = q(u, v, w, z)ge_{\rho}.$$

On the other hand, the substitution of gu, gv, gw, and gz instead of u, v, w, and z, respectively, into the same equality yields

$$[[[[e_{-\rho}, gu], gv], gw], gz] = q(gu, gv, gw, gz)e_{\rho}.$$

Comparing these two relations and taking into account the fact that $ge_{-\rho} = e_{-\rho}$ and $ge_{\rho} = e_{\rho}$, we obtain $q(u, v, w, z) = q(gu, gv, gw, gz)$, which means that q is invariant. For a bilinear form h, the proof is completely similar.

Thus, our stock of invariant four-linear forms on V includes $q(u, v, w, z)$, $h(u, v)h(w, z)$, $h(u, w)h(v, z)$, $h(u, z)h(v, w)$, and all linear combinations of these forms with coefficients in R. The following theorem asserts that there are no other invariant four-linear forms on V.

Theorem 2. *Let F be a four-linear form on V invariant with respect to the action of $E(\mathrm{E}_7, R)$, that is, a mapping $F\colon V \times V \times V \times V \longrightarrow R$ such that*

$$F(gu, gv, gw, gz) = F(u, v, w, z)$$

for all $u, v, w, z \in V$ and $g \in E(\mathrm{E}_7, R)$. Then there exist $c_1, c_2, c_3, c_4 \in R$ such that

$$F(u, v, w, z) = c_1q(u, v, w, z) + c_2h(u, v)h(w, z) + c_3h(u, w)h(v, z) + c_4h(u, z)h(v, w)$$

for all $u, v, w, z \in V$.

A similar theorem for Lie algebras was proved independently in [10].

The rest of the paper is devoted to the proof of Theorem 2. We denote the four-linear forms mentioned in the statement of the theorem by

$$h_{12}(u, v, w, z) = h(u, v)h(w, z),$$

$$h_{13}(u, v, w, z) = h(u, w)h(v, z),$$

$$h_{14}(u, v, w, z) = h(u, z)h(v, w).$$

It follows directly from the description of the forms h and q that, for an insignificant quadruple $(\alpha, \beta, \gamma, \delta)$, any linear combination of the forms q, h_{12}, h_{13}, and h_{14} vanishes at the quadruple of vectors $(e_\alpha, e_\beta, e_\gamma, e_\delta)$.

Let Σ^+ denote the set of roots in E_7 whose decompositions contain α_7 with coefficient 1. Take $\varphi \in \mathrm{E}_7$. Let $R_\varphi = \{\alpha \in \Lambda \mid \alpha + \varphi \in \Lambda\}$ be the set of "right ends" of all edges labeled by the root φ in the weight graph. Such sets can easily be extracted from the tables given in [2]; e.g., these are precisely the numbers of the columns containing nonzero matrix elements in root elements e_φ.

Lemma 1. *If $\beta, \gamma \in \Lambda \backslash \{\rho - \omega\}$, then there exists a $\varphi \in \Sigma^+ \backslash \{\alpha_7\}$ for which $\beta + \varphi \notin \Lambda$ and $\gamma + \varphi \notin \Lambda$.*

Proof. Suppose that there exists no such φ, that is, for any $\varphi \in \Sigma^+ \backslash \{\alpha_7\}$, at least one of the two sums $\beta + \varphi$ and $\gamma + \varphi$ belongs to Λ, i.e., at least one of the weights β and γ belongs to R_φ. Let us look at some values of R_φ:

$$R_{\substack{0000110\\0}} = \{3, 19, 22, 25, 28, \overline{27}, \overline{23}, \overline{20}, \overline{17}, \overline{15}, \overline{13}, \bar{1}\},$$

$$R_{\substack{0111110\\0}} = \{7, 19, 25, 28, \overline{26}, \overline{23}, \overline{18}, \overline{15}, \overline{12}, \overline{10}, \bar{6}, \bar{1}\},$$

$$R_{\substack{1122110\\1}} = \{15, 22, \overline{26}, \overline{24}, \overline{21}, \overline{16}, \overline{14}, \overline{11}, \bar{9}, \bar{7}, \bar{3}, \bar{1}\},$$

$$R_{\substack{1232210\\2}} = \{26, \overline{27}, \overline{20}, \overline{17}, \overline{15}, \overline{13}, \bar{8}, \bar{7}, \bar{5}, \bar{4}, \bar{2}, \bar{1}\},$$

$$R_{\substack{1343210\\2}} = \{\overline{22}, \overline{18}, \overline{15}, \overline{10}, \bar{8}, \bar{6}, \bar{5}, \bar{4}, \bar{3}, \bar{2}, \bar{1}\}.$$

One of the weights β and γ must belong to at least three of these five subsets; but it is easy to see that the intersection of any three weights among them equals either $\{\bar{1}\}$ or $\{\bar{1}, \overline{15}\}$. By condition, this cannot be $\bar{1}$; therefore, $\beta = \overline{15}$ or $\gamma = \overline{15}$. Suppose that $\beta = \overline{15}$. Let us look at the subsets

$$R_{\substack{000111\,0\\0}} = \{4, 19, 22, 25, \overline{26}, \overline{24}, \overline{23}, \overline{20}, \overline{14}, \overline{12}, \overline{11}, \bar{1}\},$$

$$R_{\substack{111111\,0\\1}} = \{10, 22, 25, \overline{27}, \overline{24}, \overline{20}, \overline{16}, \overline{13}, \overline{11}, \bar{7}, \bar{5}, \bar{1}\},$$

$$R_{\substack{112211\,0\\1}} = \{15, 22, \overline{26}, \overline{24}, \overline{21}, \overline{16}, \overline{14}, \overline{11}, \bar{9}, \bar{7}, \bar{3}, \bar{1}\},$$

$$R_{\substack{1234321\,0\\2}} = \{\overline{25}, \overline{21}, \overline{17}, \overline{14}, \overline{10}, \bar{9}, \bar{8}, \bar{7}, \bar{4}, \bar{3}, \bar{2}, \bar{1}\}.$$

None of them contains $\overline{15}$. Therefore, γ must belong to each of these subsets, but their intersection equals $\{\bar{1}\}$. This contradiction completes the proof. □

Lemma 2. *Let* $\varphi \in E_7$, *and let* $\alpha', \beta', \gamma', \delta' \in \Lambda$ *be weights such that* $\alpha' - \varphi \in \Lambda$, $\beta' + \varphi \notin \Lambda$, $\gamma' + \varphi \notin \Lambda$, *and* $\delta' + \varphi \notin \Lambda$. *Then* $F(e_\alpha, e_\beta, e_\gamma, e_\delta) = 0$ *for any permutation* $(\alpha, \beta, \gamma, \delta)$ *of the quadruple* $(\alpha', \beta', \gamma', \delta')$.

Proof. It is sufficient to prove the required assertion for the identity permutation, passing to another invariant form F if necessary. Take $\xi \in R$. Then

$$F(e_{\alpha-\varphi}, e_\beta, e_\gamma, e_\delta) = F(x_\varphi(\xi)e_{\alpha-\varphi}, x_\varphi(\xi)e_\beta, x_\varphi(\xi)e_\gamma, x_\varphi(\xi)e_\delta)$$
$$= F(e_{\alpha-\varphi} \pm \xi e_\alpha, e_\beta, e_\gamma, e_\delta) = F(e_{\alpha-\varphi}, e_\beta, e_\gamma, e_\delta) \pm \xi F(e_\alpha, e_\beta, e_\gamma, e_\delta).$$

The substitution of $\xi = 1$ yields $F(e_\alpha, e_\beta, e_\gamma, e_\delta) = 0$, as required. □

Lemma 3. *If F is the same as in the statement of the theorem, then* $F(e_\alpha, e_\beta, e_\gamma, e_\delta) = 0$ *for any insignificant quadruple* $(\alpha, \beta, \gamma, \delta)$.

Proof. First, let us prove that if $F(e_\alpha, e_\beta, e_\gamma, e_\delta) \neq 0$, then $\beta = \gamma = \rho - \alpha$. The group $W(E_7)$ acts transitively on the set of weights Λ; therefore, it suffices to consider the case where $\alpha = \omega$ is the highest weight. First, suppose that neither β nor γ equals $\rho - \omega$. By Lemma 1, there exists a $\varphi \in \Sigma^+\backslash\{\alpha_7\}$ such that $\beta + \varphi \notin \Lambda$ and $\gamma + \varphi \notin \Lambda$. Moreover, obviously, $\omega - \varphi \in \Lambda$ and $\omega + \varphi \notin \Lambda$. By Lemma 2, for the quadruple of weights $(\omega, \omega, \beta, \gamma)$ and the root φ, we have $F(e_\omega, e_\omega, e_\beta, e_\gamma) = 0$.

Now, consider the case where $\beta = \rho - \omega$ or $\gamma = \rho - \omega$. For definiteness, consider the former possibility. We must prove that if $F(e_\omega, e_\omega, e_{\rho-\omega}, e_\gamma) = 0$, then $\gamma = \rho - \omega$. First, suppose that $\gamma \neq \omega$ and $\gamma \neq \rho - \omega$. Then there exists a root $\varphi \in E_7$ whose decomposition contains α_7 with zero coefficient for which $\gamma - \varphi \in \Lambda$. Indeed, after the removal of all edges labeled by 7, the weight diagram decomposes into four pieces and, by assumption, γ is contained in a piece with 27 vertices. By Lemma 2, for the quadruple of weights $(\gamma, \omega, \omega, \rho - \omega)$ and the root φ, we obtain $F(e_\omega, e_\omega, e_{\rho-\omega}, e_\gamma) = 0$, as required.

It remains to consider the case where $\beta = \rho - \omega$ and $\gamma = \omega$. In this case, we can apply Lemma 2 to the quadruple $(\rho - \omega, \omega, o, \gamma)$ and the root α_7, which gives $F(e_\omega, e_\omega, e_{\rho-\omega}, e_\gamma) = 0$.

Permuting arguments of the form, we arrive at the following fact: If $F(e_\alpha, e_\beta, e_\gamma, e_\delta) = 0$ and at least two of the roots α, β, γ, and δ coincide, then there are precisely two such roots, and the remaining two roots are symmetric to them. In this case, obviously, the quadruple $(\alpha, \beta, \gamma, \delta)$ is significant.

Now, consider the case in which there are two symmetric weights among α, β, γ, and δ, and $F(e_\alpha, e_\beta, e_\gamma, e_\delta) \neq 0$. It is required to prove that the other two weights are symmetric as well. Let us assume that $\beta = \rho - \alpha$. As above, applying the action of an element of the Weyl group $W(E_7)$, we can assume that $\alpha = \omega$. We can also assume that the difference $\alpha - \gamma$ is a root (that is, the distance between α and γ equals 1); otherwise, the difference $\beta - \gamma$ is a root, and we can interchange α and β. After this, applying the action of an element of the Weyl group $W(E_6)$, we can assume that $\gamma = \omega - \alpha_7$ ($W(E_6)$ leaves α and β invariant and transitively permutes weights at a distance of 1 from ω). Thus, we have arrived at the quadruple $(\omega, \rho - \omega, \omega - \alpha_7, \delta)$. Let us try to find a root $\varphi \in E_7$ for which $\omega + \varphi$, $\rho - \omega + \varphi$, $\omega - \alpha_7 + \varphi$ are not weights, while $\delta - \varphi$ is a root. If such a root exists, then we can apply Lemma 2 to the weights δ, ω, $\rho - \omega$, and $\omega - \alpha_7$ and the root φ, which gives $F(e_\omega, e_{\rho-\omega}, e_{\omega-\alpha_7}, e_\delta) = 0$. It is easy to satisfy the conditions $\omega + \varphi \notin \Lambda$ and $\rho - \omega + \varphi \notin \Lambda$: it suffices to choose roots whose decompositions contain α_7 with zero coefficient. There remain the conditions $\omega - \alpha_7 + \varphi \notin \Lambda$ and $\delta - \varphi \in \Lambda$. If we choose positive φ, then the former condition holds automatically. It is easy to see that this can be done always except in the case where there is "no space" to the right of δ, i.e., $\delta = \overline{19}$ or $\delta = \bar{2}$. (Recall that we have already eliminated the case where two weights among α, β, γ, and δ are equal.) But in the case $\delta = \overline{19}$, $\varphi = -\alpha_1$ suits. Thus, there remains only the case $\delta = \bar{2}$, in which γ and δ are symmetric, as required.

Now, we can assume that there are neither coinciding nor symmetric weights among α, β, γ, and δ. Thus, any two weights form an angle of $\pi/3$ or $\pi/2$. Suppose that two of these weights form an angle of $\pi/3$, that is, are a distance 1 apart. As above, applying the action of the Weyl group and permuting arguments, we can assume that $\alpha = \omega$ and $\delta = \omega - \alpha_7$. By Lemma 1, there exists a $\varphi \in \Sigma^+\backslash\{\alpha_7\}$ such that $\beta + \varphi \notin \Lambda$ and $\gamma + \varphi \notin \Lambda$. Obviously, we have $\omega - \varphi \in \Lambda$ and $\omega - \alpha_7 + \varphi \notin \Lambda$. Now, applying Lemma 2 to the quadruple $(\omega, \beta, \gamma, \omega - \alpha_7)$ and the root φ, we obtain $F(e_\omega, e_\beta, e_\gamma, e_{\omega-\alpha_7}) = 0$.

There remains only the case where α, β, γ, and δ are pairwise orthogonal; but in this case, these weights form a significant quadruple (see, e.g., [11, Corollary 1.4]).

Proof of the theorem. Let us fix a nondegenerate quadruple of weights, say $(1, \bar{2}, 19, \overline{16})$. A direct calculation shows that $q(e_1, e_{\bar{2}}, e_{19}, e_{\overline{16}}) = 1$. Setting $c_1 = F(e_1, e_{\bar{2}}, e_{19}, e_{\overline{16}}))$ and replacing the form F by $F - c_1q$, we obtain a form satisfying the assumptions of the theorem and vanishing at the quadruple $(e_1, e_{\bar{2}}, e_{19}, e_{\overline{16}}))$. Let us prove that this form is a linear combination of the forms h_{12}, h_{13}, and h_{14}.

First, we show that if a quadruple of weights $(\alpha, \beta, \gamma, \delta)$ is nondegenerate, then $F(e_\alpha, e_\beta, e_\gamma, e_\delta) = 0$. We know that F vanishes at one nondegenerate quadruple; obviously, its vanishing is preserved under the action of the Weyl group E_7. Let us show that any nondegenerate quadruple can be mapped into the quadruple under consideration. Indeed, the Weyl group acts transitively on Λ; therefore, the first element of the quadruple can be mapped to the highest weight (number 1). The remaining three weights are orthogonal to it; therefore, they are among the 27 weights at a distance of 2 from the highest weight. But these weights form the weight diagram of the 27-dimensional representation of E_6, and the restriction of the action of $W(E_7)$ to the subgroup $W(E_6)$ coincides on this diagram with the standard action of $W(E_6)$ on the weights of the minimal representation. Moreover, since these three weights are pairwise orthogonal, it follows that they form a triad, and it is well known that the triads form a singe orbit under the action of $W(E_6)$ (see, e.g., [3, 1]). Therefore, the triad under consideration can be mapped to the fixed triad $(\bar{2}, \overline{19}, \overline{16})$ so that the first weight is left intact. Hence, $F = 0$ at all nondegenerate quadruples.

Now, we set

$$\begin{aligned} c_2 &= F(e_1, e_{\bar{1}}, e_{\bar{2}}, e_2), \\ c_3 &= F(e_1, e_{\bar{2}}, e_{\bar{1}}, e_2), \\ c_4 &= F(e_1, e_{\bar{2}}, e_2, e_{\bar{2}}). \end{aligned}$$

It is easy to see that h_{12}, h_{13}, and h_{14} take the value 1 or 0 at these quadruples, so that the difference $F - c_2h_{12} - c_3h_{13} - c_4h_{14}$ vanishes at these three quadruples. Let us replace F by this difference and prove that the form F identically vanishes. We already know that F vanishes at all insignificant and nondegenerate quadruples.

First, note that any degenerate quadruple of the form $(1, \bar{1}, \beta, \rho - \beta)$, where $\{\beta, \rho - \beta\} \neq \{1, \bar{1}\}$, can be reduced to the form $(1, \bar{1}, \bar{2}, 2)$ or $(1, \bar{1}, 16, \overline{16})$ by the action of the Weyl group $W(E_6)$. Indeed, $W(E_6)$ acts transitively on the pieces obtained from the weight diagram by deleting the edges labeled by 7, and the weights β and $\rho - \beta$ are in two different pieces, each containing 27 weights. But for $\xi \in R$ and $\varphi = \begin{smallmatrix}2343210\\2\end{smallmatrix}$, we have

$$\begin{aligned} F(e_1, e_{\bar{1}}, e_{\bar{2}}, e_{\overline{16}}) &= F(x_\varphi(\xi)e_1, x_\varphi(\xi)e_{\bar{1}}, x_\varphi(\xi)e_{\bar{2}}, x_\varphi(\xi)e_{\overline{16}}) \\ &= F(e_1, e_{\bar{1}} + \xi e_{19}, e_{\bar{2}} + \xi e_{16}, e_{\overline{16}} + \xi e_2) + \xi F(e_1, e_{19}, e_{\bar{2}}, e_{\overline{16}}) \\ &\quad + \xi F(e_1, e_{\bar{1}}, e_{16}, e_{\overline{16}}) + \xi F(e_1, e_{\bar{1}}, e_{\bar{2}}, e_2) + \dots \end{aligned}$$

(we have used the fact that φ is the maximal root of E_7, so that all signs of the action are positive for $x_\varphi(\xi)$). The further terms correspond to insignificant quadruples and, therefore, vanish. Moreover, the last term on the right-hand side of the expression written above vanishes by assumption, and $F(e_1, e_{19}, e_{\bar{2}}, e_{\overline{16}}) = 0$, because this term corresponds to a nondegenerate quadruple. Consequently, $F(e_1, e_{\bar{1}}, e_{16}, e_{\overline{16}}) = 0$. We conclude that any quadruple of weights of the form $(1, \bar{1}, \beta, \rho - \beta)$ with $\{\beta, \rho - \beta\} \neq \{1, \bar{1}\}$ can be reduces to one of the two quadruples at which the form vanishes. Therefore, F vanishes at all such quadruples. Permuting arguments in this consideration, we see that F also vanishes at all quadruples of the forms $(1, \beta, \bar{1}, \rho - \beta)$ and $(1, \beta, \rho - \beta, \bar{1})$. But any degenerate quadruple without repeated weights can be reduced to such a quadruple by the action of the Weyl group: it is sufficient to map the first weight to the highest weight.

It remains to consider degenerate quadruples with repeated weights. Take any weight $\alpha \in \Lambda$. Obviously, there exists a simple or a negative simple root φ for which $\alpha - \varphi$ is a weight. Then all signs of the action for $x_\varphi(\xi)$ equal 1 and, hence,

$$\begin{aligned} &F(e_\alpha, e_{\rho-\alpha}, e_{\alpha-\varphi}, e_{\rho-\alpha}) \\ &= F(x_\varphi(\xi)e_\alpha, x_\varphi(\xi)e_{\rho-\alpha}, x_\varphi(\xi)e_{\alpha-\varphi}, x_\varphi(\xi)e_{\rho-\alpha}) \\ &= F(e_\alpha, e_{\rho-\alpha} + \xi e_{\rho-\alpha+\varphi}, e_{\alpha-\varphi} + \xi e_\alpha, e_{\rho-\alpha} + \xi e_{\rho-\alpha+\varphi}) \\ &= F(e_\alpha, e_{\rho-\alpha}, e_{\alpha-\varphi}, e_{\rho-\alpha}) + \xi F(e_\alpha, e_{\rho-\alpha+\varphi}, e_{\alpha-\varphi}, e_{\rho-\alpha}) \\ &+ \xi F(e_\alpha, e_{\rho-\alpha}, e_\alpha, e_{\rho-\alpha}) + \xi F(e_\alpha, e_{\rho-\alpha}, e_{\alpha-\varphi}, e_{\rho-\alpha+\varphi}) + \dots. \end{aligned}$$

The remaining terms vanish. The second and fourth terms on the right-hand side vanish as well, because they correspond to degenerate quadruples without repeated weights. Therefore, $F(e_\alpha, e_{\rho-\alpha}, e_\alpha, e_{\rho-\alpha}) = 0$. Applying this argument to various permutations of the arguments, we see that F vanishes at all degenerate quadruples with repeated weights. Therefore, F vanishes everywhere, which completes the proof of the theorem. □

ACKNOWLEDGMENTS

This work was supported by the DAAD program “Mikhail Lomonosov,” the Russian Foundation for Basic Research (project nos. 09-01-00784, 09-01-00878, 09-01-90304, 10-01-90016, 10-01-00551, and 12-01-00947) and the Golda Meir Postdoctoral Fellowship.

REFERENCES

1. N. A. Vavilov and A. Yu. Luzgarev, “Normalizer of the Chevalley Group of Type E_6,” Algebra Analiz **19** (5), 35–62 (2007) [St. Petersburg Math. J. **19**, 699–718 (2008)].
2. N. A. Vavilov and A. Yu. Luzgarev, “Chevalley Groups of Type E_7 in the 56-Dimensional Representation,” Zap. Nauchn. Sem. POMI **386**, 5–99 (2011) [J. Math. Sci. **180** (3), 197–251 (2012)].
3. N. A. Vavilov, A. Yu. Luzgarev, and I. M. Pevzner, “Chevalley Groups of Type E_6 in the 27-Dimensional Representation,” Zap. Nauchn. Sem. POMI **338**, 5–68 (2006) [J. Math. Sci. **145** (1), 4697–4736 (2007)].
4. M. Aschbacher, “Some Multilinear Forms with Large Isometry Groups,” Geom. Dedicata **25** (1–3), 417–465 (1988).
5. R. B. Brown, “Groups of Type E_7,” J. Reine Angew. Math. **236**, 79–102 (1969).
6. B. N. Cooperstein, “The Fifty-Six-Dimensional Module for E_7, 1: A Four Form for E_7,” J. Algebra **173** (2), 361–389 (1995).
7. H. Freudenthal, “Sur le groupe exceptionnel E_7,” Proc. Nederl. Akad. Wetensch. Ser. A **56**, 81–89 (1953).
8. R. S. Garibaldi, “Groups of Type E_7 over Arbitrary Fields,” Comm. Algebra **29** (6), 2689–2710 (2001).
9. F. Helenius, *Freudenthal Triple Systems by Root System Methods*, arXiv/1005.1275. 2010.
10. J. Lurie, “On Simply Laced Lie Algebras and Their Minuscule Representations,” Comment. Math. Helv. **76** (3), 515–575 (2001).
11. G. E. Röhrle, “On Extraspecial Parabolic Subgroups,” Contemp. Math. **153**, 143–155 (1993).
12. T. A. Springer, “Some Groups of Type E_7,” Nagoya Math. J. **182**, 259–284 (2006).
13. J. Tits, “Le plan projectif des octaves et les groupes de Lie exceptionnels,” Acad. Roy. Belg. Bull. Cl. Sci. **39**, 309–329 (1953).
14. J. Tits, “Le plan projectif des octaves et les groupes exceptionnels E_6 et E_7,” Acad. Roy. Belg. Bull. Cl. Sci. **40**, 29–40 (1954).
15. N. A. Vavilov, “A Third Look at Weight Diagrams,” Rend. Sem. Mat. Univ. Padova **104**, 201–250 (2000).
16. N. A. Vavilov, “Do It Yourself Structure Constants for Lie Algebras of Type E_l,” Zap. Nauchn. Sem. POMI **281**, 60–104 (2001).