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CHEVALLEY GROUPS OF TYPE E_7 IN THE 56-DIMENSIONAL REPRESENTATION

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The present paper is devoted to a detailed computer study of the action of the Chevalley group $G(\mathrm{E}_7, R)$ on the 56-dimensional minimal module $V(\varpi_7)$. Our main objectives are an explicit choice and a tabulation of the signs of structure constants for this action, compatible with a given choice of a positive Chevalley base, construction of multilinear invariants and of equations satisfied by the matrix entries of elements from $G(\mathrm{E}_7, R)$ in this representation, and an explicit tabulation of root elements. These calculations are performed in four numberings of weights: the natural one and those compatible with the A_6-branching, the D_6-branching, and the E_6-branching. Similar tables for the action of Chevalley group $G(\mathrm{E}_6, R)$ on the 27-dimensional minimal module $V(\varpi_1)$ were published in our joint paper with Igor Pevzner. Bibliography: 142 titles.

The present work, which is a sequel of [138, 135, 18], is devoted to a detailed computer study of the action of the Chevalley group $G(\mathrm{E}_7, R)$ on the minimal module $V(\varpi_7)$. Similarly to [135, 18], the present paper is of a technical nature, and its primary objective is to serve as a reference source for subsequent more substantial works devoted to

- K-theory of exceptional groups,
- the study of overgroups of $G(\mathrm{E}_7, R)$ in the general linear group $\mathrm{GL}(56, R)$,
- description of some classes of subgroups in $G(\mathrm{E}_7, R)$,
- geometry of root subgroups,
- generation problems.

The present paper is a fragment of a much broader programme, and we refer the reader to [18, 137] for a more detailed description of the whole project, and to [133, 138, 8, 20, 10] for some possible applications.

Observe that similar calculations in the 27-dimensional module $V(\varpi_1)$ for the Chevalley group $G(\mathrm{E}_6, R)$ were performed in our joint work with Igor Pevzner [18]. That paper provided computational background for our subsequent publications [15, 136, 23, 22, 28, 29, 30, 12].

Our main tools in the present paper are

- realization of the 56-dimensional module as an internal Chevalley module in the unipotent radical of the standard parabolic subgroup of type P_8 in the Chevalley group $G(\mathrm{E}_8, R)$,
- extensive computer calculations carried through with the help of the general purpose computer algebra system `Mathematica`.

We take into account results related to the 56-dimensional Brown–Freudenthal ternary algebra $\mathbb{F}$, as part of motivation and background. However, we prefer not to invoke these results directly, but rather rely on explicit calculations with the use of the embedding $\mathrm{E}_7 < \mathrm{E}_8$. The reason is that most proofs in terms of the algebra $\mathbb{F}$ crucially depend on conditions such as $2 \in R^*$, or even $6 \in R^*$. This is completely unacceptable to us, since our broader goal in this series of papers is to state results in a form valid for arbitrary commutative rings.

The main objectives of the present paper are an explicit choice and a tabulation of the signs of structure constants for this action, compatible with the choice of a positive Chevalley base in [135], construction of multlinear invariants and equations on matrix entries for elements of $G(\mathrm{E}_7, R)$ in this representation, and an explicit tabulation of root elements in this representation. For convenience of use the resulting tables are reproduced in the following *four* numerations:

- the natural numeration,
- the numeration related to the A_6-branching,
- the numeration related to the D_6-branching,
- the numeration related to the E_6-branching.

Since even the resulting *nonsymmetric* invariant of degree 4 has 19768 nonzero monomials in 56 variables, it is impossible to reproduce here an explicit coordinate expression of this quartic form. However, 126 second partial

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derivatives of this form arise as the highest Weyl orbit of equations on highest weight vectors in this representation (see [9] and references therein). For two of the above numerations, these equations are reproduced in [16] and [103], in connection with applications therein.

1. The roots of E_7

All the notation pertaining to roots, weights, Lie algebras, algebraic groups, and representations is utterly standard and follows [2–4, 33, 35–37, 55, 56] (see also [133, 112], where one can find many further references). We do not recall definitions of Chevalley groups and their subgroups, which can be found, for example, in [1, 2, 35, 39–43, 90, 105, 125, 126, 128, 132–134, 138]. In fact, we do not even recall the basic notation used in the sequel, but refer the reader to the first four sections of our paper [18]. There one can find both the notation and specific references to the background material pertaining to

- structure constants of Chevalley groups [55, 71, 93, 72, 138, 135, 7],
- Weyl modules and weight diagrams [105, 112, 133, 134, 107],
- internal Chevalley modules [45, 113–115],
- hyperbolic realization of the root systems of type E_l [27, 86, 135, 18],
- programming in `Mathematica` language.

Also, we do not reproduce from [18, 137] a complete bibliography related to exceptional groups. Here we list exclusively

- papers that are directly cited in the text,
- papers that specifically consider the Chevalley group of type E_7 and its 56-dimensional representation.
- some further works dedicated to calculations in exceptional groups, mostly those that appeared in 2007–2010, and thus could not be included in the bibliographies of [18, 137], and those for which publication details were not available at that time.

In [135] one can find the `Mathematica` code that generates roots of E_l in the hyperbolic base and in the fundamental base. We do not reproduce this code here; it simply translates the above definitions from [18] into the `Mathematica` language. Instead, we reproduce the resulting list – in fact, in [135] positive roots are listed in the Dynkin notation, but not in the hyperbolic form. As a matter of fact, actual calculations were performed in the *hyperbolic* form, so that the main object we start with is the following list `positiveE7`:

```
{{0,-1,1,0,0,0,0,0},{1,1,1,1,0,0,0,0},{0,0,-1,1,0,0,0,0},
{0,0,0,-1,1,0,0,0},{0,0,0,0,-1,1,0,0},{0,0,0,0,0,-1,1,0},
{0,0,0,0,0,0,-1,1},{0,-1,0,1,0,0,0,0},{1,1,1,0,1,0,0,0},
{0,0,-1,0,1,0,0,0},{0,0,0,-1,0,1,0,0},{0,0,0,0,-1,0,1,0},
{0,0,0,0,0,-1,0,1},{0,-1,0,0,1,0,0,0},{1,1,0,1,1,0,0,0},
{1,1,1,0,0,1,0,0},{0,0,-1,0,0,1,0,0},{0,0,0,-1,0,0,1,0},
{0,0,0,0,-1,0,0,1},{1,0,1,1,1,0,0,0},{0,-1,0,0,0,1,0,0},
{1,1,0,1,0,1,0,0},{1,1,1,0,0,0,1,0},{0,0,-1,0,0,0,1,0},
{0,0,0,-1,0,0,0,1},{1,0,1,1,0,1,0,0},{0,-1,0,0,0,0,1,0},
{1,1,0,0,1,1,0,0},{1,1,0,1,0,0,1,0},{1,1,1,0,0,0,0,1},
{0,0,-1,0,0,0,0,1},{1,0,1,0,1,1,0,0},{1,0,1,1,0,0,1,0},
{0,-1,0,0,0,0,0,1},{1,1,0,0,1,0,1,0},{1,1,0,1,0,0,0,1},
{1,0,0,1,1,1,0,0},{1,0,1,0,1,0,1,0},{1,0,1,1,0,0,0,1},
{1,1,0,0,0,1,1,0},{1,1,0,0,1,0,0,1},{1,0,0,1,1,0,1,0},
{1,0,1,0,0,1,1,0},{1,0,1,0,1,0,0,1},{1,1,0,0,0,1,0,1},
{1,0,0,1,0,1,1,0},{1,0,0,1,1,0,0,1},{1,0,1,0,0,1,0,1},
{1,1,0,0,0,0,1,1},{1,0,0,0,1,1,1,0},{1,0,0,1,0,1,0,1},
{1,0,1,0,0,0,1,1},{2,1,1,1,1,1,1,0},{1,0,0,0,1,1,0,1},
{1,0,0,1,0,0,1,1},{2,1,1,1,1,1,0,1},{1,0,0,0,1,0,1,1},
{2,1,1,1,1,0,1,1},{1,0,0,0,0,1,1,1},{2,1,1,1,0,1,1,1},
{2,1,1,0,1,1,1,1},{2,1,0,1,1,1,1,1},{2,0,1,1,1,1,1,1}}.
```

After this list has been generated once, it is not particularly important where it came from. The initial fragment of this list of length seven

```
{{0,-1,1,0,0,0,0,0},{1,1,1,1,0,0,0,0},{0,0,-1,1,0,0,0,0},
```

```
{0,0,0,-1,1,0,0,0},{0,0,0,0,-1,1,0,0},{0,0,0,0,0,-1,1,0},
{0,0,0,0,0,0,-1,1}}
```

has been taken as the system of fundamental roots `rootbaseE7`. After that, to get the string Dynkin form, for each of the remaining roots a system of linear equations has been solved, providing their linear expressions in terms of `rootbaseE7`. We do not reproduce similar lists `positiveE8` and `rootbaseE8`, which are generated similarly, but for reader's convenience in Tables 1 and 2 we reproduce the lists of positive roots of E_7 and E_8 with respect to the `height lexicographic order`.

Table 1. Positive roots of E_7.

1	$\begin{smallmatrix}100000\\0\end{smallmatrix} = (0,-1,1,0,0,0,0,0)$	$\begin{smallmatrix}000000\\1\end{smallmatrix} = (1,1,1,1,0,0,0,0)$
	$\begin{smallmatrix}010000\\0\end{smallmatrix} = (0,0,-1,1,0,0,0,0)$	$\begin{smallmatrix}001000\\0\end{smallmatrix} = (0,0,0,-1,1,0,0,0)$
	$\begin{smallmatrix}000100\\0\end{smallmatrix} = (0,0,0,0,-1,1,0,0)$	$\begin{smallmatrix}000010\\0\end{smallmatrix} = (0,0,0,0,0,-1,1,0)$
	$\begin{smallmatrix}000001\\0\end{smallmatrix} = (0,0,0,0,0,0,-1,1)$	
2	$\begin{smallmatrix}110000\\0\end{smallmatrix} = (0,-1,0,1,0,0,0,0)$	$\begin{smallmatrix}001000\\1\end{smallmatrix} = (1,1,1,0,1,0,0,0)$
	$\begin{smallmatrix}011000\\0\end{smallmatrix} = (0,0,-1,0,1,0,0,0)$	$\begin{smallmatrix}001100\\0\end{smallmatrix} = (0,0,0,-1,0,1,0,0)$
	$\begin{smallmatrix}000110\\0\end{smallmatrix} = (0,0,0,0,-1,0,1,0)$	$\begin{smallmatrix}000011\\0\end{smallmatrix} = (0,0,0,0,0,-1,0,1)$
3	$\begin{smallmatrix}111000\\0\end{smallmatrix} = (0,-1,0,0,1,0,0,0)$	$\begin{smallmatrix}011000\\1\end{smallmatrix} = (1,1,0,1,1,0,0,0)$
	$\begin{smallmatrix}001100\\1\end{smallmatrix} = (1,1,1,0,0,1,0,0)$	$\begin{smallmatrix}011100\\0\end{smallmatrix} = (0,0,-1,0,0,1,0,0)$
	$\begin{smallmatrix}001110\\0\end{smallmatrix} = (0,0,0,-1,0,0,1,0)$	$\begin{smallmatrix}000111\\0\end{smallmatrix} = (0,0,0,0,-1,0,0,1)$
4	$\begin{smallmatrix}111000\\1\end{smallmatrix} = (1,0,1,1,1,0,0,0)$	$\begin{smallmatrix}111100\\0\end{smallmatrix} = (0,-1,0,0,0,1,0,0)$
	$\begin{smallmatrix}011100\\1\end{smallmatrix} = (1,1,0,1,0,1,0,0)$	$\begin{smallmatrix}001110\\1\end{smallmatrix} = (1,1,1,0,0,0,1,0)$
	$\begin{smallmatrix}011110\\0\end{smallmatrix} = (0,0,-1,0,0,0,1,0)$	$\begin{smallmatrix}001111\\0\end{smallmatrix} = (0,0,0,-1,0,0,0,1)$
5	$\begin{smallmatrix}111100\\1\end{smallmatrix} = (1,0,1,1,0,1,0,0)$	$\begin{smallmatrix}111110\\0\end{smallmatrix} = (0,-1,0,0,0,0,1,0)$
	$\begin{smallmatrix}012100\\1\end{smallmatrix} = (1,1,0,0,1,1,0,0)$	$\begin{smallmatrix}011110\\1\end{smallmatrix} = (1,1,0,1,0,0,1,0)$
	$\begin{smallmatrix}001111\\1\end{smallmatrix} = (1,1,1,0,0,0,0,1)$	$\begin{smallmatrix}011111\\0\end{smallmatrix} = (0,0,-1,0,0,0,0,1)$
6	$\begin{smallmatrix}112100\\1\end{smallmatrix} = (1,0,1,0,1,1,0,0)$	$\begin{smallmatrix}111110\\1\end{smallmatrix} = (1,0,1,1,0,0,1,0)$
	$\begin{smallmatrix}111111\\0\end{smallmatrix} = (0,-1,0,0,0,0,0,1)$	$\begin{smallmatrix}012110\\1\end{smallmatrix} = (1,1,0,0,1,0,1,0)$
	$\begin{smallmatrix}011111\\1\end{smallmatrix} = (1,1,0,1,0,0,0,1)$	
7	$\begin{smallmatrix}122100\\1\end{smallmatrix} = (1,0,0,1,1,1,0,0)$	$\begin{smallmatrix}112110\\1\end{smallmatrix} = (1,0,1,0,1,0,1,0)$
	$\begin{smallmatrix}111111\\1\end{smallmatrix} = (1,0,1,1,0,0,0,1)$	$\begin{smallmatrix}012210\\1\end{smallmatrix} = (1,1,0,0,0,1,1,0)$

	$\genfrac{}{}{0pt}{}{012111}{1} = (1,1,0,0,1,0,0,1)$	
8	$\genfrac{}{}{0pt}{}{122110}{1} = (1,0,0,1,1,0,1,0)$	$\genfrac{}{}{0pt}{}{112210}{1} = (1,0,1,0,0,1,1,0)$
	$\genfrac{}{}{0pt}{}{112111}{1} = (1,0,1,0,1,0,0,1)$	$\genfrac{}{}{0pt}{}{012211}{1} = (1,1,0,0,0,1,0,1)$
9	$\genfrac{}{}{0pt}{}{122210}{1} = (1,0,0,1,0,1,1,0)$	$\genfrac{}{}{0pt}{}{122111}{1} = (1,0,0,1,1,0,0,1)$
	$\genfrac{}{}{0pt}{}{112211}{1} = (1,0,1,0,0,1,0,1)$	$\genfrac{}{}{0pt}{}{012221}{1} = (1,1,0,0,0,0,1,1)$
10	$\genfrac{}{}{0pt}{}{123210}{1} = (1,0,0,0,1,1,1,0)$	$\genfrac{}{}{0pt}{}{122211}{1} = (1,0,0,1,0,1,0,1)$
	$\genfrac{}{}{0pt}{}{112221}{1} = (1,0,1,0,0,0,1,1)$	
11	$\genfrac{}{}{0pt}{}{123210}{2} = (2,1,1,1,1,1,1,0)$	$\genfrac{}{}{0pt}{}{123211}{1} = (1,0,0,0,1,1,0,1)$
	$\genfrac{}{}{0pt}{}{122221}{1} = (1,0,0,1,0,0,1,1)$	
12	$\genfrac{}{}{0pt}{}{123211}{2} = (2,1,1,1,1,1,0,1)$	$\genfrac{}{}{0pt}{}{123221}{1} = (1,0,0,0,1,0,1,1)$
13	$\genfrac{}{}{0pt}{}{123221}{2} = (2,1,1,1,1,0,1,1)$	$\genfrac{}{}{0pt}{}{123321}{1} = (1,0,0,0,0,1,1,1)$
14	$\genfrac{}{}{0pt}{}{123321}{2} = (2,1,1,1,0,1,1,1)$	
15	$\genfrac{}{}{0pt}{}{124321}{2} = (2,1,1,0,1,1,1,1)$	
16	$\genfrac{}{}{0pt}{}{134321}{2} = (2,1,0,1,1,1,1,1)$	
17	$\genfrac{}{}{0pt}{}{234321}{2} = (2,0,1,1,1,1,1,1)$	

Table 2. Positive roots of E_8.

1	$\genfrac{}{}{0pt}{}{1000000}{0} = (0,-1,1,0,0,0,0,0,0)$	$\genfrac{}{}{0pt}{}{0000000}{1} = (1,1,1,1,0,0,0,0,0)$
	$\genfrac{}{}{0pt}{}{0100000}{0} = (0,0,-1,1,0,0,0,0,0)$	$\genfrac{}{}{0pt}{}{0010000}{0} = (0,0,0,-1,1,0,0,0,0)$
	$\genfrac{}{}{0pt}{}{0001000}{0} = (0,0,0,0,-1,1,0,0,0)$	$\genfrac{}{}{0pt}{}{0000100}{0} = (0,0,0,0,0,-1,1,0,0)$
	$\genfrac{}{}{0pt}{}{0000010}{0} = (0,0,0,0,0,0,-1,1,0)$	$\genfrac{}{}{0pt}{}{0000001}{0} = (0,0,0,0,0,0,0,-1,1)$
2	$\genfrac{}{}{0pt}{}{1100000}{0} = (0,-1,0,1,0,0,0,0,0)$	$\genfrac{}{}{0pt}{}{0010000}{1} = (1,1,1,0,1,0,0,0,0)$
	$\genfrac{}{}{0pt}{}{0110000}{0} = (0,0,-1,0,1,0,0,0,0)$	$\genfrac{}{}{0pt}{}{0011000}{0} = (0,0,0,-1,0,1,0,0,0)$
	$\genfrac{}{}{0pt}{}{0001100}{0} = (0,0,0,0,-1,0,1,0,0)$	$\genfrac{}{}{0pt}{}{0000110}{0} = (0,0,0,0,0,-1,0,1,0)$
	$\genfrac{}{}{0pt}{}{0000011}{0} = (0,0,0,0,0,0,-1,0,1)$	

3	$\substack{1110000\\0} = (0,-1,0,0,1,0,0,0,0)$	$\substack{0110000\\1} = (1,1,0,1,1,0,0,0,0)$
	$\substack{0011000\\1} = (1,1,1,0,0,1,0,0,0)$	$\substack{0111000\\0} = (0,0,-1,0,0,1,0,0,0)$
	$\substack{0011100\\0} = (0,0,0,-1,0,0,1,0,0)$	$\substack{0001110\\0} = (0,0,0,0,-1,0,0,1,0)$
	$\substack{0000111\\0} = (0,0,0,0,0,-1,0,0,1)$	
4	$\substack{1110000\\1} = (1,0,1,1,1,0,0,0,0)$	$\substack{1111000\\0} = (0,-1,0,0,0,1,0,0,0)$
	$\substack{0111000\\1} = (1,1,0,1,0,1,0,0,0)$	$\substack{0011100\\1} = (1,1,1,0,0,0,1,0,0)$
	$\substack{0111100\\0} = (0,0,-1,0,0,0,1,0,0)$	$\substack{0011110\\0} = (0,0,0,-1,0,0,0,1,0)$
	$\substack{0001111\\0} = (0,0,0,0,-1,0,0,0,1)$	
5	$\substack{1111000\\1} = (1,0,1,1,0,1,0,0,0)$	$\substack{1111100\\0} = (0,-1,0,0,0,0,1,0,0)$
	$\substack{0121000\\1} = (1,1,0,0,1,1,0,0,0)$	$\substack{0111100\\1} = (1,1,0,1,0,0,1,0,0)$
	$\substack{0011110\\1} = (1,1,1,0,0,0,0,1,0)$	$\substack{0111110\\0} = (0,0,-1,0,0,0,0,1,0)$
	$\substack{0011111\\0} = (0,0,0,-1,0,0,0,0,1)$	
6	$\substack{1121000\\1} = (1,0,1,0,1,1,0,0,0)$	$\substack{1111100\\1} = (1,0,1,1,0,0,1,0,0)$
	$\substack{1111110\\0} = (0,-1,0,0,0,0,0,1,0)$	$\substack{0121100\\1} = (1,1,0,0,1,0,1,0,0)$
	$\substack{0111110\\1} = (1,1,0,1,0,0,0,1,0)$	$\substack{0011111\\1} = (1,1,1,0,0,0,0,0,1)$
	$\substack{0111111\\0} = (0,0,-1,0,0,0,0,0,1)$	
7	$\substack{1221000\\1} = (1,0,0,1,1,1,0,0,0)$	$\substack{1121100\\1} = (1,0,1,0,1,0,1,0,0)$
	$\substack{1111110\\1} = (1,0,1,1,0,0,0,1,0)$	$\substack{1111111\\0} = (0,-1,0,0,0,0,0,0,1)$
	$\substack{0122100\\1} = (1,1,0,0,0,1,1,0,0)$	$\substack{0121110\\1} = (1,1,0,0,1,0,0,1,0)$
	$\substack{0111111\\1} = (1,1,0,1,0,0,0,0,1)$	
8	$\substack{1221100\\1} = (1,0,0,1,1,0,1,0,0)$	$\substack{1122100\\1} = (1,0,1,0,0,1,1,0,0)$
	$\substack{1121110\\1} = (1,0,1,0,1,0,0,1,0)$	$\substack{1111111\\1} = (1,0,1,1,0,0,0,0,1)$
	$\substack{0122110\\1} = (1,1,0,0,0,1,0,1,0)$	$\substack{0121111\\1} = (1,1,0,0,1,0,0,0,1)$
9	$\substack{1222100\\1} = (1,0,0,1,0,1,1,0,0)$	$\substack{1221110\\1} = (1,0,0,1,1,0,0,1,0)$
	$\substack{1122110\\1} = (1,0,1,0,0,1,0,1,0)$	$\substack{1121111\\1} = (1,0,1,0,1,0,0,0,1)$
	$\substack{0122210\\1} = (1,1,0,0,0,0,1,1,0)$	$\substack{0122111\\1} = (1,1,0,0,0,1,0,0,1)$
10	$\substack{1232100\\1} = (1,0,0,0,1,1,1,0,0)$	$\substack{1222110\\1} = (1,0,0,1,0,1,0,1,0)$
	$\substack{1221111\\1} = (1,0,0,1,1,0,0,0,1)$	$\substack{1122210\\1} = (1,0,1,0,0,0,1,1,0)$
	$\substack{1122111\\1} = (1,0,1,0,0,1,0,0,1)$	$\substack{0122211\\1} = (1,1,0,0,0,0,1,0,1)$

11	$\genfrac{}{}{0pt}{}{1232100}{2} = (2,1,1,1,1,1,1,0,0)$	$\genfrac{}{}{0pt}{}{1232110}{1} = (1,0,0,0,1,1,0,1,0)$
	$\genfrac{}{}{0pt}{}{1222210}{1} = (1,0,0,1,0,0,1,1,0)$	$\genfrac{}{}{0pt}{}{1222111}{1} = (1,0,0,1,0,1,0,0,1)$
	$\genfrac{}{}{0pt}{}{1122211}{1} = (1,0,1,0,0,0,1,0,1)$	$\genfrac{}{}{0pt}{}{0122221}{1} = (1,1,0,0,0,0,0,1,1)$
12	$\genfrac{}{}{0pt}{}{1232110}{2} = (2,1,1,1,1,1,0,1,0)$	$\genfrac{}{}{0pt}{}{1232210}{1} = (1,0,0,0,1,0,1,1,0)$
	$\genfrac{}{}{0pt}{}{1232111}{1} = (1,0,0,0,1,1,0,0,1)$	$\genfrac{}{}{0pt}{}{1222211}{1} = (1,0,0,1,0,0,1,0,1)$
	$\genfrac{}{}{0pt}{}{1122221}{1} = (1,0,1,0,0,0,0,1,1)$	
13	$\genfrac{}{}{0pt}{}{1232210}{2} = (2,1,1,1,1,0,1,1,0)$	$\genfrac{}{}{0pt}{}{1232111}{2} = (2,1,1,1,1,1,0,0,1)$
	$\genfrac{}{}{0pt}{}{1233210}{1} = (1,0,0,0,0,1,1,1,0)$	$\genfrac{}{}{0pt}{}{1232211}{1} = (1,0,0,0,1,0,1,0,1)$
	$\genfrac{}{}{0pt}{}{1222221}{1} = (1,0,0,1,0,0,0,1,1)$	
14	$\genfrac{}{}{0pt}{}{1233210}{2} = (2,1,1,1,0,1,1,1,0)$	$\genfrac{}{}{0pt}{}{1232211}{2} = (2,1,1,1,1,0,1,0,1)$
	$\genfrac{}{}{0pt}{}{1233211}{1} = (1,0,0,0,0,1,1,0,1)$	$\genfrac{}{}{0pt}{}{1232221}{1} = (1,0,0,0,1,0,0,1,1)$
15	$\genfrac{}{}{0pt}{}{1243210}{2} = (2,1,1,0,1,1,1,1,0)$	$\genfrac{}{}{0pt}{}{1233211}{2} = (2,1,1,1,0,1,1,0,1)$
	$\genfrac{}{}{0pt}{}{1232221}{2} = (2,1,1,1,1,0,0,1,1)$	$\genfrac{}{}{0pt}{}{1233221}{1} = (1,0,0,0,0,1,0,1,1)$
16	$\genfrac{}{}{0pt}{}{1343210}{2} = (2,1,0,1,1,1,1,1,0)$	$\genfrac{}{}{0pt}{}{1243211}{2} = (2,1,1,0,1,1,1,0,1)$
	$\genfrac{}{}{0pt}{}{1233221}{2} = (2,1,1,1,0,1,0,1,1)$	$\genfrac{}{}{0pt}{}{1233321}{1} = (1,0,0,0,0,0,1,1,1)$
17	$\genfrac{}{}{0pt}{}{2343210}{2} = (2,0,1,1,1,1,1,1,0)$	$\genfrac{}{}{0pt}{}{1343211}{2} = (2,1,0,1,1,1,1,0,1)$
	$\genfrac{}{}{0pt}{}{1243221}{2} = (2,1,1,0,1,1,0,1,1)$	$\genfrac{}{}{0pt}{}{1233321}{2} = (2,1,1,1,0,0,1,1,1)$
18	$\genfrac{}{}{0pt}{}{2343211}{2} = (2,0,1,1,1,1,1,0,1)$	$\genfrac{}{}{0pt}{}{1343221}{2} = (2,1,0,1,1,1,0,1,1)$
	$\genfrac{}{}{0pt}{}{1243321}{2} = (2,1,1,0,1,0,1,1,1)$	
19	$\genfrac{}{}{0pt}{}{2343221}{2} = (2,0,1,1,1,1,0,1,1)$	$\genfrac{}{}{0pt}{}{1343321}{2} = (2,1,0,1,1,0,1,1,1)$
	$\genfrac{}{}{0pt}{}{1244321}{2} = (2,1,1,0,0,1,1,1,1)$	
20	$\genfrac{}{}{0pt}{}{2343321}{2} = (2,0,1,1,1,0,1,1,1)$	$\genfrac{}{}{0pt}{}{1344321}{2} = (2,1,0,1,0,1,1,1,1)$
21	$\genfrac{}{}{0pt}{}{2344321}{2} = (2,0,1,1,0,1,1,1,1)$	$\genfrac{}{}{0pt}{}{1354321}{2} = (2,1,0,0,1,1,1,1,1)$
22	$\genfrac{}{}{0pt}{}{2354321}{2} = (2,0,1,0,1,1,1,1,1)$	$\genfrac{}{}{0pt}{}{1354321}{3} = (3,2,1,1,1,1,1,1,1)$
23	$\genfrac{}{}{0pt}{}{2354321}{3} = (3,1,2,1,1,1,1,1,1)$	$\genfrac{}{}{0pt}{}{2454321}{2} = (2,0,0,1,1,1,1,1,1)$

24 $\genfrac{}{}{0pt}{}{2454321}{3} = (3,1,1,2,1,1,1,1,1)$

25 $\genfrac{}{}{0pt}{}{2464321}{3} = (3,1,1,1,2,1,1,1,1)$

26 $\genfrac{}{}{0pt}{}{2465321}{3} = (3,1,1,1,1,2,1,1,1)$

27 $\genfrac{}{}{0pt}{}{2465421}{3} = (3,1,1,1,1,1,2,1,1)$

28 $\genfrac{}{}{0pt}{}{2465431}{3} = (3,1,1,1,1,1,1,2,1)$

29 $\genfrac{}{}{0pt}{}{2465432}{3} = (3,1,1,1,1,1,1,1,2)$

TABLE 3. Weights of $V(\varpi_7)$.

27/2 $\frac{1}{2}\binom{246543}{3} = \genfrac{}{}{0pt}{}{000001}{0} = (\frac{3}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{3}{2})$

25/2 $\frac{1}{2}\binom{246541}{3} = \genfrac{}{}{0pt}{}{00001\bar{1}}{0} = (\frac{3}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{3}{2},\frac{1}{2})$

23/2 $\frac{1}{2}\binom{246521}{3} = \genfrac{}{}{0pt}{}{0001\bar{1}0}{0} = (\frac{3}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{3}{2},\frac{1}{2},\frac{1}{2})$

21/2 $\frac{1}{2}\binom{246321}{3} = \genfrac{}{}{0pt}{}{001\bar{1}00}{0} = (\frac{3}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{3}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$

19/2 $\frac{1}{2}\binom{244321}{3} = \genfrac{}{}{0pt}{}{01\bar{1}000}{1} = (\frac{3}{2},\frac{1}{2},\frac{1}{2},\frac{3}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$

17/2 $\frac{1}{2}\binom{244321}{1} = \genfrac{}{}{0pt}{}{010000}{\bar{1}} = (\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$

$\frac{1}{2}\binom{224321}{3} = \genfrac{}{}{0pt}{}{1\bar{1}0000}{1} = (\frac{3}{2},\frac{1}{2},\frac{3}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$

15/2 $\frac{1}{2}\binom{224321}{1} = \genfrac{}{}{0pt}{}{1\bar{1}1000}{\bar{1}} = (\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$

$\frac{1}{2}\binom{024321}{3} = \genfrac{}{}{0pt}{}{\bar{1}00000}{1} = (\frac{3}{2},\frac{3}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$

13/2 $\frac{1}{2}\binom{024321}{1} = \genfrac{}{}{0pt}{}{\bar{1}01000}{\bar{1}} = (\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$

$\frac{1}{2}\binom{222321}{1} = \genfrac{}{}{0pt}{}{10\bar{1}100}{0} = (\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$

11/2 $\frac{1}{2}\binom{022321}{1} = \genfrac{}{}{0pt}{}{\bar{1}1\bar{1}100}{0} = (\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$

$\frac{1}{2}\binom{222121}{1} = \genfrac{}{}{0pt}{}{100\bar{1}10}{0} = (\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2})$

9/2 $\frac{1}{2}\binom{002321}{1} = \genfrac{}{}{0pt}{}{0\bar{1}0100}{0} = (\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$

$\frac{1}{2}\binom{022121}{1} = \genfrac{}{}{0pt}{}{\bar{1}10\bar{1}10}{0} = (\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2})$

$\frac{1}{2}\binom{222101}{1} = \genfrac{}{}{0pt}{}{1000\bar{1}1}{0} = (\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2})$

7/2

$$\tfrac{1}{2}\binom{002121}{1} = \begin{matrix}0\overline{1}1\overline{1}10\\0\end{matrix} = (\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2})$$

$$\tfrac{1}{2}\binom{022101}{1} = \begin{matrix}\overline{1}100\overline{1}1\\0\end{matrix} = (\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{2},\tfrac{1}{2})$$

$$\tfrac{1}{2}\binom{22210\overline{1}}{1} = \begin{matrix}10000\overline{1}\\0\end{matrix} = (\tfrac{1}{2},-\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{2})$$

5/2

$$\tfrac{1}{2}\binom{000121}{1} = \begin{matrix}00\overline{1}010\\1\end{matrix} = (\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2})$$

$$\tfrac{1}{2}\binom{002101}{1} = \begin{matrix}0\overline{1}10\overline{1}1\\0\end{matrix} = (\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{2},\tfrac{1}{2})$$

$$\tfrac{1}{2}\binom{02210\overline{1}}{1} = \begin{matrix}\overline{1}1000\overline{1}\\0\end{matrix} = (\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{2})$$

3/2

$$\tfrac{1}{2}\binom{000121}{\overline{1}} = \begin{matrix}000010\\\overline{1}\end{matrix} = (-\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2})$$

$$\tfrac{1}{2}\binom{000101}{1} = \begin{matrix}00\overline{1}1\overline{1}1\\1\end{matrix} = (\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{2},\tfrac{1}{2})$$

$$\tfrac{1}{2}\binom{00210\overline{1}}{1} = \begin{matrix}0\overline{1}100\overline{1}\\0\end{matrix} = (\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{2})$$

1/2

$$\tfrac{1}{2}\binom{000101}{\overline{1}} = \begin{matrix}0001\overline{1}1\\\overline{1}\end{matrix} = (-\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{2},\tfrac{1}{2})$$

$$\tfrac{1}{2}\binom{000\overline{1}01}{1} = \begin{matrix}000\overline{1}01\\1\end{matrix} = (\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2},\tfrac{1}{2})$$

$$\tfrac{1}{2}\binom{00010\overline{1}}{1} = \begin{matrix}00\overline{1}10\overline{1}\\1\end{matrix} = (\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{2})$$

−1/2

$$\tfrac{1}{2}\binom{000\overline{1}01}{\overline{1}} = \begin{matrix}001\overline{1}01\\\overline{1}\end{matrix} = (-\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2},\tfrac{1}{2})$$

$$\tfrac{1}{2}\binom{00010\overline{1}}{\overline{1}} = \begin{matrix}00010\overline{1}\\\overline{1}\end{matrix} = (-\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{2})$$

$$\tfrac{1}{2}\binom{000\overline{1}0\overline{1}}{1} = \begin{matrix}000\overline{1}1\overline{1}\\1\end{matrix} = (\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{2})$$

−3/2

$$\tfrac{1}{2}\binom{00\overline{21}01}{\overline{1}} = \begin{matrix}01\overline{1}001\\0\end{matrix} = (-\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2},\tfrac{1}{2})$$

$$\tfrac{1}{2}\binom{000\overline{1}0\overline{1}}{\overline{1}} = \begin{matrix}001\overline{1}1\overline{1}\\\overline{1}\end{matrix} = (-\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{2})$$

$$\tfrac{1}{2}\binom{000\overline{121}}{1} = \begin{matrix}0000\overline{1}0\\1\end{matrix} = (\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2})$$

−5/2

$$\tfrac{1}{2}\binom{0\overline{221}01}{\overline{1}} = \begin{matrix}1\overline{1}0001\\0\end{matrix} = (-\tfrac{1}{2},-\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2},\tfrac{1}{2})$$

$$\tfrac{1}{2}\binom{00\overline{21}0\overline{1}}{\overline{1}} = \begin{matrix}01\overline{1}01\overline{1}\\0\end{matrix} = (-\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{2})$$

$$\tfrac{1}{2}\binom{000\overline{121}}{\overline{1}} = \begin{matrix}0010\overline{1}0\\\overline{1}\end{matrix} = (-\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2})$$

−7/2

$$\tfrac{1}{2}\binom{\overline{2221}01}{\overline{1}} = \begin{matrix}\overline{1}00001\\0\end{matrix} = (-\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2},\tfrac{1}{2})$$

$$\tfrac{1}{2}\binom{0\overline{221}0\overline{1}}{\overline{1}} = \begin{matrix}1\overline{1}001\overline{1}\\0\end{matrix} = (-\tfrac{1}{2},-\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{2})$$

$$\tfrac{1}{2}\binom{00\overline{2121}}{\overline{1}} = \begin{matrix}01\overline{1}1\overline{1}0\\0\end{matrix} = (-\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2})$$

−9/2

$$\tfrac{1}{2}\binom{\overline{2221}0\overline{1}}{\overline{1}} = \begin{matrix}\overline{1}0001\overline{1}\\0\end{matrix} = (-\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{2})$$

$$\tfrac{1}{2}\binom{0\overline{22121}}{\overline{1}} = \begin{matrix}1\overline{1}01\overline{1}0\\0\end{matrix} = (-\tfrac{1}{2},-\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2})$$

$$\tfrac{1}{2}\binom{00\overline{2321}}{\overline{1}} = \begin{matrix}010\overline{1}00\\0\end{matrix} = (-\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2})$$

−11/2

$$\tfrac{1}{2}\binom{\overline{222121}}{\overline{1}} = \begin{matrix}\overline{1}001\overline{1}0\\0\end{matrix} = (-\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2})$$

$$\tfrac{1}{2}\binom{0\overline{22321}}{\overline{1}} = \begin{matrix}1\overline{1}1\overline{1}00\\0\end{matrix} = (-\tfrac{1}{2},-\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2})$$

$\mathbf{-13/2}$	$\frac{1}{2}\begin{pmatrix}\overline{222321}\\ \bar{1}\end{pmatrix} = \begin{matrix}\bar{1}01\bar{1}00\\ 0\end{matrix} = (-\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})$
	$\frac{1}{2}\begin{pmatrix}0\overline{24321}\\ \bar{1}\end{pmatrix} = \begin{matrix}10\bar{1}000\\ 1\end{matrix} = (-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})$
$\mathbf{-15/2}$	$\frac{1}{2}\begin{pmatrix}\overline{224321}\\ \bar{1}\end{pmatrix} = \begin{matrix}\bar{1}1\bar{1}000\\ 1\end{matrix} = (-\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})$
	$\frac{1}{2}\begin{pmatrix}0\overline{24321}\\ \bar{3}\end{pmatrix} = \begin{matrix}100000\\ \bar{1}\end{matrix} = (-\frac{3}{2}, -\frac{3}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})$
$\mathbf{-17/2}$	$\frac{1}{2}\begin{pmatrix}\overline{224321}\\ \bar{3}\end{pmatrix} = \begin{matrix}\bar{1}10000\\ \bar{1}\end{matrix} = (-\frac{3}{2}, -\frac{1}{2}, -\frac{3}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})$
	$\frac{1}{2}\begin{pmatrix}\overline{244321}\\ \bar{1}\end{pmatrix} = \begin{matrix}0\bar{1}0000\\ 1\end{matrix} = (-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})$
$\mathbf{-19/2}$	$\frac{1}{2}\begin{pmatrix}\overline{244321}\\ \bar{3}\end{pmatrix} = \begin{matrix}0\bar{1}1000\\ \bar{1}\end{matrix} = (-\frac{3}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})$
$\mathbf{-21/2}$	$\frac{1}{2}\begin{pmatrix}\overline{246321}\\ \bar{3}\end{pmatrix} = \begin{matrix}00\bar{1}100\\ 0\end{matrix} = (-\frac{3}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})$
$\mathbf{-23/2}$	$\frac{1}{2}\begin{pmatrix}\overline{246521}\\ \bar{3}\end{pmatrix} = \begin{matrix}000\bar{1}10\\ 0\end{matrix} = (-\frac{3}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}, -\frac{1}{2}, -\frac{1}{2})$
$\mathbf{-25/2}$	$\frac{1}{2}\begin{pmatrix}\overline{246541}\\ \bar{3}\end{pmatrix} = \begin{matrix}0000\bar{1}1\\ 0\end{matrix} = (-\frac{3}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}, -\frac{1}{2})$
$\mathbf{-27/2}$	$\frac{1}{2}\begin{pmatrix}\overline{246543}\\ \bar{3}\end{pmatrix} = \begin{matrix}00000\bar{1}\\ 0\end{matrix} = (-\frac{3}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{3}{2})$

TABLE 4. Weights of $V(\varpi_7)$ as roots of E_8.

1	$\begin{matrix}0000001\\ 0\end{matrix} = (0,0,0,0,0,0,0,-1,1)$	
2	$\begin{matrix}0000011\\ 0\end{matrix} = (0,0,0,0,0,0,-1,0,1)$	
3	$\begin{matrix}0000111\\ 0\end{matrix} = (0,0,0,0,0,-1,0,0,1)$	
4	$\begin{matrix}0001111\\ 0\end{matrix} = (0,0,0,0,-1,0,0,0,1)$	
5	$\begin{matrix}0011111\\ 0\end{matrix} = (0,0,0,-1,0,0,0,0,1)$	
6	$\begin{matrix}0011111\\ 1\end{matrix} = (1,1,1,0,0,0,0,0,1)$	$\begin{matrix}0111111\\ 0\end{matrix} = (0,0,-1,0,0,0,0,0,1)$
7	$\begin{matrix}1111111\\ 0\end{matrix} = (0,-1,0,0,0,0,0,0,1)$	$\begin{matrix}0111111\\ 1\end{matrix} = (1,1,0,1,0,0,0,0,1)$
8	$\begin{matrix}1111111\\ 1\end{matrix} = (1,0,1,1,0,0,0,0,1)$	$\begin{matrix}0121111\\ 1\end{matrix} = (1,1,0,0,1,0,0,0,1)$
9	$\begin{matrix}1121111\\ 1\end{matrix} = (1,0,1,0,1,0,0,0,1)$	$\begin{matrix}0122111\\ 1\end{matrix} = (1,1,0,0,0,1,0,0,1)$
10	$\begin{matrix}1221111\\ 1\end{matrix} = (1,0,0,1,1,0,0,0,1)$	$\begin{matrix}1122111\\ 1\end{matrix} = (1,0,1,0,0,1,0,0,1)$
	$\begin{matrix}0122211\\ 1\end{matrix} = (1,1,0,0,0,0,1,0,1)$	
11	$\begin{matrix}1222111\\ 1\end{matrix} = (1,0,0,1,0,1,0,0,1)$	$\begin{matrix}1122211\\ 1\end{matrix} = (1,0,1,0,0,0,1,0,1)$
	$\begin{matrix}0122221\\ 1\end{matrix} = (1,1,0,0,0,0,0,1,1)$	
12	$\begin{matrix}1232111\\ 1\end{matrix} = (1,0,0,0,1,1,0,0,1)$	$\begin{matrix}1222211\\ 1\end{matrix} = (1,0,0,1,0,0,1,0,1)$
	$\begin{matrix}1122221\\ 1\end{matrix} = (1,0,1,0,0,0,0,1,1)$	

13 $\substack{1232111\\2} = (2,1,1,1,1,1,0,0,1)$ $\substack{1232211\\1} = (1,0,0,0,1,0,1,0,1)$
$\substack{1222221\\1} = (1,0,0,1,0,0,0,1,1)$

14 $\substack{1232211\\2} = (2,1,1,1,1,0,1,0,1)$ $\substack{1233211\\1} = (1,0,0,0,0,1,1,0,1)$
$\substack{1232221\\1} = (1,0,0,0,1,0,0,1,1)$

15 $\substack{1233211\\2} = (2,1,1,1,0,1,1,0,1)$ $\substack{1232221\\2} = (2,1,1,1,1,0,0,1,1)$
$\substack{1233221\\1} = (1,0,0,0,0,1,0,1,1)$

16 $\substack{1243211\\2} = (2,1,1,0,1,1,1,0,1)$ $\substack{1233221\\2} = (2,1,1,1,0,1,0,1,1)$
$\substack{1233321\\1} = (1,0,0,0,0,0,1,1,1)$

17 $\substack{1343211\\2} = (2,1,0,1,1,1,1,0,1)$ $\substack{1243221\\2} = (2,1,1,0,1,1,0,1,1)$
$\substack{1233321\\2} = (2,1,1,1,0,0,1,1,1)$

18 $\substack{2343211\\2} = (2,0,1,1,1,1,1,0,1)$ $\substack{1343221\\2} = (2,1,0,1,1,1,0,1,1)$
$\substack{1243321\\2} = (2,1,1,0,1,0,1,1,1)$

19 $\substack{2343221\\2} = (2,0,1,1,1,1,0,1,1)$ $\substack{1343321\\2} = (2,1,0,1,1,0,1,1,1)$
$\substack{1244321\\2} = (2,1,1,0,0,1,1,1,1)$

20 $\substack{2343321\\2} = (2,0,1,1,1,0,1,1,1)$ $\substack{1344321\\2} = (2,1,0,1,0,1,1,1,1)$

21 $\substack{2344321\\2} = (2,0,1,1,0,1,1,1,1)$ $\substack{1354321\\2} = (2,1,0,0,1,1,1,1,1)$

22 $\substack{2354321\\2} = (2,0,1,0,1,1,1,1,1)$ $\substack{1354321\\3} = (3,2,1,1,1,1,1,1,1)$

23 $\substack{2354321\\3} = (3,1,2,1,1,1,1,1,1)$ $\substack{2454321\\2} = (2,0,0,1,1,1,1,1,1)$

24 $\substack{2454321\\3} = (3,1,1,2,1,1,1,1,1)$

25 $\substack{2464321\\3} = (3,1,1,1,2,1,1,1,1)$

26 $\substack{2465321\\3} = (3,1,1,1,1,2,1,1,1)$

27 $\substack{2465421\\3} = (3,1,1,1,1,1,2,1,1)$

28 $\substack{2465431\\3} = (3,1,1,1,1,1,1,2,1)$

2. The weights of E_7

The fundamental weights of the root system of type E_7 are now generated in exactly the same way as in [18, §5], for the case of E_6. By definition, the base of fundamental weights `weightbaseE7` is dual to the base of fundamental roots `rootbaseE7`. As in [18], the only subtlety is that in the hyperbolic realization the roots of E_7 live in an 8-dimensional rather than in a 7-dimensional space. To get rid of an extra-dimension, in constructing the weights it is necessary to check the orthogonality — with respect to the usual *Euclidean* inner product! — of the resulting vectors to the following test vector:

```
testvectorE7={3,-1,-1,-1,-1,-1,-1,-1}.
```

Thus, the fundamental weights $\varpi_1, \ldots, \varpi_7$ are solutions of the following systems of linear equations:

```
omegaE7[i_]:=LinearSolve[Append[rootbasebisE7,testvectorE7],
              Table[If[j==i,1,0],{j,1,8}]] /; 1<=i<=7.
```

The list `rootbasebisE7` differs from the list `rootbaseE7` in exactly one position. Namely, its second component is `{-1,1,1,1,0,0,0,0}`. The objective is to ensure that the solution is orthogonal to α_2 in the sense of the *hyperbolic* inner product. Since all other α_i's do not contain e_0, for them it does not matter whether we take the Euclidean or the hyperbolic inner product, so we do not have to modify them. Now the evaluation of

```
weightbaseE7=Table[omegaE7[i],{i,7}]
```

returns the coordinates of the fundamental weights in the orthonormal base of the space $\mathbb{R}^{7,1}$:

```
{{2,0,1,1,1,1,1,1},{7/2,3/2,3/2,3/2,3/2,3/2,3/2,3/2},
{4,1,1,2,2,2,2,2},{6,2,2,2,3,3,3,3},
{9/2,3/2,3/2,3/2,3/2,5/2,5/2,5/2},{3,1,1,1,1,1,2,2}},
{3/2,1/2,1/2,1/2,1/2,1/2,1/2,3/2}}.
```

A dominant (integer) weight is precisely a linear combination of fundamental weights with nonnegative integer coefficients. Equivalently, one could stipulate that all of its inner products with fundamental roots are nonnegative integers. The following function gives a test for a weight to be dominant:

```
dominantE7Q[u_]:=And@@Table[Block[
                        {xxx=hip[u,rootbaseE7[[i]]]},
                        xxx>=0&&IntegerQ[xxx]],{i,7}].
```

Here, as in [18], the function `hip` denotes the hyperbolic inner product in $\mathbb{R}^{l,1}$.

As in [18], the set of weights $\overline{\Lambda}(\omega)$ of the representation with the highest weight ω is generated by the following clumsy code:

```
weightsE7[u_]:=Block[{i,j,list={u},len=hip[u],we},
        For[j=1,j<=Length[list],j++,
        For[i=1,i<=7,i++,we=list[[j]]-rootbaseE7[[i]];
                If[hip[we]<=len&&!MemberQ[list,we],
                        list=Append[list,we]]]];
        Return[list]] /; dominantE7Q[u]
```

As we noted in [18], such a procedural algorithm is dreadfully inefficient, so that the generation of 56 weights of the representation $V(\omega_7)$ requires preposterously long time of 0.05 seconds. Since we only do it once, we decided to leave it as is, for the time being.

3. The 56-dimensional module for E_7

Now the evaluation of

```
minimalE7=weightsE7[omegaE7[7]]
```

returns the list of 56 weights of the minimal module. These weights are listed in Table 3, together with their expression in Dynkin form with respect to the base of fundamental roots and to the base of fundamental weights. These expressions are obtained as solutions of the following systems of linear equations:

```
rootformE7[u_]:=LinearSolve[Transpose[rootbaseE7],u]
weightformE7[u_]:=LinearSolve[Transpose[weightbaseE7],u].
```

Observe that, as opposed to the positive roots, the weights of $V(\varpi_7)$ are listed in this table not in increasing order, but in the *decreasing* order, starting with the highest weight ϖ_7, which has height 27/2, and ending with $-\varpi_7$, which has height $-27/2$.

However, usually it is more convenient to use for calculations the realization of $V(\varpi_7)$ as an internal Chevalley module in the group $G(\mathrm{E}_8, R)$. Namely, consider the standard parabolic subgroup $P_8 \leq G(\mathrm{E}_8, R)$. The commutator subgroup $G_8 = [L_8, L_8]$ of its Levi subgroup is the simply connected Chevalley group of type E_7, and the conjugation action of G_8 on the abelianization $U_8/[U_8, U_8]$ of the unipotent radical U_8 makes $U_8/[U_8, U_8]$ into a $G(\mathrm{E}_7, R)$-module $V(\varpi_7)$. Now, Theorem 1 of [134] implies that the elements $x_\alpha(1)[U_8, U_8]$, $\alpha \in \Sigma = \Sigma_8(1)$, corresponding to a positive Chevalley base, form a crystal base of the module $V(\varpi_7)$. It follows that we can read off the action structure constants simply by picking out the rows corresponding to the roots of E_7, and the columns corresponding to the roots from Σ, from the structure constants table for Lie algebra of type E_8.

Choose among positive roots of E_8 those belonging to E_7, and those belonging to Σ:

```
positiveE7insideE8=Select[positiveE8,
Last[rootformE8[#]]==0&]
```

```
minimalE7insideE8=Select[positiveE8,Last[rootformE8[#]]==1&]
```

Observe, though, that the roots of E_7 inside E_8 *in the same order* could be obtained simply by appending 0 to the roots of E_7, in the usual order:

```
positiveE7insideE8=Map[Append[#,0]&,positiveE7]
```

The weights of $V(\varpi_7)$ as roots of E_8 are listed, in accordance with increasing height, in Table 4. Clearly, this table is much more convenient for further use, since now both in their expansion in the fundamental base of E_8, and in the corresponding hyperbolic base all the coefficients are nonnegative integers.

TABLE 5. Structure constants of $G(\mathrm{E}_7, R)$.

	1 0 0 0 0 0 1 0 0 0 0 0 1 0 0 0 0 0 1 0 0 0 0 0 1 0 0 0 0 0 0 0 0 0 0	0 0 1 0 0 0 0 0 1 0 0 0 1 0 1 0 0 0 1 1 0 0 0 0 0 1 0 0 0 0 0 1 0 0 0	0 0 0 1 0 0 0 0 0 1 0 0 0 1 1 1 0 0 1 1 1 1 0 0 0 0 1 1 0 0 0 0 1 0 0	0 0 0 0 1 1 0 0 0 1 0 1 0 0 1 1 1 1 0 1 1 1 1 1 0 0 0 1 1 0 0 0 0 1 0	1 0 0 0 0 0 1 1 0 0 1 1 0 1 0 1 1 1 1 1 1 1 1 1 1 0 0 1 1 1 0 0 0 0 1	1 1 0 0 0 1 0 1 1 1 1 1 1 1 0 1 1 2 1 1 1 1 1 1 1 0 1 0 1 1 0 0 0 0 1	0 1 0 1 1 1 1 2 1 1 1 0 1 0
1000000	0 0 + 0 0	0 0 0 0 +	0 0 0 0 +	0 + 0 0 0	0 + 0 + 0	0 0 + + 0	+ 0
0100000	0 0 0 + 0	0 0 0 0 +	+ 0 0 + 0	0 + + 0 0	+ 0 0 + +	0 + 0 0 0	+ 0
0010000	− 0 0 + 0	0 0 0 + 0	+ 0 0 0 0	+ 0 + 0 0	0 0 + 0 +	0 0 0 0 +	0 +
0001000	0 − − 0 +	0 0 − 0 0	0 + 0 0 0	0 0 0 + 0	0 + 0 0 0	+ 0 0 + 0	0 0
0000100	0 0 0 − 0	+ 0 0 − −	0 0 + − −	0 0 0 0 −	0 0 0 0 0	0 0 0 0 0	0 0
0000010	0 0 0 0 −	0 + 0 0 0	− 0 0 0 0	− − 0 0 0	− − 0 0 0	− 0 − 0 0	0 −
0000001	0 0 0 0 0	− 0 0 0 0	0 − 0 0 0	0 0 − 0 0	0 0 − − 0	0 − 0 − 0	0 0
1010000	0 0 0 + 0	0 0 0 + 0	+ 0 0 0 0	+ 0 + 0 0	0 0 + 0 +	0 0 − 0 +	0 0
0101000	0 0 − 0 +	0 0 − 0 0	0 + 0 0 0	0 − 0 + 0	− 0 0 − 0	0 − 0 0 0	− 0
0011000	− − 0 0 +	0 0 0 0 0	0 + 0 0 0	− 0 0 + 0	0 0 − 0 0	+ 0 0 0 −	0 0
0001100	0 − − 0 0	+ 0 − 0 0	0 0 + 0 −	0 0 0 0 −	0 0 0 0 0	0 0 0 − 0	0 0
0000110	0 0 0 − 0	0 + 0 − −	0 0 0 − −	0 0 0 0 −	0 0 0 0 0	0 0 − 0 0	0 −
0000011	0 0 0 0 −	0 0 0 0 0	− 0 0 0 0	− − 0 0 0	− − 0 0 0	− 0 − 0 0	0 −
1011000	0 − 0 0 +	0 0 0 0 0	0 + 0 0 0	− 0 0 + 0	0 − − 0 0	0 0 0 − −	0 0
0111000	− 0 0 0 +	0 0 0 0 0	+ + 0 0 0	0 0 + + 0	− 0 0 0 +	0 − 0 0 0	0 0
0101100	0 0 − 0 0	+ 0 − 0 +	0 0 + + 0	0 0 0 0 0	0 0 0 + 0	0 + 0 0 0	+ 0
0011100	− − 0 0 0	+ 0 0 + 0	0 0 + 0 0	0 0 0 0 −	0 0 + 0 0	0 0 0 0 +	0 0
0001110	0 − − 0 0	0 + − 0 0	0 0 0 0 −	0 0 0 0 −	0 + 0 0 0	+ 0 0 0 0	0 0
0000111	0 0 0 − 0	0 0 0 − −	0 0 0 − −	0 0 0 0 −	0 0 0 0 0	0 0 − 0 0	0 −
1111000	0 0 0 0 +	0 0 0 0 0	+ + 0 0 0	0 + + + 0	0 0 0 + +	0 0 0 0 0	+ 0
1011100	0 − 0 0 0	+ 0 0 + 0	0 0 + 0 +	0 0 0 0 0	0 0 + 0 0	0 0 0 + +	0 0
0111100	− 0 0 − 0	+ 0 0 0 0	0 0 + + 0	0 0 − 0 0	0 0 0 0 −	0 + 0 0 0	0 0
0101110	0 0 − 0 0	0 + − 0 +	0 0 0 + 0	0 − 0 0 0	− 0 0 0 0	0 0 0 0 0	− 0
0011110	− − 0 0 0	0 + 0 + 0	0 0 0 0 0	− 0 0 0 −	0 0 0 0 0	+ 0 0 0 −	0 +
0001111	0 − − 0 0	0 0 − 0 0	0 0 0 0 −	0 0 0 0 −	0 + 0 0 0	+ 0 0 − 0	0 0
1111100	0 0 0 − 0	+ 0 0 0 −	0 0 + 0 0	0 0 − 0 0	0 0 0 − −	0 0 0 0 0	− 0
1011110	0 − 0 0 0	0 + 0 + 0	0 0 0 0 +	− 0 0 0 0	0 − 0 0 0	0 0 − 0 −	0 0
0112100	− 0 0 0 0	+ 0 + 0 0	0 + + 0 0	0 0 0 + 0	0 0 0 0 0	0 + 0 0 0	0 0
0111110	− 0 0 − 0	0 + 0 0 0	+ 0 0 + 0	0 0 0 0 0	− 0 0 0 +	0 0 0 0 0	0 +
0101111	0 0 − 0 0	0 0 − 0 +	0 0 0 + 0	0 − 0 0 0	− 0 0 + 0	0 + 0 0 0	0 0
0011111	− − 0 0 0	0 0 0 + 0	0 0 0 0 0	− 0 0 0 −	0 0 + 0 0	+ 0 0 0 0	0 +
1112100	0 0 − 0 0	+ 0 0 0 0	0 + + 0 0	0 0 0 + 0	0 0 0 − 0	0 0 0 − 0	− 0

Table 5 (continued).

	1111110	1011111	0112110	0111111	1122100	1112110	1111111	0112210	0112111	1122110	1112210	1112111	0112211	1122210	1122111	1112211	0112221	1123210	1122211	1112221	1223210	1123211	1122221	1223211	1123221	1223221	1123321	1223321	1224321	1234321	2234321
1000000	0	0	+	+	0	0	0	+	+	0	0	0	+	0	0	0	+	0	0	0	0	0	0	0	0	0	0	0	0	+	0
0100000	0	+	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	+	0	0	0	+	0	0	+	0	+	0	0	0	0
0010000	0	0	0	0	0	+	0	0	0	0	+	+	0	0	0	+	0	0	0	+	0	0	0	0	0	0	0	0	+	0	0
0001000	+	0	0	+	0	0	+	0	0	0	0	0	0	+	0	0	0	0	+	0	0	0	+	0	0	0	0	+	0	0	0
0000100	0	0	+	0	0	+	0	0	+	+	0	+	0	0	+	0	0	0	0	0	0	0	0	0	+	+	0	0	0	0	0
0000010	0	0	0	0	−	0	0	0	0	0	0	0	+	0	0	+	0	0	+	0	0	+	0	+	0	0	0	0	0	0	0
0000001	−	0	−	0	0	−	0	−	0	−	−	0	0	−	0	0	0	−	0	0	−	0	0	0	0	0	0	0	0	0	0
1010000	0	0	−	0	0	0	0	−	−	0	0	0	−	0	0	0	−	0	0	0	0	0	0	0	0	0	0	0	+	0	0
0101000	0	−	0	0	0	0	0	0	0	0	0	0	0	+	0	0	0	0	+	0	0	0	+	0	0	0	−	0	0	0	0
0011000	+	0	0	0	0	0	+	0	0	0	−	0	0	0	0	−	0	0	0	−	0	0	0	0	0	0	0	+	0	0	0
0001100	−	0	0	−	0	0	−	0	0	+	0	0	0	0	+	0	0	0	0	0	0	0	−	0	0	+	0	0	0	0	0
0000110	0	0	0	0	−	0	0	0	−	0	0	−	0	0	−	0	0	0	0	0	0	+	0	+	0	0	0	0	0	0	0
0000011	0	0	0	0	−	0	0	−	0	0	−	0	0	−	0	0	0	−	0	0	−	0	0	0	0	0	0	0	0	0	0
1011000	0	0	0	−	0	0	0	+	0	0	0	0	+	0	0	0	+	0	0	0	0	0	0	0	0	0	0	+	0	0	0
0111000	0	-1	0	0	0	0	0	0	0	0	−	0	0	0	0	−	0	0	0	−	0	0	0	0	0	0	−	0	0	0	0
0101100	0	+	0	0	0	0	0	0	0	+	0	0	0	0	+	0	0	0	0	0	0	0	−	0	−	0	0	0	0	0	0
0011100	−	0	0	0	0	−	−	0	0	0	0	−	0	0	0	0	0	0	0	+	0	0	0	0	0	+	0	0	0	0	0
0001110	0	0	0	+	−	0	+	0	0	0	0	0	0	0	−	0	0	0	−	0	0	0	0	+	0	0	0	0	0	0	0
0000111	0	0	+	0	−	+	0	0	0	+	0	0	0	0	0	0	0	−	0	0	−	0	0	0	0	0	0	0	0	0	0
1111000	0	0	0	0	0	0	0	+	0	0	0	0	+	0	0	0	+	0	0	0	0	0	0	0	0	0	−	0	0	0	0
1011100	0	0	+	+	0	0	0	0	+	0	0	0	0	0	0	0	−	0	0	0	0	0	0	0	0	+	0	0	0	0	0
0111100	0	+	0	0	0	−	0	0	0	0	0	−	0	0	0	0	0	0	0	+	0	0	0	0	−	0	0	0	0	0	0
0101110	0	−	0	0	−	0	0	0	0	0	0	0	0	0	−	0	0	0	−	0	0	−	0	0	0	0	0	0	0	0	0
0011110	0	0	0	0	0	0	+	0	0	0	0	+	0	0	0	+	0	0	0	0	0	0	0	+	0	0	0	0	0	0	0
0001111	−	0	0	0	−	0	0	0	0	+	0	0	0	+	0	0	0	0	0	0	−	0	0	0	0	0	0	0	0	0	0
1111100	0	0	+	0	0	0	0	0	+	0	0	0	0	0	0	0	−	0	0	0	0	0	0	0	−	0	0	0	0	0	0
1011110	0	0	0	−	0	0	0	0	−	0	0	0	−	0	0	0	0	0	0	0	0	0	0	+	0	0	0	0	0	0	0
0112100	+	+	0	0	0	0	+	0	0	0	0	0	0	0	0	0	0	0	0	+	0	0	+	0	0	0	0	0	0	0	0
0111110	0	−	0	0	0	0	0	0	0	0	0	+	0	0	0	+	0	0	0	0	0	−	0	0	0	0	0	0	0	0	0
0101111	0	0	0	0	−	0	0	0	0	+	0	0	0	+	0	0	0	+	0	0	0	0	0	0	0	0	0	0	0	0	0
0011111	−	0	0	0	0	−	0	0	0	0	−	0	0	0	0	0	0	0	0	0	−	0	0	0	0	0	0	0	0	0	0
1112100	0	0	0	−	0	0	0	0	0	0	0	0	0	0	0	0	−	0	0	0	0	0	+	0	0	0	0	0	0	0	0

As a matter of fact, we performed all calculations precisely in this realization. In particular, to construct the action structure constants it suffices to pick out the rows with indices corresponding to the positions of members of `positiveE7insideE8`, and the columns with indices corresponding to the positions of members of `minimalE7insideE8`, in the list `positiveE8`, from the structure constant table for E_8, constructed in [135]. The result is reproduced in Table 6. Since $c_{\lambda+\alpha,-\alpha} = c_{\lambda,\alpha}$, the action structure constants for negative roots are also read off from this table.

As it happens, in many practical calculations it is more convenient to use not the table of action structure constants, but rather the **matrix of signs** of the representation $V(\varpi_7)$, whose rows and columns are indexed by weights of this representation. The entry of this matrix in the position (λ, μ) equals the constant $c_{\lambda,\mu-\lambda}$, with which the root element $x_{\mu-\lambda}(1)$ adds v^λ to v^μ, or 0 if $\mu - \lambda$ is not a root. The matrix of signs for the representation $V(\varpi_7)$ for the numbering of weights as roots of E_8 is reproduced in Table 7.

What we call the *natural* numbering of weights is obtained from the list `positiveE7insideE8` by an application of `Reverse`:

```
minimalE7natural = Reverse[minimalE7insideE8];
```

The reason is that we number weights starting with the highest one, as is customary in algebra, rather than starting with the smallest one, as is typical of `Computer Science`.

Table 5 (continued).

	1 0 0 0 0 0 1 0 0 0 0 0 1 0 0 0 0 0 1 0 0 0 0 0 1 0 0 0 0 0 0 0 0 0 0	0 0 1 0 0 0 0 0 1 0 0 0 1 0 1 0 0 0 1 1 0 0 0 0 0 1 0 0 0 0 0 1 0 0 0	0 0 0 1 0 0 0 0 0 1 0 0 0 1 1 1 0 0 1 1 1 1 0 0 0 0 1 1 0 0 0 0 1 0 0	0 0 0 0 1 1 0 0 0 1 0 1 0 0 1 1 1 1 0 1 1 1 1 1 0 0 0 1 1 0 0 0 0 1 0	1 0 0 0 0 0 1 1 0 0 1 1 0 1 0 1 1 1 1 1 1 1 1 1 1 0 0 1 1 1 0 0 0 0 1	1 1 0 0 0 1 0 1 1 1 1 1 1 1 0 1 1 2 1 1 1 1 1 1 1 0 1 0 1 1 0 0 0 0 1	0 1 0 1 1 1 1 2 1 1 1 0 1 0
1111110	0 0 0 − 0	0 + 0 0 −	+ 0 0 0 0	0 + 0 0 0	0 0 0 0 +	0 0 − 0 0	+ 0
1011111	0 − 0 0 0	0 0 0 + 0	0 0 0 0 +	− 0 0 0 0	0 − + 0 0	0 0 − + 0	0 0
0112110	− 0 0 0 −	0 + + 0 0	0 0 0 0 0	0 0 0 − 0	− 0 0 0 0	− 0 0 0 0	0 0
0111111	− 0 0 − 0	0 0 0 0 0	+ 0 0 + 0	0 0 − 0 0	− 0 0 0 0	0 + 0 0 0	0 1
1122100	0 0 0 0 0	+ 0 0 0 0	0 + + 0 0	0 0 + + 0	0 0 + 0 +	0 0 0 0 +	0 0
1112110	0 0 − 0 −	0 + 0 0 0	0 0 0 0 0	0 + 0 − 0	0 + 0 0 0	0 0 0 0 0	+ 0
1111111	0 0 0 − 0	0 0 0 0 −	+ 0 0 0 0	0 + − 0 0	0 0 0 − 0	0 0 − 0 0	0 0
0112210	− 0 0 0 0	0 + + 0 0	0 0 + − 0	0 0 0 0 −	0 0 0 0 0	0 0 0 0 0	0 0
0112111	− 0 0 0 −	0 0 + 0 0	0 + 0 0 0	0 0 0 0 0	− 0 0 0 0	− + 0 0 0	0 0
1122110	0 0 0 0 −	0 + 0 0 0	− 0 0 0 0	− 0 0 − 0	0 0 0 0 −	0 0 0 0 −	0 0
1112210	0 0 − 0 0	0 + 0 0 +	0 0 + 0 +	0 0 0 0 0	0 0 0 0 0	0 0 0 0 0	+ 0
1112111	0 0 − 0 −	0 0 0 0 0	0 + 0 0 0	0 + 0 0 0	0 + 0 − 0	0 0 0 − 0	0 0
0112211	− 0 0 0 0	− 0 + 0 0	0 0 0 − 0	0 0 0 0 −	0 0 0 0 0	0 + 0 0 0	0 0
1122210	0 0 0 − 0	0 + 0 − 0	0 0 + 0 0	0 0 0 0 0	0 0 0 0 −	0 0 0 0 −	0 0
1122111	0 0 0 0 −	0 0 0 0 0	− + 0 0 0	− 0 + 0 0	0 0 + 0 0	0 0 0 0 0	0 0
1112211	0 0 − 0 0	− 0 0 0 +	0 0 0 0 +	0 0 0 0 0	0 0 0 − 0	0 0 0 − 0	0 0
0112221	− 0 0 0 0	0 0 + 0 0	0 0 0 − 0	0 0 0 0 −	+ 0 0 0 0	+ 0 0 0 0	0 +
1123210	0 − 0 0 0	0 + 0 0 0	0 0 + 0 0	0 0 0 + 0	0 0 0 0 0	0 0 0 0 −	0 0
1122211	0 0 0 − 0	− 0 0 − 0	0 0 0 0 0	0 0 + 0 0	0 0 + 0 0	0 0 0 0 0	0 0
1112221	0 0 − 0 0	0 0 0 0 +	0 0 0 0 +	0 − 0 0 0	0 − 0 0 0	0 0 − 0 0	0 0
1223210	0 0 0 0 0	0 + 0 0 0	0 0 + 0 0	0 0 0 + 0	0 0 0 0 +	0 0 0 0 0	+ 0
1123211	0 − 0 0 0	− 0 0 0 0	0 − 0 0 0	0 0 0 0 0	0 0 + 0 0	0 0 0 + 0	0 0
1122221	0 0 0 − 0	0 0 0 − 0	+ 0 0 0 0	+ 0 0 0 0	0 0 0 0 0	0 0 − 0 0	0 −
1223211	0 0 0 0 0	− 0 0 0 0	0 − 0 0 0	0 0 − 0 0	0 0 0 − 0	0 − 0 0 0	0 0
1123221	0 − 0 0 −	0 0 0 0 0	0 0 0 0 0	+ 0 0 0 0	0 + 0 0 0	+ 0 0 0 0	0 0
1223221	0 0 0 0 −	0 0 0 0 0	− 0 0 0 0	0 − 0 0 0	− 0 0 0 0	0 0 0 0 0	0 0
1123321	0 − 0 0 0	0 0 0 + 0	0 0 0 0 +	0 0 0 0 +	0 0 0 0 0	0 0 0 0 0	0 0
1223321	0 0 0 − 0	0 0 0 0 −	0 0 0 − 0	0 0 0 0 0	0 0 0 0 0	0 0 0 0 0	0 0
1224321	0 0 − 0 0	0 0 − 0 0	0 0 0 0 0	0 0 0 0 0	0 0 0 0 0	0 0 0 0 0	0 0
1234321	− 0 0 0 0	0 0 0 0 0	0 0 0 0 0	0 0 0 0 0	0 0 0 0 0	0 0 0 0 0	0 0
2234321	0 0 0 0 0	0 0 0 0 0	0 0 0 0 0	0 0 0 0 0	0 0 0 0 0	0 0 0 0 0	0 0

Table 5 (continued).

	1111110	1011111	0112110	0111111	1122100	1112110	1111111	0112210	0112111	1122110	1112210	1112111	0112211	1122210	1122111	1112211	0112221	1123210	1122211	1112221	1223210	1123211	1122221	1223211	1123221	1223221	1123321	1223321	1224321	1234321	2234321
1111110	0	0	0	0	0	0	0	0	−	0	0	0	−	0	0	0	0	0	0	0	0	−	0	0	0	0	0	0	0	0	0
1011111	0	0	+	0	0	0	0	+	0	0	0	0	0	0	0	0	0	0	0	0	−	0	0	0	0	0	0	0	0	0	0
0112110	0	−	0	0	0	0	−	0	0	0	0	0	0	0	0	+	0	0	+	0	0	0	0	0	0	0	0	0	0	0	0
0111111	0	0	0	0	0	−	0	0	0	0	−	0	0	0	0	0	0	+	0	0	0	0	0	0	0	0	0	0	0	0	0
1122100	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	−	0	0	−	0	0	0	0	0	0	0	0	0	0	0
1112110	0	0	0	+	0	0	0	0	0	0	0	0	−	0	0	0	0	0	+	0	0	0	0	0	0	0	0	0	0	0	0
1111111	0	0	+	0	0	0	0	+	0	0	0	0	0	0	0	0	0	+	0	0	0	0	0	0	0	0	0	0	0	0	0
0112210	0	−	0	0	0	0	−	0	0	0	0	−	0	0	−	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0112111	+	0	0	0	0	0	0	0	0	0	−	0	0	−	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1122110	0	0	0	0	0	0	0	0	0	0	0	0	−	0	0	−	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1112210	0	0	0	+	0	0	0	0	+	0	0	0	0	0	−	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1112111	0	0	0	0	0	0	0	+	0	0	0	0	0	−	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0112211	+	0	0	0	0	+	0	0	0	+	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1122210	0	0	0	0	0	0	0	0	+	0	0	+	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1122111	0	0	0	0	0	0	0	+	0	0	+	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1112211	0	0	−	0	0	0	0	0	0	+	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0112221	0	0	0	0	+	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1123210	0	0	0	−	0	0	−	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1122211	0	0	−	0	0	−	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1112221	0	0	0	0	+	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1223210	0	+	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1123211	+	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1122221	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1223211	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1123221	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1223221	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1123321	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1223321	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1224321	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1234321	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2234321	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

4. The weight diagram of $V(\varpi_7)$

An important tool in the study and use of representations of exceptional groups are their *weight diagrams*. A detailed discussion of weight diagrams and many further references can be found in [133, 112, 134, 105, 126].

Recall that the **weight diagram** of a representation π is a marked oriented graph – or, in Kashiwara's terminology, a *color* graph, constructed as follows.

- Its vertices correspond to the weights of the representation π, usually with multiplicities.
- Two vertices λ and μ are joined by an arrow with **mark** $i =$ of **color** i directed from μ to λ, provided that $\lambda - \mu = \alpha_i$ is the ith fundamental root.

The arrowheads are usually not set, but the whole diagram is read in positive direction, usually from right to left, and bottom-up.

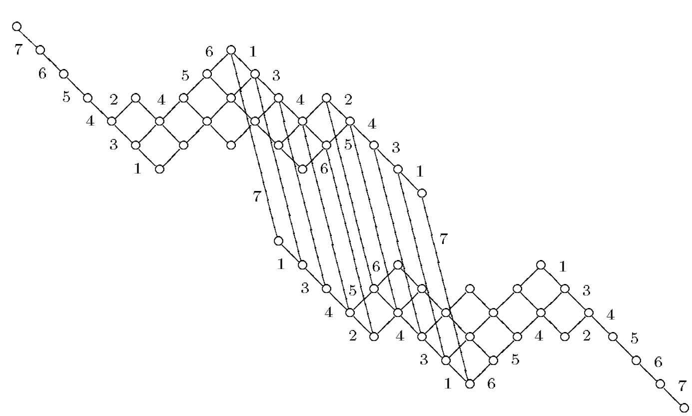
Table 6. Structure constants of $V(\varpi_7)$.

	00000001	00000011	00000111	00001111	00011111	01011111	00111111	10111111	01111111	11111111	01121111	11121111	01122111	11221111	11122111	01122211	11222111	11122211	01122221	11232111	11222211	11122221	12232111	11232211	11222221	12232211	11233211	11232221
1000000	0	0	0	0	0	0	+	0	+	0	+	0	+	0	0	+	0	0	+	0	0	0	0	0	0	0	0	0
0100000	0	0	0	0	+	0	+	+	0	0	0	0	0	0	0	0	0	0	0	+	0	0	0	+	0	0	+	+
0010000	0	0	0	0	+	+	0	0	0	0	0	+	0	0	+	0	0	+	0	0	0	+	0	0	0	0	0	0
0001000	0	0	0	+	0	0	0	0	+	+	0	0	0	0	0	0	+	0	0	0	+	0	0	0	+	0	0	0
0000100	0	0	+	0	0	0	0	0	0	0	+	+	0	+	0	0	0	0	0	0	0	0	0	+	0	+	0	+
0000010	0	+	0	0	0	0	0	0	0	0	0	0	+	0	+	0	+	0	0	+	0	0	+	0	0	0	0	0
0000001	+	0	0	0	0	0	0	0	0	0	0	0	0	0	0	+	0	+	0	0	+	0	0	+	0	+	+	0
1010000	0	0	0	0	+	+	0	0	0	0	−	0	−	0	0	−	0	0	−	0	0	0	0	0	0	0	0	0
0101000	0	0	0	+	0	0	−	−	0	0	0	0	0	0	0	0	+	0	0	0	+	0	0	0	+	0	−	0
0011000	0	0	0	+	0	−	0	0	0	+	0	0	0	0	−	0	0	−	0	0	0	−	0	0	0	0	0	0
0001100	0	0	+	0	0	0	0	0	−	−	0	0	0	+	0	0	0	0	0	0	−	0	0	0	−	+	0	0
0000110	0	+	0	0	0	0	0	0	0	0	−	−	0	−	0	0	0	0	0	+	0	0	+	0	0	0	0	−
0000011	+	0	0	0	0	0	0	0	0	0	0	0	−	0	−	0	−	0	0	−	0	0	−	0	0	0	+	0
1011000	0	0	0	+	0	−	0	0	−	0	0	0	+	0	0	+	0	0	+	0	0	0	0	0	0	0	0	0
0111000	0	0	0	+	+	0	0	−	0	0	0	0	0	0	−	0	0	−	0	0	0	−	0	0	0	0	−	0
0101100	0	0	+	0	0	0	+	+	0	0	0	0	0	+	0	0	0	0	0	0	−	0	0	−	−	0	0	−
0011100	0	0	+	0	0	+	0	0	0	−	0	−	0	0	0	0	0	+	0	0	0	+	0	0	0	+	0	0
0001110	0	+	0	0	0	0	0	0	+	+	0	0	0	−	0	0	−	0	0	0	0	0	+	0	+	0	0	0
0000111	+	0	0	0	0	0	0	0	0	0	+	+	0	+	0	0	0	0	0	−	0	0	−	−	0	−	0	0
1111000	0	0	0	+	+	0	+	0	0	0	0	0	+	0	0	+	0	0	+	0	0	0	0	0	0	0	−	0
1011100	0	0	+	0	0	+	0	0	+	0	+	0	0	0	0	−	0	0	−	0	0	0	0	0	0	+	0	0
0111100	0	0	+	0	−	0	0	+	0	0	0	−	0	0	0	0	0	+	0	0	0	+	0	−	0	0	0	−
0101110	0	+	0	0	0	0	−	−	0	0	0	0	0	−	0	0	−	0	0	−	0	0	0	0	+	0	0	+
0011110	0	+	0	0	0	−	0	0	0	+	0	+	0	0	+	0	0	0	0	0	0	−	+	0	0	0	0	0
0001111	+	0	0	0	0	0	0	0	−	−	0	0	0	+	0	0	+	0	0	0	+	0	−	0	0	−	0	0
1111100	0	0	+	0	−	0	−	0	0	0	+	0	0	0	0	−	0	0	−	0	0	0	0	−	0	0	0	−
1011110	0	+	0	0	0	−	0	0	−	0	−	0	−	0	0	0	0	0	+	0	0	0	+	0	0	0	0	0
0112100	0	0	+	+	0	0	0	+	0	+	0	0	0	0	0	0	0	+	0	0	+	+	0	0	+	0	0	0
0111110	0	+	0	0	+	0	0	−	0	0	0	+	0	0	+	0	0	0	0	−	0	−	0	0	0	0	0	+
0101111	+	0	0	0	0	0	+	+	0	0	0	0	0	+	0	0	+	0	0	+	+	0	0	+	0	0	+	0
0011111	+	0	0	0	0	+	0	0	0	−	0	−	0	0	−	0	0	−	0	0	0	0	−	0	0	−	0	0
1112100	0	0	+	+	0	0	−	0	−	0	0	0	0	0	0	−	0	0	−	0	+	0	0	0	+	0	0	0

Obviously, for multiple weights λ and/or μ, one must give precise sense to what exactly is meant by the above equality. Generally the solution of the multiplicity problem is *highly nontrivial* and was obtained only in the famous works by Lusztig and Kashiwara in the context of quantum groups. Soon thereafter Littelmann invented a strikingly beautiful elementary but rather ingenious approach to the construction of these graphs – the *path model*. However, for microweight representations all multiplicities are equal to 1, so that there are no further hurdles.

Table 6 (continued).

	12233211	12232221	11233221	12243211	12233221	11233321	12343211	12243221	12233321	22343211	12343221	12243321	22343221	12343321	12244321	22343321	12344321	22344321	12354321	22354321	13354321	23354321	22454321	23454321	23464321	23465321	23465421	23465431
1000000	0	0	0	0	0	0	+	0	0	0	+	0	0	+	0	0	+	0	+	0	+	0	0	0	0	0	0	0
0100000	0	0	+	0	0	+	0	0	0	0	0	0	0	0	0	0	0	0	+	+	0	0	+	0	0	0	0	0
0010000	0	0	0	+	0	0	0	+	0	0	0	+	0	0	+	0	0	0	0	+	0	+	0	0	0	0	0	0
0001000	+	0	0	0	+	0	0	0	+	0	0	0	0	0	0	0	+	+	0	0	0	0	0	+	0	0	0	0
0000100	0	+	0	0	0	0	0	0	0	0	0	+	0	+	0	+	0	0	0	0	0	0	0	0	+	0	0	0
0000010	0	0	+	0	+	0	0	+	0	0	+	0	+	0	0	0	0	0	0	0	0	0	0	0	0	+	0	0
0000001	+	0	0	+	0	0	+	0	0	+	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	+	0
1010000	0	0	0	+	0	0	0	+	0	0	0	+	0	0	+	0	0	0	−	0	−	0	0	0	0	0	0	0
0101000	0	0	−	0	0	−	0	0	0	0	0	0	0	0	0	0	+	+	0	0	0	0	−	0	0	0	0	0
0011000	+	0	0	0	+	0	0	0	+	0	0	0	0	0	−	0	0	+	0	0	0	−	0	0	0	0	0	0
0001100	0	+	0	0	0	0	0	0	−	0	0	0	0	+	0	+	0	0	0	0	0	0	0	−	0	0	0	0
0000110	0	−	0	0	0	0	0	+	0	0	+	0	+	0	0	0	0	0	0	0	0	0	0	0	−	0	0	0
0000011	+	0	0	+	0	0	+	0	0	+	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	−	0	0
1011000	+	0	0	0	+	0	0	0	+	0	0	0	0	0	−	0	−	0	0	0	+	0	0	0	0	0	0	0
0111000	0	0	−	0	0	−	0	0	0	0	0	0	0	0	−	0	0	+	0	+	0	0	0	0	0	0	0	0
0101100	0	0	0	0	0	+	0	0	0	0	0	0	0	+	0	+	0	0	0	0	0	0	+	0	0	0	0	0
0011100	0	+	0	0	0	0	0	0	−	0	0	−	0	0	0	+	0	0	0	0	0	+	0	0	0	0	0	0
0001110	0	−	0	0	−	0	0	0	0	0	+	0	+	0	0	0	0	0	0	0	0	0	0	+	0	0	0	0
0000111	0	0	0	+	0	0	+	0	0	+	0	0	0	0	0	0	0	0	0	0	0	0	0	0	+	0	0	0
1111000	0	0	−	0	0	−	0	0	0	0	0	0	0	0	−	0	−	0	−	0	0	0	0	0	0	0	0	0
1011100	0	+	0	0	0	0	0	0	−	0	0	−	0	−	0	0	0	0	0	0	−	0	0	0	0	0	0	0
0111100	0	0	0	0	0	+	0	0	0	0	0	−	0	0	0	+	0	0	0	−	0	0	0	0	0	0	0	0
0101110	0	0	+	0	0	0	0	0	0	0	+	0	+	0	0	0	0	0	0	0	0	0	−	0	0	0	0	0
0011110	0	−	0	0	−	0	0	−	0	0	0	0	+	0	0	0	0	0	0	0	0	−	0	0	0	0	0	0
0001111	−	0	0	0	0	0	+	0	0	+	0	0	0	0	0	0	0	0	0	0	0	0	0	−	0	0	0	0
1111100	0	0	0	0	0	+	0	0	0	0	0	−	0	−	0	0	0	0	+	0	0	0	0	0	0	0	0	0
1011110	0	−	0	0	−	0	0	−	0	0	−	0	0	0	0	0	0	0	0	0	+	0	0	0	0	0	0	0
0112100	0	0	0	0	0	+	0	0	+	0	0	0	0	0	0	+	0	+	0	0	0	0	0	0	0	0	0	0
0111110	0	0	+	0	0	0	0	−	0	0	0	0	+	0	0	0	0	0	0	+	0	0	0	0	0	0	0	0
0101111	0	0	0	0	0	0	+	0	0	+	0	0	0	0	0	0	0	0	0	0	0	0	+	0	0	0	0	0
0011111	−	0	0	−	0	0	0	0	0	+	0	0	0	0	0	0	0	0	0	0	0	+	0	0	0	0	0	0
1112100	0	0	0	0	0	+	0	0	+	0	0	0	0	−	0	0	−	0	0	0	0	0	0	0	0	0	0	0



Weight diagrams are often used as a shorthand graphical imprint of the corresponding **weight graph**. The vertices of the weight graph are again the weights of the representation π, but now the arrows correspond to *all* positive roots and not just to the fundamental ones, as in the case of a weight diagram. In other words, the weights λ and μ are joined by an arrow with mark $\alpha \in \Phi^+$, directed from μ to λ, provided that $\lambda - \mu = \alpha$. Weight graphs possess some very strong regularity properties and crop up in an immense number of publications in combinatorics, finite geometries, sphere packings, and all that (see [49] and references therein).

Table 6 (continued).

	00000001	00000011	00000111	00001111	00011111	01011111	00111111	10111111	01111111	11111111	01121111	11121111	01122111	11221111	11122111	01122211	11222111	11122211	01122221	11232111	11122221	11222221	12232111	11232211	11222221	12232211	11233211	11232221
1111110	0	+	0	0	+	0	+	0	0	0	−	0	−	0	0	0	0	0	+	−	0	0	0	0	0	0	0	+
1011111	+	0	0	0	0	+	0	0	+	0	+	0	+	0	0	+	0	0	0	0	0	0	−	0	0	−	0	0
0112110	0	+	0	−	0	0	0	−	0	−	0	0	0	0	+	0	+	0	0	0	0	−	0	0	−	0	0	0
0111111	+	0	0	0	−	0	0	+	0	0	0	−	0	0	−	0	0	−	0	+	0	0	0	+	0	0	+	0
1122100	0	0	+	+	+	+	0	0	0	0	0	0	0	0	0	−	0	−	−	0	0	−	0	0	0	0	0	0
1112110	0	+	0	−	0	0	+	0	+	0	0	0	−	0	0	0	+	0	+	0	0	0	0	0	−	0	0	0
1111111	+	0	0	0	−	0	−	0	0	0	+	0	+	0	0	+	0	0	0	+	0	0	0	+	0	0	+	0
0112210	0	+	+	0	0	0	0	−	0	−	0	−	0	−	0	0	0	0	0	0	0	−	0	0	−	0	0	−
0112111	+	0	0	+	0	0	0	+	0	+	0	0	0	0	−	0	−	−	0	0	−	0	0	0	0	0	+	0
1122110	0	+	0	−	−	−	0	0	0	0	0	0	−	0	−	0	0	0	+	0	0	+	0	0	0	0	0	0
1112210	0	+	+	0	0	0	+	0	+	0	+	0	0	−	0	0	0	0	+	0	0	0	0	0	−	0	0	−
1112111	+	0	0	+	0	0	−	0	−	0	0	0	+	0	0	+	−	0	0	0	−	0	0	0	0	0	+	0
0112211	+	0	−	0	0	0	0	+	0	+	0	+	0	+	0	0	0	−	0	0	−	0	0	−	0	−	0	0
1122210	0	+	+	0	−	−	0	0	0	0	+	+	0	0	0	0	0	0	+	0	0	+	0	0	0	0	0	−
1122111	+	0	0	+	+	+	0	0	0	0	0	0	+	0	+	+	0	+	0	0	0	0	0	0	0	0	+	0
1112211	+	0	−	0	0	0	−	0	−	0	−	0	0	+	0	+	0	0	0	0	−	0	0	−	0	−	0	0
0112221	+	+	0	0	0	0	0	+	0	+	0	+	0	+	+	0	+	0	0	+	0	0	+	0	0	0	0	0
1123210	0	+	+	+	0	−	0	0	−	−	0	0	0	0	0	0	0	0	+	0	0	+	0	0	+	0	0	0
1122211	+	0	−	0	+	+	0	0	0	0	−	−	0	0	0	+	0	+	0	0	0	0	0	−	0	−	0	0
1112221	+	+	0	0	0	0	−	0	−	0	−	0	−	+	0	0	+	0	0	+	0	0	+	0	0	0	0	0
1223210	0	+	+	+	+	0	+	+	0	0	0	0	0	0	0	0	0	0	+	0	0	+	0	0	+	0	0	+
1123211	+	0	−	−	0	+	0	0	+	+	0	0	0	0	0	+	0	+	0	0	+	0	0	0	0	−	0	0
1122221	+	+	0	0	+	+	0	0	0	0	−	−	−	0	−	0	0	0	0	+	0	0	+	0	0	0	0	0
1223211	+	0	−	−	−	0	−	−	0	0	0	0	0	0	0	+	0	+	0	0	+	0	0	+	0	0	+	0
1123221	+	+	0	−	0	+	0	0	+	+	0	0	−	0	−	0	−	0	0	0	0	0	+	0	0	0	0	0
1223221	+	+	0	−	−	0	−	−	0	0	0	0	−	0	−	0	−	0	0	−	0	0	0	0	0	0	+	0
1123321	+	+	+	0	0	+	0	0	+	+	+	+	0	+	0	0	0	0	0	0	0	0	+	0	0	+	0	0
1223321	+	+	+	0	−	0	−	−	0	0	+	+	0	+	0	0	0	0	0	−	0	0	0	−	0	0	0	−
1224321	+	+	+	+	0	0	−	−	−	−	0	0	0	+	0	0	+	0	0	0	+	0	0	0	+	0	0	0
1234321	+	+	+	+	+	+	0	−	0	−	0	−	0	0	−	0	0	−	0	0	0	−	0	0	0	0	0	0
2234321	+	+	+	+	+	+	+	0	+	0	+	0	+	0	0	+	0	0	+	0	0	0	0	0	0	0	0	0

Table 6 (continued).

	12233211	12232221	11233221	12243211	12233221	11233321	12343211	12243221	12233321	22343211	12343221	12243321	22343221	12343321	12244321	22343321	12344321	22344321	12354321	22354321	13354321	23354321	22454321	23454321	23464321	23465321	23465421	23465431
1111110	0	0	+	0	0	0	0	−	0	0	−	0	0	0	0	0	0	0	−	0	0	0	0	0	0	0	0	0
1011111	−	0	0	−	0	0	−	0	0	0	0	0	0	0	0	0	0	0	0	0	−	0	0	0	0	0	0	0
0112110	0	0	+	0	+	0	0	0	0	0	0	0	+	0	0	0	0	−	0	0	0	0	0	0	0	0	0	0
0111111	0	0	0	−	0	0	0	0	0	+	0	0	0	0	0	0	0	0	0	−	0	0	0	0	0	0	0	0
1122100	0	0	0	0	0	+	0	0	+	0	0	+	0	0	+	0	0	0	0	0	0	0	0	0	0	0	0	0
1112110	0	0	+	0	+	0	0	0	0	0	−	0	0	0	0	0	+	0	0	0	0	0	0	0	0	0	0	0
1111111	0	0	0	−	0	0	−	0	0	0	0	0	0	0	0	0	0	0	+	0	0	0	0	0	0	0	0	0
0112210	0	−	0	0	0	0	0	0	0	0	0	0	+	0	0	+	0	0	0	0	0	0	0	0	0	0	0	0
0112111	+	0	0	0	0	0	0	0	0	+	0	0	0	0	0	0	0	+	0	0	0	0	0	0	0	0	0	0
1122110	0	0	+	0	+	0	0	+	0	0	0	0	0	0	−	0	0	0	0	0	0	0	0	0	0	0	0	0
1112210	0	−	0	0	0	0	0	0	0	0	−	0	0	−	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1112111	+	0	0	0	0	0	−	0	0	0	0	0	0	0	0	0	−	0	0	0	0	0	0	0	0	0	0	0
0112211	0	0	0	0	0	0	0	0	0	+	0	0	0	0	0	−	0	0	0	0	0	0	0	0	0	0	0	0
1122210	0	−	0	0	0	0	0	+	0	0	0	+	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1122111	+	0	0	+	0	0	0	0	0	0	0	0	0	0	+	0	0	0	0	0	0	0	0	0	0	0	0	0
1112211	0	0	0	0	0	0	−	0	0	0	0	0	0	+	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0112221	0	0	0	0	0	0	0	0	0	+	0	0	+	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1123210	0	−	0	0	−	0	0	0	−	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1122211	0	0	0	+	0	0	0	0	0	0	0	−	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1112221	0	0	0	0	0	0	−	0	0	0	−	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1223210	0	0	+	0	0	+	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1123211	−	0	0	0	0	0	0	0	+	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1122221	0	0	0	+	0	0	0	+	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1223211	0	0	0	0	0	−	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1123221	−	0	0	0	−	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1223221	0	0	+	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1123321	0	+	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1223321	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1224321	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1234321	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2234321	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

The weight graph of type (E_7, ϖ_7) – or its complement! – is usually called the *Gosset graph*. It first appeared in the study of algebraic surfaces. Namely, consider the algebraic surface obtained from the complex projective plane $\mathbb{P}^2$ by blow-up of l points in general position [38]. The Gosset graph describes a configuration of 56 nonsingular curves with self-intersection -1, occurring on this surface in the case $l = 7$.

5. Restrictions to A_6, D_6, and E_6

The case of E_7 is somewhat harder than that of E_6. In particular, this is due to the fact that in the process of reduction to smaller ranks, subgroups of *three* distinct types E_6, D_6, and A_6 naturally arise, rather than of two types, as was the case for E_6.

Therefore, in many calculations it is necessary to understand how the 56-dimensional E_6-module branches upon restriction to subsystems of types A_6, D_6, and E_6. How the restrictions of $V(\varpi_7)$ to subgroups of these types decompose into irreducibles, is well known, and is easy to compute. As is explained in [112], it suffices to strike out the edges marked by 2, 1, or 7, respectively, in the weight diagram.

Hence,

$$\begin{aligned} V(\mathrm{E}_7,\varpi_7)\downarrow \mathrm{A}_6 &= V(\mathrm{A}_6,\varpi_6)\oplus V(\mathrm{A}_6,\varpi_2)\oplus V(\mathrm{A}_6,\varpi_5)\oplus V(\mathrm{A}_6,\varpi_1),\\ V(\mathrm{E}_7,\varpi_7)\downarrow \mathrm{D}_6 &= V(\mathrm{D}_6,\varpi_1)\oplus V(\mathrm{D}_6,\varpi_5)\oplus V(\mathrm{D}_6,\varpi_1),\\ V(\mathrm{E}_7,\varpi_7)\downarrow \mathrm{E}_6 &= V(\mathrm{E}_6,0)\oplus V(\mathrm{E}_6,\varpi_1)\oplus V(\mathrm{E}_6,\varpi_6)\oplus V(\mathrm{E}_6,0). \end{aligned}$$

Note that for the first two cases, the numbering of the fundamental roots and weights on the right-hand side is their usual numbering in the root systems of types A_6 or D_6, respectively, rather than their numbering as the fundamental roots of E_7. Expessing this in a more down-to-earth fashion, the 56-dimensional module $V(\varpi_7)$ decomposes as follows.

TABLE 7. The matrix of signs of $V(\varpi_7)$.

	00000001	00000011	00000111	00001111	00011111	01011111	00111111	10111111	01111111	11111111	01121111	11121111	01122111	11221111	11122111	01122211	11222111	11122211	01122221	11232111	11222211	11122221	12232111	11232211	11222221	12232211	11233211	11232221
00000001	0	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	0	+	+	0	+	+	0	+	+	0
00000011	+	0	+	+	+	+	+	+	+	+	+	+	+	+	+	0	+	0	+	+	0	+	+	0	+	0	0	+
00000111	+	+	0	+	+	+	+	+	+	+	+	+	0	+	0	+	0	+	−	0	+	−	0	+	−	+	0	−
00001111	+	+	+	0	+	+	+	+	+	+	0	0	+	0	+	−	+	−	+	0	−	+	0	0	+	0	+	0
00011111	+	+	+	+	0	+	+	+	0	0	+	+	−	0	−	+	0	+	−	+	0	−	0	−	0	0	−	+
01011111	+	+	+	+	+	0	0	0	+	+	−	−	+	0	+	−	0	−	+	0	0	+	+	0	0	−	0	0
00111111	+	+	+	+	+	0	0	+	+	0	−	0	+	+	0	−	−	0	+	−	+	0	0	+	−	0	+	−
10111111	+	+	+	+	+	0	+	0	0	+	0	−	0	−	+	0	+	−	0	+	−	+	0	−	+	0	−	+
01111111	+	+	+	+	0	+	+	0	0	+	+	0	−	−	0	+	+	0	−	0	−	0	−	0	+	+	0	0
11111111	+	+	+	+	0	+	0	+	+	0	0	+	0	+	−	0	−	+	0	0	+	−	+	0	−	−	0	0
01121111	+	+	+	0	+	−	−	0	+	0	0	+	+	−	0	−	0	0	+	+	0	0	+	−	0	−	0	+
11121111	+	+	+	0	+	−	0	−	0	+	+	0	0	+	+	0	0	−	0	−	0	+	−	+	0	+	0	−
01122111	+	+	0	+	−	+	+	0	−	0	+	0	0	0	+	+	−	0	−	+	0	0	+	0	0	0	−	0
11221111	+	+	+	0	0	0	+	−	−	+	−	+	0	0	0	0	+	0	0	+	−	0	+	−	+	−	0	+
11122111	+	+	0	+	−	+	0	+	0	−	0	+	+	0	0	0	+	+	0	−	0	−	−	0	0	0	+	0
01122211	+	0	+	−	+	−	−	0	+	0	−	0	+	0	0	0	0	+	+	0	−	0	0	+	0	+	−	0
11222111	+	+	0	+	0	0	−	+	+	−	0	0	−	+	+	0	0	0	0	+	+	0	+	0	−	0	−	0
11122211	+	0	+	−	+	−	0	−	0	+	0	−	0	0	+	+	0	0	0	0	+	+	0	−	0	−	+	0
01122221	0	+	−	+	−	+	+	0	−	0	+	0	−	0	0	+	0	0	0	0	0	+	0	0	−	0	0	+
11232111	+	+	0	0	+	0	−	+	0	0	+	−	+	+	−	0	+	0	0	0	0	0	+	+	0	0	+	−
11222211	+	0	+	−	0	0	+	−	−	+	0	0	0	−	0	−	+	+	0	0	0	0	0	+	+	+	−	0
11122221	0	+	−	+	−	+	0	+	0	−	0	+	0	0	−	0	0	+	+	0	0	0	0	0	+	0	0	−
12232111	+	+	0	0	0	+	0	0	−	+	+	−	+	+	−	0	+	0	0	+	0	0	0	0	0	+	0	0
11232211	+	0	+	0	−	0	+	−	0	0	−	+	0	−	0	+	0	−	0	+	+	0	0	0	0	+	+	+
11222221	0	+	−	+	0	0	−	+	+	−	0	0	0	+	0	0	−	0	−	0	+	+	0	0	0	0	0	+
12232211	+	0	+	0	0	−	0	0	+	−	−	+	0	−	0	+	0	−	0	0	+	0	+	+	0	0	0	0
11233211	+	0	0	+	−	0	+	−	0	0	0	0	−	0	+	−	−	+	0	+	−	0	0	+	0	0	0	0
11232221	0	+	−	0	+	0	−	+	0	0	+	−	0	+	0	0	0	0	+	−	0	−	0	+	+	0	0	0

Table 7 (continued).

	12233211	12232221	11233221	12243211	12233221	11233321	12343211	12243221	12233321	22343211	12343221	12243321	22343221	12343321	12244321	22343321	12344321	22344321	12354321	22354321	13354321	23354321	22454321	23454321	23464321	23465321	23465421	23465431
00000001	+	0	0	+	0	0	+	0	0	+	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
00000011	0	+	+	0	+	0	0	+	0	0	+	0	+	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
00000111	0	−	0	0	0	+	0	0	+	0	0	+	0	+	0	+	0	0	0	0	0	0	0	0	0	0	0	0
00001111	+	0	−	0	−	−	0	0	−	0	0	0	0	0	+	0	+	+	0	0	0	0	0	0	0	0	0	0
00011111	0	0	+	+	0	+	0	−	0	0	0	−	0	0	−	0	0	0	+	+	0	0	0	0	0	0	0	0
01011111	−	+	0	−	+	0	0	+	+	0	0	+	0	0	+	0	0	0	0	0	+	+	0	0	0	0	0	0
00111111	0	0	−	0	0	−	+	0	0	0	−	0	0	−	0	0	−	0	−	0	0	0	+	0	0	0	0	0
10111111	0	0	+	0	0	+	0	0	0	+	0	0	−	0	0	−	0	−	0	−	0	0	−	0	0	0	0	0
01111111	+	−	0	0	−	0	−	0	−	0	+	0	0	+	0	0	+	0	0	0	−	0	0	+	0	0	0	0
11111111	−	+	0	0	+	0	0	0	+	−	0	0	+	0	0	+	0	+	0	0	0	−	0	−	0	0	0	0
01121111	0	+	0	+	0	0	+	−	0	0	−	−	0	−	0	0	0	0	+	0	+	0	0	0	+	0	0	0
11121111	0	−	0	−	0	0	0	+	0	+	0	+	−	0	0	−	0	0	0	+	0	+	0	0	−	0	0	0
01122111	−	0	+	−	+	0	−	+	0	0	+	0	0	0	−	0	−	0	−	0	−	0	0	0	0	+	0	0
11221111	0	+	0	0	0	0	−	0	0	−	+	0	+	+	0	+	0	0	0	0	0	0	+	+	+	0	0	0
11122111	+	0	−	+	−	0	0	−	0	−	0	0	+	0	+	0	0	−	0	−	0	−	0	0	0	−	0	0
01122211	−	0	0	−	0	+	−	0	+	0	0	+	0	+	+	0	+	0	+	0	+	0	0	0	0	0	+	0
11222111	−	0	+	0	+	0	+	0	0	+	−	0	−	0	0	0	+	+	0	0	0	0	−	−	0	+	0	0
11122211	+	0	0	+	0	−	0	0	−	−	0	−	0	0	−	+	0	+	0	+	0	+	0	0	0	0	−	0
01122221	0	+	−	0	−	+	0	−	+	0	−	+	0	+	+	0	+	0	+	0	+	0	0	0	0	0	0	+
11232111	0	0	−	−	0	0	−	+	0	−	+	0	+	0	0	0	0	0	+	+	0	0	+	0	−	−	0	0
11222211	−	0	0	0	0	+	+	0	+	+	0	0	0	−	0	−	−	−	0	0	0	0	+	+	0	0	+	0
11122221	0	−	+	0	+	−	0	+	−	0	0	−	−	0	−	+	0	+	0	+	0	+	0	0	0	0	0	−
12232111	+	−	0	+	−	0	+	−	0	+	−	0	−	0	0	0	0	0	0	0	+	+	0	+	+	+	0	0
11232211	0	0	0	−	0	−	−	0	0	−	0	+	0	+	0	+	0	0	−	−	0	0	−	0	+	0	−	0
11222221	0	+	−	0	−	+	0	0	+	0	+	0	+	−	0	−	−	−	0	0	0	0	+	+	0	0	0	+
12232211	+	+	0	+	0	0	+	0	−	+	0	−	0	−	0	−	0	0	0	0	−	−	0	−	−	0	+	0
11233211	+	0	+	−	0	+	−	0	0	−	0	0	0	0	+	0	+	+	+	+	0	0	+	0	0	+	+	0
11232221	0	+	+	0	0	−	0	−	0	0	−	+	−	+	0	+	0	0	−	−	0	0	−	0	+	0	0	−

• By restriction to A_6 it decomposes into four irreducible summands: the 7-dimensional covector representation, the 21-dimensional representation, which is the exterior square of the vector one, the 21-dimensional representation, which is the exterior square of the covector one, and, finally, the 7-dimensional covector representation.

• By restriction to D_6 it decomposes into three irreducible summands: two 12-dimensional covector representations and the 32-dimensional semispinor representation.

• By restriction to E_6 it decomposes into four irreducible summands: two trivial ones and two contragradient minimal ones.

Table 7 (continued).

	00000001	00000011	00000111	00001111	00011111	01011111	00111111	10111111	01111111	11111111	01121111	11121111	01122111	11221111	11122111	01122211	11222111	11122211	01122221	11232111	11222211	11122221	12232111	11232211	11222221	12232211	11233211	11232221
12233211	+	0	0	+	0	−	0	0	+	−	0	0	−	0	+	−	−	+	0	0	−	0	+	0	0	+	+	0
12232221	0	+	−	0	0	+	0	0	−	+	+	−	0	+	0	0	0	0	+	0	0	−	−	0	+	+	0	+
11233221	0	+	0	−	+	0	−	+	0	0	0	0	+	0	−	0	+	0	−	−	0	+	0	0	−	0	+	+
12243211	+	0	0	0	+	−	0	0	0	0	+	−	−	0	+	−	0	+	0	−	0	0	+	−	0	+	−	0
12233221	0	+	0	−	0	+	0	0	−	+	0	0	+	0	−	0	+	0	−	0	0	+	−	0	−	0	0	0
11233321	0	0	+	−	+	0	−	+	0	0	0	0	0	0	0	+	0	−	+	0	+	−	0	−	+	0	+	−
12343211	+	0	0	0	0	0	+	0	−	0	+	0	−	−	0	−	+	0	0	−	+	0	+	−	0	+	−	0
12243221	0	+	0	0	−	+	0	0	0	0	−	+	+	0	−	0	0	0	−	+	0	+	−	0	0	0	0	−
12233321	0	0	+	−	0	+	0	0	−	+	0	0	0	0	0	+	0	−	+	0	+	−	0	0	+	−	0	0
22343211	+	0	0	0	0	0	0	+	0	−	0	+	0	−	−	0	+	−	0	−	+	0	+	−	0	+	−	0
12343221	0	+	0	0	0	0	−	0	+	0	−	0	+	+	0	0	−	0	−	+	0	0	−	0	+	0	0	−
12243321	0	0	+	0	−	+	0	0	0	0	−	+	0	0	0	+	0	−	+	0	0	−	0	+	0	−	0	+
22343221	0	+	0	0	0	0	0	−	0	+	0	−	0	+	+	0	−	0	0	+	0	−	−	0	+	0	0	−
12343321	0	0	+	0	0	0	−	0	+	0	−	0	0	+	0	+	0	0	+	0	−	0	0	+	−	−	0	+
12244321	0	0	0	+	−	+	0	0	0	0	0	0	−	0	+	+	0	−	+	0	0	−	0	0	0	0	+	0
22343321	0	0	+	0	0	0	0	−	0	+	0	−	0	+	0	0	0	+	0	0	−	+	0	+	−	−	0	+
12344321	0	0	0	+	0	0	−	0	+	0	0	0	−	0	0	+	+	0	+	0	−	0	0	0	−	0	+	0
22344321	0	0	0	+	0	0	0	−	0	+	0	0	0	0	−	0	+	+	0	0	−	+	0	0	−	0	+	0
12354321	0	0	0	0	+	0	−	0	0	0	+	0	−	0	0	+	0	0	+	+	0	0	0	−	0	0	+	−
22354321	0	0	0	0	+	0	0	−	0	0	0	+	0	0	−	0	0	+	0	+	0	+	0	−	0	0	+	−
13354321	0	0	0	0	0	+	0	0	−	0	+	0	−	0	0	+	0	0	+	0	0	0	+	0	0	−	0	0
23354321	0	0	0	0	0	+	0	0	0	−	0	+	0	0	−	0	0	+	0	0	0	+	+	0	0	−	0	0
22454321	0	0	0	0	0	0	+	−	0	0	0	0	0	+	0	0	−	0	0	+	+	0	0	−	+	0	+	−
23454321	0	0	0	0	0	0	0	0	+	−	0	0	0	+	0	0	−	0	0	0	+	0	+	0	+	−	0	0
23464321	0	0	0	0	0	0	0	0	0	0	+	−	0	+	0	0	0	0	0	−	0	0	+	+	0	−	0	+
23465321	0	0	0	0	0	0	0	0	0	0	0	0	+	0	−	0	+	0	0	−	0	0	+	0	0	0	+	0
23465421	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	+	0	−	0	0	+	0	0	−	0	+	+	0
23465431	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	+	0	0	−	0	0	+	0	0	−

Considering $V(\varpi_1)$ as an internal Chevalley module in the Chevalley group of type E_8, we can detect these decompositions as follows. The A_6, D_6, E_6-components can be told apart by the value of the coefficient at α_2, α_1, or α_7, in the expansion with respect to the fundamental roots. For the weights of $V(\varpi_7)$, the coefficient at α_2 can only take the values 0, 1, 2, 3; the coefficient at α_1 can only take the values 0, 1, 2; the coefficient at α_7 can only take the values 0, 1, 2, 3. Thus, to get the weights in an order compatible with the required branching, it is sufficient to select weights with a prescribed value of the respective components of its `rootformE8`, from the list of weights, combine the resulting lists again, and finally invert the resulting list using `Reverse`, so that the highest weight comes first:

```
minimalE7branchA6=Reverse[Apply[Join,Table[
        Select[minimalE7insideE8,rootformE8[#][[2]]==i&],
                {i,0,3}]]]
```

```
minimalE7branchD6=Reverse[Apply[Join,Table[
        Select[minimalE7insideE8,rootformE8[#][[1]]==i&],
                {i,0,2}]]]
```

```
minimalE7branchE6=Reverse[Apply[Join,Table[
     Select[minimalE7insideE8,rootformE8[#][[7]]==i&],
                     {i,0,3}]]]
```

Table 7 (continued).

	12233211	12232221	11233221	12243211	12233221	11233321	12343211	12243221	12233321	22343211	12343221	12243321	22343221	12343321	12244321	22343321	12344321	22344321	12354321	22354321	13354321	23354321	22454321	23454321	23464321	23465321	23465421	23465431
12233211	0	0	0	+	+	0	+	0	+	+	0	0	0	0	−	0	−	−	0	0	+	+	0	+	0	−	−	0
12232221	0	0	0	0	+	0	0	+	−	0	+	−	+	−	0	−	0	0	0	0	−	−	0	−	−	0	0	+
11233221	0	0	0	0	+	+	0	−	0	0	−	0	−	0	+	0	+	+	+	+	0	0	+	0	0	+	0	+
12243211	+	0	0	0	0	0	+	+	0	+	0	+	0	0	+	0	0	0	−	−	−	−	0	0	+	+	+	0
12233221	+	+	+	0	0	0	0	+	+	0	+	0	+	0	−	0	−	−	0	0	+	+	0	+	0	−	0	−
11233321	0	0	+	0	0	0	0	0	+	0	0	−	0	−	+	−	+	+	+	+	0	0	+	0	0	0	+	−
12343211	+	0	0	+	0	0	0	0	0	+	+	0	0	+	0	0	+	0	+	0	+	0	−	−	−	−	−	0
12243221	0	+	−	+	+	0	0	0	0	0	+	+	+	0	+	0	0	0	−	−	−	−	0	0	+	+	0	+
12233321	+	−	0	0	+	+	0	0	0	0	0	+	0	+	−	+	−	−	0	0	+	+	0	+	0	0	−	+
22343211	+	0	0	+	0	0	+	0	0	0	0	0	+	0	0	+	0	+	0	+	0	+	+	+	+	+	+	0
12343221	0	+	−	0	+	0	+	+	0	0	0	0	+	+	0	0	+	0	+	0	+	0	−	−	−	−	0	−
12243321	0	−	0	+	0	−	0	+	+	0	0	0	0	+	+	+	0	0	−	−	−	−	0	0	+	0	+	−
22343221	0	+	−	0	+	0	0	+	0	+	+	0	0	0	0	+	0	+	0	+	0	+	+	+	+	+	0	+
12343321	0	−	0	0	0	−	+	0	+	0	+	+	0	0	0	+	+	0	+	0	+	0	−	−	−	0	−	+
12244321	−	0	+	+	−	+	0	+	−	0	0	+	0	0	0	0	+	+	−	−	−	−	0	0	0	+	−	+
22343321	0	−	0	0	0	−	0	0	+	+	0	+	+	+	0	0	0	+	0	+	0	+	+	+	+	0	+	−
12344321	−	0	+	0	−	+	+	0	−	0	+	0	0	+	+	0	0	+	+	0	+	0	−	−	0	−	+	−
22344321	−	0	+	0	−	+	0	0	−	+	0	0	+	0	+	+	+	0	0	+	0	+	+	+	0	+	−	+
12354321	0	0	+	−	0	+	+	−	0	0	+	−	0	+	−	0	+	0	0	+	+	0	−	0	−	+	−	+
22354321	0	0	+	−	0	+	0	−	0	+	0	−	+	0	−	+	0	+	+	0	0	+	+	0	+	−	+	−
13354321	+	−	0	−	+	0	+	−	+	0	+	−	0	+	−	0	+	0	+	0	0	+	0	−	+	−	+	−
23354321	+	−	0	−	+	0	0	−	+	+	0	−	+	0	−	+	0	+	0	+	+	0	0	+	−	+	−	+
22454321	0	0	+	0	0	+	−	0	0	+	−	0	+	−	0	+	−	+	−	+	0	0	0	+	−	+	−	+
23454321	+	−	0	0	+	0	−	0	+	+	−	0	+	−	0	+	−	+	0	0	−	+	+	0	+	−	+	−
23464321	0	−	0	+	0	0	−	+	0	+	−	+	+	−	0	+	0	0	−	+	+	−	−	+	0	+	−	+
23465321	−	0	+	+	−	0	−	+	0	+	−	0	+	0	+	0	−	+	+	−	−	+	+	−	+	0	+	−
23465421	−	0	0	+	0	+	−	0	−	+	0	+	0	−	−	+	+	−	−	+	+	−	−	+	−	+	0	+
23465431	0	+	+	0	−	−	0	+	+	0	−	−	+	+	+	−	−	+	+	−	−	+	+	−	+	−	+	0

From Table 3 one sees that the set of weights of $V(\varpi_7)$ is symmetric. This observation is transcribed into the language of internal Chevalley modules as follows. Every weight λ has a *symmetric* counterpart $\overline{\lambda}$ such that their sum $\lambda+\overline{\lambda}$ equals the maximal root of E_8. Thus, it will be more convenient to us to number the weights not by the numbers from 1 to 56, but by the numbers $1, 2, \dots, 28, -28, \dots, -2, -1$. For notational convenience we often write $\overline{n}$ instead of $-n$. The resulting weight numberings are collected in Table 8.

6. The matrix of signs

Now we are all set to explicitly calculate the tables of action structure constants for the represnetation $V(\varpi_7)$ and the shape of root elements in this representation. In [135], we calculated the structure constants of the Lie algebra of type E_l twice, via an inductive algorithm `mu[l,i,j]` and by means of the Frenkel–Kac cocycle `kac[l,i,j]`. The resulting table of structure constants for E_7 is reproduced as Table 5. As was noted in Sec. 3, the table of action structure constants for the module $V(\varpi_7)$ is obtained simply by extracting the requisite part from the table of structure constants for E_8:

```
minimalE7insideE8sign=Table[
nu[8,positiveE7insideE8[[i]],minimalE7insideE8[[j]]],
                             {i,1,63},{j,1,56}]]
```

The function nu[8,x,y] defined in [135] expresses the structure constants for E_8. It differs from the function mu in that its arguments are the roots themselves rather than their positions on the list of positive roots. The resulting table is reproduced as Table 6.

As in the process of our work on [18], we have not directly borrowed tables from an electronic version of [135]. Instead, we calculated all necessary signs afresh. This was dozens of times faster than the same calculations in the 1990s. We gather that such an acceleration cannot be plausibly explained by a higher clock speed of the hardware itself, and owes to the increased performance and functionality of computer algebra systems.

Actually, usually it is more convenient to use the table of signs in a slightly different form, not as the table of action structure constants but rather as the *matrix of signs*:

```
signtable=Table[
       nu[8,minimalE7insideE8[[j]]-minimalE7insideE8[[i]],
       minimalE7insideE8[[i]]],
       {i,1,56},{j,1,56}]]
```

As opposed to the table of action structure constants the rows of which are indexed by roots, and the columns are indexed by weights, here both the rows and the columns are indexed by weights of the module $V(\varpi_7)$. The entry of this matrix in the position (λ, μ) equals the coefficient with which the root element e_α, where $\alpha = \lambda - \mu$, adds the number of the column to the number of the row. The matrix of signs is reproduced in Table 7; all subsequent tables rely on this one.

Below we reproduce the results in four numerations of weights: the *natural* one, the one related to the A_6-branching, the one related to the D_6-branching, and the one related to the E_6-branching. As we have already mentioned, Table 8 establishes the correspondence of these numerations.

Recall that we number weights starting with the highest one. This means that – apart from the numbering of rows and columns! – the matrix of signes reproduced in Table 9 is obtained from the matrix in Table 7 by passing to the transpose with regard to the *skew* diagonal. Therefore, with the exception of some typographic dainties – various \vrule, \tablerule, etc. – Table 9 is the table form of the following matrix:

```
minimalE7naturalsigntable=Reverse[Transpose[Reverse[signtable]]]
```

Tables 11, 13, and 15 are obtained thereof by renumbering weights in accordance with the A_6-numeration, D_6-numeration, and E_6-numeration, respectively. This can be done in many ways, for example, by calculating the corresponding permutations. However, since we work with *tiny* lists, and the entire calculation takes fractions of a second, it does not make any sense to strive for efficience. The following function returns the *position* of y on the list x:

```
search[x_,y_]:=Nest[First,Position[x,y],2]
```

Recall that the built-in function Position returns the *list* of positions formatted as a list, so that we must get rid of *two* pairs of braces. It remains only to pass from the natural numeration to the A_5-numeration. This can be done, for example, as follows:

```
minimalE7branchA6signtable=Table[minimalE7naturalsigntable
       [search[minimalE7natural,minimalE7branchA6[[i]]],
       search[minimalE7natural,minimalE7branchA6[[j]]]]],
                            {i,1,56},{j,1,56}]
```

For the D_6-numeration and the E_6-numeration one does exactly the same, but the weights are taken from the lists minimalE7branchD6 and minimalE7branchE6, respectively.

We decompose the resulting matrices into blocks in accordance with the A_6-branching, the D_6-branching, or the E_6-branching, respectively, to emphasize their structure patterns.

For instance, in the North-West and South-East corners of Table 11 one clearly sees two 7-dimensional representations of $\mathrm{SL}(7, R)$ corresponding to the subsystem $\langle \alpha_1, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7 \rangle$, the vector and the covector ones. On the other hand, in the North-East and South-West corners one sees the action of $\mathrm{SL}(2, R)$ corresponding to the highest root δ.

In turn, in Table 15 one clearly sees that, in the E_6-numeration, for any element of the Lie algebra L of type E_7 the last 28 components of the first column are all equal to 0. It is precisely this fact that underlies the PROOF FROM THE BOOK for this case (see [13, 14]).

7. Elementary root unipotents

Now we are all set to calculate root unipotents for the four root numberations above. First, we calculate the root elements e_α of the corresponding Lie algebra:

```
rooteE7natural[h_]:=Table[
        If[minimalE7natural[[i]]-minimalE7natural[[j]]==
                               positiveE7insideE8[[h]],
                       minimalE7naturalsigntable[[h,j]],0],
                                       {i,1,56},{j,1,56}]
```

For the natural numbering the resulting elements e_α are listed in Table 10. Observe that here $1 \leq h \leq 63$ is the number of a *positive* root. However, we know that the root element ε_{-a} corresponding to a negative root is just the transpose ε_α^t. Hence, the remaining 63 root elements are just the transposes of 63 elements listed in the table.

The matrices

`rooteE7branchA6[h_]`, `rooteE7branchD6[h_]`, `rooteE7brandhE6[h_]`

are defined similarly. The results are reproduced in Tables 12, 14, and 16.

Now, we can easily describe the remaining root elements as matrices. For example, since $V(\varpi_7)$ is a microweight representation, one has $e_\alpha^2 = 0$. Hence, $x_\alpha(\xi) = e + \xi e_\alpha$. Thus, root unipotents can be defined as follows:

```
rootxE7natural[h_,x_]:=IdentityMatrix[56]+
                               x*rooteE7naturalE8[h]
```

Obviously, defining the elements

```
rootxE7branchA6[h_,x_], rootxE7branchD6[h_,x_],
                               rootxE7branchE6[h_,x_]
```

one should use

```
rooteE7branchA6[h_], rooteE7branchD6[h_], rooteE7branchE6[h_],
```

respectively, instead of `rooteE7natural[h]`.

As usual, we set $w_\alpha(\varepsilon) = x_\alpha(\varepsilon)x_{-\alpha}(-\varepsilon^{-1})x_\alpha(\varepsilon)$, where $\varepsilon \in R^*$. Thus, we can now define $w_\alpha(\varepsilon)$ as a matrix:

```
rootwE7natural[h_,x_]:=rootxE7natural[h,x].
        Transpose[rootxE7natural[h,-Power[x,-1]].
        rootxE7natural[h,x]
```

Finally, the *semisimple* root element $h_\alpha(\varepsilon) = w_\alpha(\varepsilon)w_\alpha(1)^{-1}$, where $\varepsilon \in R^*$, can be expressed in terms of $w_\alpha(\varepsilon)$,

```
roothE7natural[h_,x_]:=rootwE7natural[h,x].
          rootwE7natural[h,-1]
```

or, at wish, directly in terms of root unipotents $x_\alpha(\xi)$ themselves.

However, in the calculations that invoke only a few of root elements, we prefer to store and use e_α and $x_\alpha(\xi)$ as *sparse* matrices in the format `SparseArray`. In that case, we draw 56×56-matrices only at the very last stage of a calculation, and only when we actually need such a matrix. We do not reproduce the codes needed for such calculations. Since at this point we are already in the possession of the matrix of signs, anyone familiar with the basics of `Mathematica` language, can easily write such codes for herself.

8. An invariant quartic form

Another paramount tool in the study of Chevalley groups of type E_7 in the 56-dimensional representation is the invariant quartic form Q and its partial derivatives. That the group of type E_7 in characteristic 0 preserves a form of degree 4 in 56 variables, was first noticed by Elie Cartan[1]. This form was first explicitly described by Leonhard Dickson in 1901, [64], in the context of 28 bitangents and, thus, of the Weyl group of type $W(\mathrm{E}_7)$. Apparently, Dickson failed to notice a connection with the Chevalley group of type E_7 itself. Or else, Chevalley groups would have been discovered 50 years earlier.

[1] As it stands, the explicit construction of the form in Cartan's paper contains an error, but most probably this is merely a misprint.

Later the form Q was studied in depth by Hans Freudenthal, Jacque Tits, Robert Brown, Michael Aschbacher, Bruce Cooperstein, Tony Springer, Skip Garibaldi, and others (see, in particular, [73, 74, 75, 76, 129, 130, 51, 52, 44, 62, 122, 78, 79] and references therein). Usually, one assumed that $2 \in R^*$ – and oftentimes even that $6 \in R^*$.

An extraordinarily elegant explicit construction of the form Q over a field K of characteristic distinct from 2 was proposed by Hans Freudenthal. Namely, he interprets the 56-dimensional space V as the space $A(8,K)^2$, where $A(8,K)$ is the space of antisymmetric 8×8-matrices. On this space V, he considers the following symplectic inner product:

$$h((a_1,b_1),(a_2,b_2)) = \frac{1}{2}\Big(\operatorname{tr}(a_1 b_2^t) - \operatorname{tr}(a_2 b_1^t)\Big).$$

Further, he introduces the following quartic form Q on this space:

$$Q((a,b)) = \operatorname{pf}(a) + \operatorname{pf}(b) - \frac{1}{4}\operatorname{tr}\left((ab)^2\right) + \frac{1}{16}\operatorname{tr}\left(ab\right)^2.$$

Then in all characteristics distinct from 2 one can identify the isometry group of this pair with the simply connected Chevalley group G of type E_7 over K (see [44, 62]).

The construction of the form Q in papers [44, 62] is somewhat different. In fact, Michael Aschbacher in [44] constructs the form Q in terms of A_6. The essence of this construction is expressed by the partition $56 = 7 + 21 + 21 + 7$. More precisely, the space V is expressed as the direct sum

$$V = U \oplus \wedge^2(U^*) \oplus \wedge^2(U) \oplus U^*$$

for a 7-dimensional space U, while the form Q is constructed in terms of exterior products, and duality.

On the other hand, the construction of the form by Bruce Cooperstein [62] is closer in spirit to the original Freudenthal's construction, and is phrased in terms of A_7, where $56 = 28 + 28$. The isometry group of Q alone is spanned by G and a diagonal element of order 2 (see [62]). In characteristics $p \geq 5$ everything works smoothly, whereas some additional complications occur in characteristic 3.

However, this approach runs into *insurmountable* difficulties in characteristic 2 – the above constructions do not work, and apparently in characteristic 2 there are no interesting G-invariant symmetric four-linear forms on V at all (see [44]). This is due to the fact that in characteristic 2, the four-linear form

$$f(u,v,x,y) = h(u,v)h(x,y) + h(u,x)h(v,y) + h(u,y)h(v,x),$$

obtained by squaring the symplectic form h, becomes symmetric. Clearly, this is not the case in characteristics ≥ 3. Since the dimension of the space of invariant symmetric four-linear forms equals 1, independent of the characteristic, this uninteresting form f takes the place of the genuine four-linear invariant F. As a matter of fact, Aschbacher [44] constructs a four-linear G-invariant form F in characteristic 2, which is symmetric with respect to *even* permutations.

There are further constructions of the form Q, notably the celebrated construction by Robert Brown [50–52], which works in characteristics $\neq 2, 3$. Let V be a space that supports a nondegenerate inner product. Then to define a three-linear form on V is essentially the same as to give V an algebra structure. By the same token, to define a four-linear form on V is essentially the same as to give V the structure of a *ternary* algebra. Indeed, there exists a remarkable ternary algebra $\mathbb{F}$ of dimension 56, constructed in terms of the exceptional Jordan algebra $\mathbb{J}$ (see [52, 68] and references therein). The algebra consists of 2×2-matrices over $\mathbb{J}$ with scalar diagonal entries, $56 = 1 + 27 + 27 + 1$.

It is natural to ask whether one can give an *elementary* and *characteristic-free* construction of the quartic form Q, where Q is from the outset invariant with respect to the extended Weyl group $\widetilde{W}(\mathrm{E}_7)$. Such a construction of the invariant cubic form for E_6 is discussed in our papers [133, 134, 15, 22, 23]. In that case, everything works smoothly, independently of the characteristic. A similar construction of the form Q is indeed possible, but it is noticeably more complicated, and still breaks down in characteristic 2. Let us very briefly explain what goes on here, according to [134, 16].

The *cubic* form for E_6 was constructed in terms of the *quadratic* form for D_5 on the 10-dimensional space. Supposedly, E_7 stands very much in the same relation to E_6, as E_6 itself does to D_5. This philosophy immediately suggests that one should try to construct the *quartic* form Q in a similar fashion, in terms of the *cubic* form for E_6 on the 27-dimensional space.

Let us fix a base vector $v^\lambda \in V$. Then the base vectors v^μ, $d(\lambda,\mu)=2$, span a free module U of rank 27, which carries the cubic form related to E_6. Let us define *tetrades* as quadruples $(\lambda_1,\lambda_2,\lambda_3,\lambda_4)$ of pairwise orthogonal weights. Let Θ and Θ_0 be the sets of ordered and unordered tetrades, respectively. Clearly, $|\Theta| = 56\cdot 27\cdot 10$, whereas $|\Theta_0| = |\Theta|/24 = 630$.

Tentatively, one can define a quartic form Q_{tent} as

$$Q_{\mathrm{tent}}(x) = \sum \pm x_{\lambda_1}x_{\lambda_2}x_{\lambda_3}x_{\lambda_4},$$

where the sum is taken over $\{\lambda_1,\lambda_2,\lambda_3,\lambda_4\}\in\Theta_0$, and the signs are defined by the condition that the resulting form is invariant under the action of the extended Weyl group $\widetilde{W}(\mathrm{E}_7)$.

From the outset one should be *slightly* more cautious than for the case of E_6. In addition to the two cases occurring in a similar construction of the invariant cubic form for E_6, the next case may occur. Namely, w_α may move all four weights of a tetrade, two in positive and two in negative direction, in which case the sign of the monomial does not change.

However the expression for the sign in terms of the distances $h(\lambda_i,\mu_i)$ in the weight diagram still works. This is essentially the same as to define a four-linear form F_{tent} by setting

$$F_{\mathrm{tent}}(v^{\lambda_1},v^{\lambda_2},v^{\lambda_3},v^{\lambda_4}) = (-1)^{h(\lambda_1,\lambda_2,\lambda_3,\lambda_4)}, \quad (\lambda_1,\lambda_2,\lambda_3,\lambda_4)\in\Theta,$$

and $F_{\mathrm{tent}}(v^{\lambda_1},v^{\lambda_2},v^{\lambda_3},v^{\lambda_4})=0$ otherwise. Here, $h(\lambda_1,\lambda_2,\lambda_3,\lambda_4)$ denotes the sum of pairwise distances of these weights in the weight diagram (see [134] for precise definitions).

By construction, the form is invariant under the action of the extended Weyl group $\widetilde{W}(\mathrm{E}_7)$, and it remains only to check that it is preserved by the root subgroup X_α, for *some* root $\alpha\in\Phi$.

Indeed, it is easy to see that for any *tetrade* $(\lambda_1,\lambda_2,\lambda_3,\lambda_4)$ and *any* elementary root unipotent $g = x_\alpha(\xi)$, one has

$$F_{\mathrm{tent}}(gv^{\lambda_1},gv^{\lambda_2},gv^{\lambda_3},gv^{\lambda_4}) = F_{\mathrm{tent}}(v^{\lambda_1},v^{\lambda_2},v^{\lambda_3},v^{\lambda_4}).$$

Unfortunately, there exist quadruples – not tetrades! – of weights, for which the right-hand side is zero, whereas the left-hand side is not. For instance, take four weights $\lambda_1,\lambda_2,\lambda_3,\lambda_4$ such that $\lambda_1+\alpha,\lambda_2+\alpha,\lambda_3+\alpha,\lambda_4-\alpha$ are weights, and the above eight weights form a cube. In other words, the corresponding weight diagram is the tensor product of three copies of (A_1,ϖ_1) (see [112]). Then one of the weights $\lambda_1,\lambda_2,\lambda_3$ is adjacent with the other two weights, say, $d(\lambda_1,\lambda_2)=d(\lambda_1,\lambda_3)=1$, so that

$$F_{\mathrm{tent}}(v^{\lambda_1},v^{\lambda_2},v^{\lambda_3},v^{\lambda_4}) = 0.$$

On the other hand, expanding $F_{\mathrm{tent}}(gv^{\lambda_1},gv^{\lambda_2},gv^{\lambda_3},gv^{\lambda_4})$ by linearity, we get 8 summands. Exactly one of these summands, namely, $F_{\mathrm{tent}}(v^{\lambda_1+\alpha},v^{\lambda_2},v^{\lambda_3},v^{\lambda_4})$, corresponds to a tetrade and is equal to ±1. Thus, the form F_{tent} is not invariant under the action of X_α, contrary to the expectations.

In itself this is not tragic. One could hope to repair the situation by throwing in another Weyl orbit of monomials. This is precisely where the *real* trouble starts. In the above example, throwing in another orbit immediately begets *two* additional nonzero summands, which either cancel, or amount to *twice* something. Thus, we could not have started with the value of the invariant form on tetrades to be ±1, but should set ±2 instead. Clearly, in characteristic 2 this does not make sense.

Nevertheless, in characteristic $\neq 2$ the above construction is *essentially* correct, in the sense that it describes the *relevant* part of the quartic form, the one that accounts for reduction to E_6. Let us fix a base vector v^λ. Then $F(v^\lambda,*,*,*)$ consists of two parts, the form F_{tent}, as defined above, and another summand thrown in to make the resulting form G-invariant. This summand has the form $F(v^\lambda,v^{\lambda^*},*,*)$ and reveals nothing new as compared with the fact that the group G preserves a symplectic inner product.

9. An invariant quartic form, continued

The difficulties occurring in characteristic 2 have been completely surmounted only recently, in the works by Jacob Lurie [102], and the second author [24]. The basic idea is not to require the invariant form to be symmetric. Instead, one should consider the whole 4-dimensional space of invariants of degree 4 on V. These invariants only differ by summands of the form $h(u,v)h(x,y)$, where h is the invariant symplectic inner product on V. Thus, *inside the symplectic group* $\mathrm{Sp}(56,R)$, the Chevalley group of type E_7 can be characterized as the stabilizer of any such quartic invariant. We refer the reader to [24, 16] for detailed constructions, precise statements, and proofs.

Here, we reproduce one such construction of a *nonsymmetric* four-linear invariant form on the module $V(\varpi_7)$, which works without any restrictions on the characteristic of the ground ring R. In the case of a field of characteristic $\neq 2$, the quartic form associated with the *symmetrization* of our form coincides with the form constructed by Cartan, up to a scalar multiple.

Let $\mathfrak{g}$ be the Lie algebra of type E_8. Recall that the α_8*-height* of a root is the coefficient at α_8, in the expansion of the root with respect to the base of fundamental roots of E_8. Since the highest root of E_8 equals

$$\rho = \begin{smallmatrix} 2465432 \\ 3 \end{smallmatrix},$$

the α_8-height can only take the values $-2, -1, 0, 1, 2$, and thus defines a $\mathbb{Z}$-grading of length 5 on $\mathfrak{g}$:

$$\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2.$$

Namely, the subspace of $\mathfrak{g}$ spanned by e_α falls into $\mathfrak{g}_i$ if the coefficient of α at α_8 equals i. Furthermore, $\mathfrak{g}_0$ contains the Cartan subalgebra $\mathfrak{h}$. Observe that the 56-dimensional space $\mathfrak{g}_1$ has the base of root elements e_α, where α runs over all roots of α_8-height 1, or, what is the same, the weights of the representation $V(\varpi_7)$. Moreover, the spaces $\mathfrak{g}_{-2}$ and $\mathfrak{g}_2$ are one-dimensional, and generated by $e_{-\rho}$ and e_ρ, respectively.

Take any four weights $\alpha, \beta, \gamma, \delta$ of $V(\varpi_7)$ and consider the commutator $[[[[e_{-\rho}, e_\alpha], e_\beta], e_\gamma], e_\delta]$. The root $-\rho$ has α_8-height -2, whereas the roots $\alpha, \beta, \gamma, \delta$ have α_8-height 1. It follows that the resulting element lies in $\mathfrak{g}_2$, and, thus, is a multiple of e_ρ. Define $c(\alpha, \beta, \gamma, \delta)$ by

$$[[[[e_{-\rho}, e_\alpha], e_\beta], e_\gamma], e_\delta] = c(\alpha, \beta, \gamma, \delta) e_\rho.$$

It is easy to see that $c(\alpha, \beta, \gamma, \delta) \in \mathbb{Z}$. The coefficients $c(\alpha, \beta, \gamma, \delta)$ define a four-linear form q on $V(\varpi_7)$ as follows:

$$q(u, v, w, z) = \sum_{\alpha, \beta, \gamma, \delta \in \Lambda} c(\alpha, \beta, \gamma, \delta) u^\alpha v^\beta w^\gamma z^\delta.$$

By construction, this form is invariant under the action of E_7 on the module $V(\varpi_7)$. In other words, $q(u, v, w, z) = q(gu, gv, gw, gz)$ for all $u, v, w, z \in V(\varpi_7)$ and all $g \in G(\mathrm{E}_7, R)$.

Denote by F the symmetrization of the form q,

$$F(u_1, u_2, u_3, u_4) = \sum_{\sigma \in S_4} q(u_{\sigma(1)}, u_{\sigma(2)}, u_{\sigma(3)}, u_{\sigma(4)}),$$

and by Q the associated quartic form,

$$Q(u) = F(u, u, u, u).$$

Clearly, all the coefficients of the form Q are divisible by 6. Formally dividing the form Q by $4! = 24$, we get a form whose coefficients may possibly contain 2 or 4 in the denominators. This is precisely why the characteristic 3 does not lead to serious problems in the analysis of *symmetric* quartic invariants.

Recall that

$$[e_\alpha, e_\beta] = \begin{cases} h_\alpha & \text{if } \alpha + \beta = 0; \\ N_{\alpha\beta} e_{\alpha+\beta} & \text{if } \alpha + \beta \in \mathrm{E}_8; \ [h_\alpha, e_\beta] = \langle \alpha, \beta \rangle e_\beta = 2\frac{(\alpha,\beta)}{(\beta,\beta)} e_\beta; \\ 0 & \text{otherwise.} \end{cases}$$

Observe that if a quadruple of weights $(\alpha, \beta, \gamma, \delta)$ is such that $c(\alpha, \beta, \gamma, \delta) \neq 0$, then $\alpha + \beta + \gamma + \delta = 2\rho$. Let us call quadruples such that

$$\alpha + \beta + \gamma + \delta = 2\rho, \qquad \alpha + \beta \neq \rho,$$

regular. For a regular quadruple, the semisimple elements h_α do not occur in the process of calculating the commutator used to define $c(\alpha, \beta, \gamma, \delta)$:

$$\begin{aligned} [[[[e_{-\rho}, e_\alpha], e_\beta], e_\gamma], e_\delta] &= [[[N_{-\rho,\alpha} e_{-\rho+\alpha}, e_\beta], e_\gamma], e_\delta] \\ &= [[N_{-\rho,\alpha} N_{-\rho+\alpha,\beta} e_{-\rho+\alpha+\beta}, e_\gamma], e_\delta] \\ &= [N_{-\rho,\alpha} N_{-\rho+\alpha,\beta} N_{-\rho+\alpha+\beta,\gamma} e_{-\rho+\alpha+\beta+\gamma}, e_\delta] \\ &= N_{-\rho,\alpha} N_{-\rho+\alpha,\beta} N_{-\rho+\alpha+\beta,\gamma} N_{-\rho+\alpha+\beta+\gamma,\delta} e_\rho. \end{aligned}$$

It follows that in this case,

$$c(\alpha,\beta,\gamma,\delta)=N_{-\rho,\alpha}N_{-\rho+\alpha,\beta}N_{-\rho+\alpha+\beta,\gamma}N_{-\rho+a+b+g,\delta}.$$

Quadruples of weights such that

$$\alpha+\beta+\gamma+\delta=2\rho, \qquad \alpha+\beta=\rho,$$

will be called *irregular*. In other words, an irregular quadruple has the form $(\alpha,\rho-\alpha,\gamma,\rho-\gamma)$. We perform a similar calculation for irregular quadruples:

$$\begin{aligned}[[[[e_{-\rho},e_\alpha],e_{\rho-\alpha}],e_\gamma],e_{\rho-\gamma}] &= [[[N_{-\rho,\alpha}e_{-\rho+\alpha},e_{\rho-\alpha}],e_\gamma],e_{\rho-\gamma}]\\ &= [[N_{-\rho,\alpha}h_{-\rho+\alpha},e_\gamma],e_{\rho-\gamma}]\\ &= [N_{-\rho,\alpha}\langle-\rho+\alpha,\gamma\rangle e_\gamma],e_{\rho-\gamma}]\\ &= N_{-\rho,\alpha}\langle-\rho+\alpha,\gamma\rangle N_{\gamma,\rho-\gamma}e_\rho.\end{aligned}$$

It follows that for irregular quadruples, one has

$$c(\alpha,\beta,\gamma,\delta)=N_{-\rho,\alpha}\langle-\rho+\alpha,\gamma\rangle N_{\gamma,\rho-\gamma}=N_{-\rho,\alpha}N_{-\rho,\gamma}\langle-\rho+\alpha,\gamma\rangle,$$

Now, we can start computing the coefficients of the form q. To this end we generate the list of all roots of E_8, both positive and negative, and the corresponding membership test:

```
negativeE8=Map[-#&,positiveE8]
rootsE8=Join[positiveE8,negativeE8]
isrootE8:=MemberQ[rootsE8,#]&;
```

Also, we will have to compute Cartan numbers $\langle\alpha,\beta\rangle=2(\alpha,\beta)/(\beta,\beta)$:

```
cartanmatrixE8={{2,0,-1,0,0,0,0,0},{0,2,0,-1,0,0,0,0},
    {-1,0,2,-1,0,0,0,0},{0,-1,-1,2,-1,0,0,0},{0,0,0,-1,2,-1,0,0},
    {0,0,0,0,-1,2,-1,0},{0,0,0,0,0,-1,2,-1},{0,0,0,0,0,0,-1,2}}
cartannumberE8[u_,v_]:=rootformE8[u].cartanmatrixE8.
                                        rootformE8[v]
```

The position of the root ρ on our list `rootE8` is 120, whereas the position of $-\rho$ is 240:

```
rho=rootsE8[[120]]
minusrho=rootsE8[[240]]
```

Finally, we are in a position to compute the complete table of coefficients

```
coefftable=SparseArray[{},{56,56,56,56}]
```

We start filling this table with the coefficients corresponding to the regular quadruples:

```
For[a=1,a<=56,a++,
    alpha=minimalE7natural[[a]];
    For[b=1,b<=56,b++,
        beta=minimalE7natural[[b]];
        If[Not[isrootE8[-rho+alpha+beta]],Continue[]];
        For[c=1,c<=56,c++,
            gamma=minimalE7natural[[c]];
            delta=2*rho-alpha-beta-gamma;
            If[Not[isrootE8[delta]],Continue[]];
            d=search[minimalE7natural,delta];
            coefftable[[a,b,c,d]]=nu[8,minusrho,alpha]
              *nu[8,minusrho+alpha,beta]
              *nu[8,minusrho+alpha+beta,gamma]
              *nu[8,minusrho+alpha+beta+gamma,delta]]]]
```

Next, we fill in the coefficients corresponding to the irregular quadruples:

```
For[a=1,a<=56,a++,
    alpha=minimalE7natural[[a]];
    For[c=1,c<=56,c++,
        gamma=minimalE7natural[[c]];
        coefftable[[a,57-a,c,57-c]]=nu[8,-rho,alpha]
             *nu[8,-rho,gamma]
             *cartannumberE8[-rho+alpha,gamma]]]
```

The resulting form has 19768 nonzero coefficients, 18144 of which correspond to regular quadruples and 1624 correspond to the irregular ones. Obviously, we cannot reproduce the explicit coordinate expression of this form here, since – for each of the numerations we use! – this would take about 130 printed pages.

In many applications, it is essential to have an explicit coordinate expression of the second partial derivatives of the form Q, which define the highest Weyl orbit of equations determining the highest weight orbit in the 56-dimensional representation, see [99]. In particular, these equations naturally occur in the context of the theory of standard monomials [120, 97, 48, 96, 100, 101]. One can find a detailed bibliography in our papers [134, 11, 25].

As observed in [60, 84], these quadratic equations are intrinsically related to Groebner bases. Some of these equations are reproduced in [111, 142]. In connection with the study of Freudenthal varieties, in [16] and [103] we reproduce an explicit form of the second partial derivatives of the form Q for two of the above numerations.

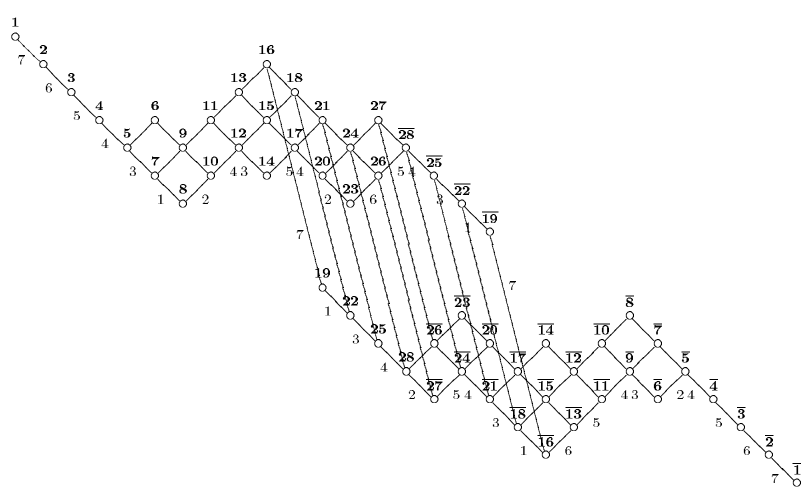
TABLE 8. Weight numerations: the natural one, A_6, D_6, and E_6.

	natural	A_6-branching	D_6-branching	E_6-branching
1	2465431 3	2465431 3	2465431 3	2465431 3
2	2465421 3	2465421 3	2465421 3	2465421 3
3	2465321 3	2465321 3	2465321 3	2465321 3
4	2464321 3	2464321 3	2464321 3	2464321 3
5	2454321 3	2454321 3	2454321 3	2454321 3
6	2454321 2	2354321 3	2454321 2	2454321 2
7	2354321 3	1354321 3	2354321 3	2354321 3
8	1354321 3	2454321 2	2354321 2	1354321 3
9	2354321 2	2354321 2	2344321 2	2354321 2
10	1354321 2	1354321 2	2343321 2	1354321 2
11	2344321 2	2344321 2	2343221 2	2344321 2
12	1344321 2	1344321 2	2343211 2	1344321 2
13	2343321 2	2343321 2	1354321 3	2343321 2
14	1244321 2	1244321 2	1354321 2	1244321 2

15	1343321 2	1343321 2	1344321 2	1343321 2
16	2343221 2	2343221 2	1244321 2	2343221 2
17	1243321 2	1243321 2	1343321 2	1243321 2
18	1343221 2	1343221 2	1243321 2	1343221 2
19	2343211 2	2343211 2	1343221 2	1233321 2
20	1233321 2	1233321 2	1233321 2	1243221 2
21	1243221 2	1243221 2	1243221 2	1233321 1
22	1343211 2	1343211 2	1343211 2	1233221 2
23	1233321 1	1233221 2	1233321 1	1233221 1
24	1233221 2	1243211 2	1233221 2	1232221 2
25	1243211 2	1232221 2	1243211 2	1232221 1
26	1233221 1	1233211 2	1233221 1	1222221 1
27	1232221 2	1232211 2	1232221 2	1122221 1
28	1233211 2	1232111 2	1233211 2	0122221 1
−28	1232221 1	1233321 1	1232221 1	2343211 2
−27	1233211 1	1233221 1	1233211 1	1343211 2
−26	1232211 2	1232221 1	1232211 2	1243211 2
−25	1222221 1	1233211 1	1222221 1	1233211 2
−24	1232211 1	1222221 1	1232211 1	1233211 1
−23	1232111 2	1232211 1	1232111 2	1232211 2
−22	1122221 1	1122221 1	1122221 1	1232211 1
−21	1222211 1	1222211 1	1222211 1	1232111 2
−20	1232111 1	1232111 1	1232111 1	1222211 1
−19	0122221 1	0122221 1	1122211 1	1232111 1
−18	1122211 1	1122211 1	1222111 1	1122211 1
−17	1222111 1	1222111 1	1122111 1	1222111 1
−16	0122211 1	0122211 1	1221111 1	0122211 1

−15	1122111 1	1122111 1	1121111 1	1122111 1
−14	1221111 1	1221111 1	1111111 1	1221111 1
−13	0122111 1	0122111 1	1111111 0	0122111 1
−12	1121111 1	1121111 1	0122221 1	1121111 1
−11	0121111 1	0121111 1	0122211 1	0121111 1
−10	1111111 1	1111111 1	0122111 1	1111111 1
−9	0111111 1	0111111 1	0121111 1	0111111 1
−8	1111111 0	0011111 1	0111111 1	1111111 0
−7	0111111 0	1111111 0	0111111 0	0111111 0
−6	0011111 1	0111111 0	0011111 1	0011111 1
−5	0011111 0	0011111 0	0011111 0	0011111 0
−4	0001111 0	0001111 0	0001111 0	0001111 0
−3	0000111 0	0000111 0	0000111 0	0000111 0
−2	0000011 0	0000011 0	0000011 0	0000011 0
−1	0000001 0	0000001 0	0000001 0	0000001 0

TABLE 9. The matrix of signs of $V(\varpi_7)$: the natural numeration.



	1 2 3 4 5	6 7 8 9 10	11 12 13 14 15	16 17 18 19 20	21 22 23 24 25	26 27 28
1	0 + − + −	+ + − − +	+ − − + +	+ − − 0 +	+ 0 − − 0	+ + 0
2	+ 0 + − +	− − + + −	− + + − −	0 + 0 + −	0 − + 0 +	0 0 −
3	− + 0 + −	+ + − − +	+ − 0 + 0	+ 0 − + 0	+ − 0 − +	+ 0 −
4	+ − + 0 +	− − + + −	0 0 + 0 −	+ + − + 0	+ − 0 0 +	0 − 0
5	− + − + 0	+ + − 0 0	+ − + 0 −	+ 0 − + +	0 − 0 + 0	0 − +
6	+ − + − +	0 0 0 + −	+ − + 0 −	+ 0 − + 0	0 − + 0 0	+ 0 0
7	+ − + − +	0 0 + + 0	+ 0 + − 0	+ − 0 + +	− 0 0 + −	0 − +
8	− + − + −	0 + 0 0 +	0 + 0 − +	0 − + 0 +	− + 0 + −	0 − +
9	− + − + 0	+ + 0 0 +	+ 0 + − 0	+ − 0 + 0	− 0 + 0 −	+ 0 0
10	+ − + − 0	− 0 + + 0	0 + 0 − +	0 − + 0 0	− + + 0 −	+ 0 0
11	+ − + 0 +	+ + 0 + 0	0 + + + 0	+ 0 0 + −	0 0 + − 0	+ 0 −
12	− + − 0 −	− 0 + 0 +	+ 0 0 + +	0 0 + 0 −	0 + + − 0	+ 0 −
13	− + 0 + +	+ + 0 + 0	+ 0 0 0 +	+ + 0 + +	0 0 − 0 0	0 − 0
14	+ − + 0 0	0 − − − −	+ + 0 0 0	0 + 0 0 −	+ 0 + − +	+ 0 −
15	+ − 0 − −	− 0 + 0 +	0 + + 0 0	0 + + 0 +	0 + − 0 0	0 − 0
16	+ 0 + + +	+ + 0 + 0	+ 0 + 0 0	0 0 + + 0	+ 0 0 + 0	− + 0
17	− + 0 + 0	0 − − − −	0 0 + + +	0 0 0 0 +	+ 0 − 0 +	0 − 0
18	− 0 − − −	− 0 + 0 +	0 + 0 0 +	+ 0 0 0 0	+ + 0 + 0	− + 0
19	0 + + + +	+ + 0 + 0	+ 0 + 0 0	+ 0 0 0 0	0 + 0 0 +	0 0 +
20	+ − 0 0 +	0 + + 0 0	− − + − +	0 + 0 0 0	0 0 + + 0	0 − +
21	+ 0 + + 0	0 − − − −	0 0 0 + 0	+ + + 0 0	0 0 0 + +	− + 0
22	0 − − − −	− 0 + 0 +	0 + 0 0 +	0 0 + + 0	0 0 0 0 +	0 0 +
23	− + 0 0 0	+ 0 0 + +	+ + − + −	0 − 0 0 +	0 0 0 0 0	+ 0 0
24	− 0 − 0 +	0 + + 0 0	− − 0 − 0	+ 0 + 0 +	+ 0 0 0 0	+ + +
25	0 + + + 0	0 − − − −	0 0 0 + 0	0 + 0 + 0	+ + 0 0 0	0 0 +
26	+ 0 + 0 0	+ 0 0 + +	+ + 0 + 0	− 0 − 0 0	− 0 + + 0	0 0 0
27	+ 0 0 − −	0 − − 0 0	0 0 − 0 −	+ − + 0 −	+ 0 0 + 0	0 0 0
28	0 − − 0 +	0 + + 0 0	− − 0 − 0	0 0 0 + +	0 + 0 + +	0 0 0

	$\overline{28}$ $\overline{27}$ $\overline{26}$ $\overline{25}$ $\overline{24}$	$\overline{23}$ $\overline{22}$ $\overline{21}$ $\overline{20}$ $\overline{19}$	$\overline{18}$ $\overline{17}$ $\overline{16}$ $\overline{15}$ $\overline{14}$	$\overline{13}$ $\overline{12}$ $\overline{11}$ $\overline{10}$ $\overline{9}$	$\overline{8}$ $\overline{7}$ $\overline{6}$ $\overline{5}$ $\overline{4}$	$\overline{3}$ $\overline{2}$ $\overline{1}$
1	− 0 0 + 0	0 − 0 0 +	0 0 0 0 0	0 0 0 0 0	0 0 0 0 0	0 0 0
2	0 + + 0 −	0 0 + 0 0	− 0 + 0 0	0 0 0 0 0	0 0 0 0 0	0 0 0
3	0 + 0 0 0	+ 0 0 − 0	0 + 0 − 0	+ 0 0 0 0	0 0 0 0 0	0 0 0
4	+ 0 − 0 +	+ 0 0 − 0	0 0 0 0 +	0 − + 0 0	0 0 0 0 0	0 0 0
5	0 0 − + 0	+ 0 + 0 0	0 − 0 0 +	0 0 0 − +	0 0 0 0 0	0 0 0
6	− + 0 + −	0 0 + + 0	0 − 0 0 +	0 0 0 0 0	− + 0 0 0	0 0 0
7	0 0 − 0 0	+ + 0 0 0	+ 0 0 − 0	0 + 0 − 0	0 0 + 0 0	0 0 0
8	0 0 − 0 0	+ 0 0 0 +	0 0 + 0 0	− 0 + 0 −	0 0 + 0 0	0 0 0
9	− + 0 0 −	0 + 0 + 0	+ 0 0 − 0	0 + 0 0 0	− 0 0 + 0	0 0 0
10	− + 0 0 −	0 0 0 + +	0 0 + 0 0	− 0 + 0 0	0 − 0 + 0	0 0 0
11	0 + 0 − 0	0 + − 0 0	+ + 0 − 0	0 0 0 + 0	− 0 0 0 +	0 0 0
12	0 + 0 − 0	0 0 − 0 +	0 + + 0 0	− 0 0 0 +	0 − 0 0 +	0 0 0
13	+ 0 − − +	0 + − 0 0	+ 0 0 0 +	0 − 0 + 0	− 0 0 0 0	+ 0 0
14	0 + 0 0 0	0 − 0 0 +	− 0 + + 0	− 0 0 0 0	0 0 + − +	0 0 0
15	+ 0 − − +	0 0 − 0 +	0 0 + 0 +	0 0 − 0 +	0 − 0 0 0	+ 0 0
16	− 0 0 + 0	− − 0 + 0	0 − 0 + +	0 − 0 + 0	− 0 0 0 0	0 + 0
17	+ 0 − 0 +	0 − 0 0 +	− 0 + 0 0	0 + − 0 0	0 0 + − 0	+ 0 0
18	− 0 0 + 0	− 0 0 + −	0 − 0 0 +	+ 0 − 0 +	0 − 0 0 0	0 + 0
19	0 − + 0 −	+ 0 + − 0	− + 0 − −	0 + 0 − 0	+ 0 0 0 0	0 0 +
20	0 0 − + 0	0 − + 0 +	− 0 + 0 0	0 0 0 + −	0 0 + 0 −	+ 0 0
21	− 0 0 0 0	− + 0 + −	0 0 0 − 0	+ + − 0 0	0 0 + − 0	0 + 0
22	0 − + 0 −	+ 0 + − 0	0 + − 0 −	− 0 + 0 −	0 + 0 0 0	0 0 +
23	− + 0 + −	0 − + 0 +	− 0 + 0 0	0 0 0 0 0	+ − 0 + −	+ 0 0
24	0 0 0 − 0	− + 0 0 −	0 + 0 − 0	+ 0 0 + −	0 0 + 0 −	0 + 0
25	0 − + 0 −	+ 0 0 − 0	+ 0 − + 0	− − + 0 0	0 0 − + 0	0 0 +
26	+ + 0 − 0	0 + 0 − −	0 + 0 − 0	+ 0 0 0 0	+ − 0 + −	0 + 0
27	+ 0 + + 0	− − 0 0 +	0 0 0 0 +	0 − + + −	0 0 + 0 0	− + 0
28	0 + + 0 0	+ 0 − 0 0	+ − − + 0	− 0 0 − +	0 0 − 0 +	0 0 +

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28
$\overline{28}$	−	0	0	+	0	−	0	0	−	−	0	0	+	0	+	−	+	−	0	0	−	0	−	0	0	+	+	0
$\overline{27}$	0	+	+	0	0	+	0	0	+	+	+	+	0	+	0	0	0	0	−	0	0	−	+	0	−	+	0	+
$\overline{26}$	0	+	0	−	−	0	−	−	0	0	0	0	−	0	−	0	−	0	+	−	0	+	0	0	+	0	+	+
$\overline{25}$	+	0	0	0	+	+	0	0	0	0	−	−	−	0	−	+	0	+	0	+	0	0	+	−	0	−	+	0
$\overline{24}$	0	−	0	+	0	−	0	0	−	−	0	0	+	0	+	0	+	0	−	0	0	−	−	0	−	0	0	0
$\overline{23}$	0	0	+	+	+	0	+	+	0	0	0	0	0	0	0	−	0	−	+	0	−	+	0	−	+	0	−	+
$\overline{22}$	−	0	0	0	0	0	+	0	+	0	+	0	+	−	0	−	−	0	0	−	+	0	−	+	0	+	−	0
$\overline{21}$	0	+	0	0	+	+	0	0	0	0	−	−	−	0	−	0	0	0	+	+	0	+	+	0	0	0	0	−
$\overline{20}$	0	0	−	−	0	+	0	0	+	+	0	0	0	0	0	+	0	+	−	0	+	−	0	0	−	−	0	0
$\overline{19}$	+	0	0	0	0	0	0	+	0	+	0	+	0	+	+	0	+	−	0	+	−	0	+	−	0	−	+	0
$\overline{18}$	0	−	0	0	0	0	+	0	+	0	+	0	+	−	0	0	−	0	−	−	0	0	−	0	+	0	0	+
$\overline{17}$	0	0	+	0	−	−	0	0	0	0	+	+	0	0	0	−	0	−	+	0	0	+	0	+	0	+	0	−
$\overline{16}$	0	+	0	0	0	0	0	+	0	+	0	+	0	+	+	0	+	0	0	+	0	−	+	0	−	0	0	−
$\overline{15}$	0	0	−	0	0	0	−	0	−	0	−	0	0	+	0	+	0	0	−	0	−	0	0	−	+	−	0	+
$\overline{14}$	0	0	0	+	+	+	0	0	0	0	0	0	+	0	+	+	0	+	−	0	0	−	0	0	0	0	+	0
$\overline{13}$	0	0	+	0	0	0	0	−	0	−	0	−	0	−	0	0	0	+	0	0	+	−	0	+	−	+	0	−
$\overline{12}$	0	0	0	−	0	0	+	0	+	0	0	0	−	0	0	−	+	0	+	0	+	0	0	0	−	0	−	0
$\overline{11}$	0	0	0	+	0	0	0	+	0	+	0	0	0	0	−	0	−	−	0	0	−	+	0	0	+	0	+	0
$\overline{10}$	0	0	0	0	−	0	−	0	0	0	+	0	+	0	0	+	0	0	−	+	0	0	0	+	0	0	+	−
$\overline{9}$	0	0	0	0	+	0	0	−	0	0	0	+	0	0	+	0	0	+	0	−	0	−	0	−	0	0	−	+
$\overline{8}$	0	0	0	0	0	−	0	0	−	0	−	0	−	0	0	−	0	0	+	0	0	0	+	0	0	+	0	0
$\overline{7}$	0	0	0	0	0	+	0	0	0	−	0	−	0	0	−	0	0	−	0	0	0	+	−	0	0	−	0	0
$\overline{6}$	0	0	0	0	0	0	+	+	0	0	0	0	0	+	0	0	+	0	0	+	+	0	0	+	−	0	+	−
$\overline{5}$	0	0	0	0	0	0	0	0	+	+	0	0	0	−	0	0	−	0	0	0	−	0	+	0	+	+	0	0
$\overline{4}$	0	0	0	0	0	0	0	0	0	0	+	+	0	+	0	0	0	0	0	−	0	0	−	−	0	−	0	+
$\overline{3}$	0	0	0	0	0	0	0	0	0	0	0	0	+	0	+	0	+	0	0	+	0	0	+	0	0	0	−	0
$\overline{2}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	+	0	+	0	0	+	0	0	+	0	+	+	0
$\overline{1}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	+	0	0	+	0	0	+	0	0	+

	$\overline{28}$	$\overline{27}$	$\overline{26}$	$\overline{25}$	$\overline{24}$	$\overline{23}$	$\overline{22}$	$\overline{21}$	$\overline{20}$	$\overline{19}$	$\overline{18}$	$\overline{17}$	$\overline{16}$	$\overline{15}$	$\overline{14}$	$\overline{13}$	$\overline{12}$	$\overline{11}$	$\overline{10}$	$\overline{9}$	$\overline{8}$	$\overline{7}$	$\overline{6}$	$\overline{5}$	$\overline{4}$	$\overline{3}$	$\overline{2}$	$\overline{1}$
$\overline{28}$	0	0	0	+	+	0	−	0	−	+	0	0	0	0	+	0	−	+	0	0	+	−	0	+	0	−	+	0
$\overline{27}$	0	0	0	0	+	0	0	−	+	0	+	−	−	+	0	−	0	0	0	0	−	+	0	−	+	0	0	+
$\overline{26}$	0	0	0	0	+	+	0	+	0	0	−	0	+	0	−	0	+	−	−	+	0	0	−	0	0	+	0	+
$\overline{25}$	+	0	0	0	0	0	+	+	0	−	0	−	0	0	+	0	0	0	−	+	+	−	0	0	+	−	+	0
$\overline{24}$	+	+	+	0	0	0	0	+	+	0	−	0	+	0	−	0	+	−	0	0	−	+	0	−	0	+	0	+
$\overline{23}$	0	0	+	0	0	0	0	0	+	0	0	+	0	−	+	+	−	+	+	−	0	0	+	0	0	0	+	+
$\overline{22}$	−	0	0	+	0	0	0	0	0	+	+	0	0	−	0	0	+	0	−	0	+	0	+	−	+	−	+	0
$\overline{21}$	0	−	+	+	+	0	0	0	0	0	+	+	−	0	−	0	0	0	+	−	−	+	0	0	−	+	0	+
$\overline{20}$	−	+	0	0	+	+	0	0	0	0	0	+	0	−	+	+	−	+	0	0	+	−	0	+	0	0	+	+
$\overline{19}$	+	0	0	−	0	0	+	0	0	0	0	0	+	0	0	−	0	+	0	−	0	+	+	−	+	−	+	0
$\overline{18}$	0	+	−	0	−	0	+	+	0	0	0	0	+	+	0	0	−	0	+	0	−	0	−	+	−	+	0	+
$\overline{17}$	0	−	0	−	0	+	0	+	+	0	0	0	0	+	+	−	0	0	−	+	+	−	0	0	+	0	+	+
$\overline{16}$	0	−	+	0	+	0	0	−	0	+	+	0	0	0	0	+	0	−	0	+	0	−	−	+	−	+	0	+
$\overline{15}$	0	+	0	0	0	−	−	0	−	0	+	+	0	0	0	+	+	0	−	0	+	0	+	−	+	0	+	+
$\overline{14}$	+	0	−	+	−	+	0	−	+	0	0	+	0	0	0	0	+	−	+	−	−	+	0	0	0	+	+	+
$\overline{13}$	0	−	0	0	0	+	0	0	+	−	0	−	+	+	0	0	0	+	0	−	0	+	+	−	+	0	+	+
$\overline{12}$	−	0	+	0	+	−	+	0	−	0	−	0	0	+	+	0	0	+	+	0	−	0	−	+	0	+	+	+
$\overline{11}$	+	0	−	0	−	+	0	0	+	+	0	0	−	0	−	+	+	0	0	+	0	−	−	+	0	+	+	+
$\overline{10}$	0	0	−	−	0	+	−	+	0	0	+	−	0	−	+	0	+	0	0	+	+	0	+	0	+	+	+	+
$\overline{9}$	0	0	+	+	0	−	0	−	0	−	0	+	+	0	−	−	0	+	+	0	0	+	+	0	+	+	+	+
$\overline{8}$	+	−	0	+	−	0	+	−	+	0	−	+	0	+	−	0	−	0	+	0	0	+	0	+	+	+	+	+
$\overline{7}$	−	+	0	−	+	0	0	+	−	+	0	−	−	0	+	+	0	−	0	+	+	0	0	+	+	+	+	+
$\overline{6}$	0	0	−	0	0	+	+	0	0	+	−	0	−	+	0	+	−	−	+	+	0	0	0	+	+	+	+	+
$\overline{5}$	+	−	0	0	−	0	−	0	+	−	+	0	+	−	0	−	+	+	0	0	+	+	+	0	+	+	+	+
$\overline{4}$	0	+	0	+	0	0	+	−	0	+	−	+	−	+	0	+	0	0	+	+	+	+	+	+	0	+	+	+
$\overline{3}$	−	0	+	−	+	0	−	+	0	−	+	0	+	0	+	0	+	+	+	+	+	+	+	+	+	0	+	+
$\overline{2}$	+	0	0	+	0	+	+	0	+	+	0	+	0	+	+	+	+	+	+	+	+	+	+	+	+	+	0	+
$\overline{1}$	0	+	+	0	+	+	0	+	+	0	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	0

Table 10. Root elements: the natural numeration.

$$e_{\substack{100000\\0}} = e_{7,8} + e_{9,10} + e_{11,12} + e_{13,15} + e_{16,18} + e_{19,22} + e_{\overline{22},\overline{19}} + e_{\overline{18},\overline{16}} + e_{\overline{15},\overline{13}} + e_{\overline{12},\overline{11}} + e_{\overline{10},\overline{9}} + e_{\overline{87}}$$

$$e_{\substack{000000\\1}} = e_{5,6} + e_{7,9} + e_{8,10} + e_{20,23} + e_{24,26} + e_{28,\overline{27}} + e_{27,\overline{28}} + e_{\overline{26},\overline{24}} + e_{\overline{23},\overline{20}} + e_{\overline{10},\overline{8}} + e_{\overline{9},\overline{7}} + e_{\overline{6},\overline{5}}$$

$$e_{\substack{010000\\0}} = e_{5,7} + e_{6,9} + e_{12,14} + e_{15,17} + e_{18,21} + e_{22,25} + e_{\overline{25},\overline{22}} + e_{\overline{21},\overline{18}} + e_{\overline{17},\overline{15}} + e_{\overline{14},\overline{12}} + e_{\overline{9},\overline{6}} + e_{\overline{7},\overline{5}}$$

$$e_{\substack{001000\\0}} = e_{4,5} + e_{9,11} + e_{10,12} + e_{17,20} + e_{21,24} + e_{25,28} + e_{\overline{28},\overline{25}} + e_{\overline{24},\overline{21}} + e_{\overline{20},\overline{17}} + e_{\overline{12},\overline{10}} + e_{\overline{11},\overline{9}} + e_{\overline{5},\overline{4}}$$

$$e_{\substack{000100\\0}} = e_{3,4} + e_{11,13} + e_{12,15} + e_{14,17} + e_{24,27} + e_{28,\overline{26}} + e_{26,\overline{28}} + e_{\overline{27},\overline{24}} + e_{\overline{17},\overline{14}} + e_{\overline{15},\overline{12}} + e_{\overline{13},\overline{11}} + e_{\overline{4},\overline{3}}$$

$$e_{\substack{000010\\0}} = e_{2,3} + e_{13,16} + e_{15,18} + e_{17,21} + e_{20,24} + e_{\overline{26},\overline{23}} + e_{23,26} + e_{\overline{24},\overline{20}} + e_{\overline{21},\overline{17}} + e_{\overline{18},\overline{15}} + e_{\overline{16},\overline{13}} + e_{\overline{3},\overline{2}}$$

$$e_{\substack{000001\\0}} = e_{1,2} + e_{16,19} + e_{18,22} + e_{21,25} + e_{24,28} + e_{27,\overline{26}} + e_{26,\overline{27}} + e_{\overline{28},\overline{24}} + e_{\overline{25},\overline{21}} + e_{\overline{22},\overline{18}} + e_{\overline{19},\overline{16}} + e_{\overline{2},\overline{1}}$$

$$e_{\substack{110000\\0}} = -e_{5,8} - e_{6,10} + e_{11,14} + e_{13,17} + e_{16,21} + e_{19,25} - e_{\overline{25},\overline{19}} - e_{\overline{21},\overline{16}} - e_{\overline{17},\overline{13}} - e_{\overline{14},\overline{11}} + e_{\overline{10},\overline{6}} + e_{\overline{85}}$$

$$e_{\substack{001000\\1}} = -e_{4,6} + e_{7,11} + e_{8,12} - e_{17,23} - e_{21,26} - e_{25,\overline{27}} + e_{27,\overline{25}} + e_{\overline{26},\overline{21}} + e_{\overline{23},\overline{17}} - e_{\overline{12},\overline{8}} - e_{\overline{11},\overline{7}} + e_{\overline{6},\overline{4}}$$

$$e_{\substack{011000\\0}} = -e_{4,7} + e_{6,11} - e_{10,14} + e_{15,20} + e_{18,24} + e_{22,28} - e_{\overline{28},\overline{22}} - e_{\overline{24},\overline{18}} - e_{\overline{20},\overline{15}} + e_{\overline{14},\overline{10}} - e_{\overline{11},\overline{6}} + e_{\overline{7},\overline{4}}$$

$$e_{\substack{001100\\0}} = -e_{3,5} + e_{9,13} + e_{10,15} - e_{14,20} + e_{21,27} + e_{25,\overline{26}} - e_{26,\overline{25}} - e_{\overline{27},\overline{21}} + e_{\overline{20},\overline{14}} - e_{\overline{15},\overline{10}} - e_{\overline{13},\overline{9}} + e_{\overline{5},\overline{3}}$$

$$e_{\substack{000110\\0}} = -e_{2,4} + e_{11,16} + e_{12,18} + e_{14,21} - e_{20,27} + e_{28,\overline{23}} - e_{23,\overline{28}} + e_{\overline{27},\overline{20}} - e_{\overline{21},\overline{14}} - e_{\overline{18},\overline{12}} - e_{\overline{16},\overline{11}} + e_{\overline{4},\overline{2}}$$

$$e_{\substack{000011\\0}} = -e_{1,3} + e_{13,19} + e_{15,22} + e_{17,25} + e_{20,28} - e_{27,\overline{23}} + e_{23,\overline{27}} - e_{\overline{28},\overline{20}} - e_{\overline{25},\overline{17}} - e_{\overline{22},\overline{15}} - e_{\overline{19},\overline{13}} + e_{\overline{3},\overline{1}}$$

$$e_{\substack{111000\\0}} = e_{4,8} - e_{6,12} - e_{9,14} + e_{13,20} + e_{16,24} + e_{19,28} + e_{\overline{28},\overline{19}} + e_{\overline{24},\overline{16}} + e_{\overline{20},\overline{13}} - e_{\overline{14},\overline{9}} - e_{\overline{12},\overline{6}} + e_{\overline{8},\overline{4}}$$

$$e_{\substack{011000\\1}} = e_{4,9} + e_{5,11} - e_{8,14} - e_{15,23} - e_{18,26} - e_{22,\overline{27}} - e_{27,\overline{22}} - e_{\overline{26},\overline{18}} - e_{\overline{23},\overline{15}} - e_{\overline{14},\overline{8}} + e_{\overline{11},\overline{5}} + e_{\overline{9},\overline{4}}$$

$$e_{\substack{001100\\1}} = e_{3,6} + e_{7,13} + e_{8,15} + e_{14,23} - e_{21,\overline{28}} - e_{25,\overline{24}} - e_{24,\overline{25}} - e_{28,\overline{21}} + e_{\overline{23},\overline{14}} + e_{\overline{15},\overline{8}} + e_{\overline{13},\overline{7}} + e_{\overline{6},\overline{3}}$$

$$e_{\substack{011100\\0}} = e_{3,7} + e_{6,13} - e_{10,17} - e_{12,20} + e_{18,27} + e_{22,\overline{26}} + e_{26,\overline{22}} + e_{\overline{27},\overline{18}} - e_{\overline{20},\overline{12}} - e_{\overline{17},\overline{10}} + e_{\overline{13},\overline{6}} + e_{\overline{7},\overline{3}}$$

$$e_{\substack{001110\\0}} = e_{2,5} + e_{9,16} + e_{10,18} - e_{14,24} - e_{17,27} + e_{25,\overline{23}} + e_{23,\overline{25}} - e_{\overline{27},\overline{17}} - e_{\overline{24},\overline{14}} + e_{\overline{18},\overline{10}} + e_{\overline{16},\overline{9}} + e_{\overline{5},\overline{2}}$$

$$e_{\substack{000111\\0}} = e_{1,4} + e_{11,19} + e_{12,22} + e_{14,25} - e_{20,\overline{26}} - e_{24,\overline{23}} - e_{23,\overline{24}} - e_{26,\overline{20}} + e_{\overline{25},\overline{14}} + e_{\overline{22},\overline{12}} + e_{\overline{19},\overline{11}} + e_{\overline{4},\overline{1}}$$

$$e_{\substack{111000\\1}} = -e_{4,10} - e_{5,12} - e_{7,14} - e_{13,23} - e_{16,26} - e_{19,\overline{27}} + e_{27,\overline{19}} + e_{\overline{26},\overline{16}} + e_{\overline{23},\overline{13}} + e_{\overline{14},\overline{7}} + e_{\overline{12},\overline{5}} + e_{\overline{10},\overline{4}}$$

$$e_{\substack{111100\\0}} = -e_{3,8} - e_{6,15} - e_{9,17} - e_{11,20} + e_{16,27} + e_{19,\overline{26}} - e_{26,\overline{19}} - e_{\overline{27},\overline{16}} + e_{\overline{20},\overline{11}} + e_{\overline{17},\overline{9}} + e_{\overline{15},\overline{6}} + e_{\overline{8},\overline{3}}$$

$$e_{\substack{011100\\1}} = -e_{3,9} + e_{5,13} - e_{8,17} + e_{12,23} - e_{18,\overline{28}} - e_{22,\overline{24}} + e_{24,\overline{22}} + e_{28,\overline{18}} - e_{\overline{23},\overline{12}} + e_{\overline{17},\overline{8}} - e_{\overline{13},\overline{5}} + e_{\overline{9},\overline{3}}$$

$$e_{\substack{001110\\1}} = -e_{2,6} + e_{7,16} + e_{8,18} + e_{14,26} + e_{17,\overline{28}} - e_{25,\overline{20}} + e_{20,\overline{25}} - e_{28,\overline{17}} - e_{\overline{26},\overline{14}} - e_{\overline{18},\overline{8}} - e_{\overline{16},\overline{7}} + e_{\overline{6},\overline{2}}$$

$$e_{\substack{011110\\0}} = -e_{2,7} + e_{6,16} - e_{10,21} - e_{12,24} - e_{15,27} + e_{22,\overline{23}} - e_{23,\overline{22}} + e_{\overline{27},\overline{15}} + e_{\overline{24},\overline{12}} + e_{\overline{21},\overline{10}} - e_{\overline{16},\overline{6}} + e_{\overline{7},\overline{2}}$$

$$e_{\substack{001111\\0}} = -e_{1,5} + e_{9,19} + e_{10,22} - e_{14,28} - e_{17,\overline{26}} - e_{21,\overline{23}} + e_{23,\overline{21}} + e_{26,\overline{17}} + e_{\overline{28},\overline{14}} - e_{\overline{22},\overline{10}} - e_{\overline{19},\overline{9}} + e_{\overline{5},\overline{1}}$$

$$e_{\substack{111100\\1}} = e_{3,10} - e_{5,15} - e_{7,17} + e_{11,23} - e_{16,\overline{28}} - e_{19,\overline{24}} - e_{24,\overline{19}} - e_{28,\overline{16}} + e_{\overline{23},\overline{11}} - e_{\overline{17},\overline{7}} - e_{\overline{15},\overline{5}} + e_{\overline{10},\overline{3}}$$

$$e_{\substack{111110\\0}} = e_{2,8} - e_{6,18} - e_{9,21} - e_{11,24} - e_{13,27} + e_{19,\overline{23}} + e_{23,\overline{19}} - e_{\overline{27},\overline{13}} - e_{\overline{24},\overline{11}} - e_{\overline{21},\overline{9}} - e_{\overline{18},\overline{6}} + e_{\overline{8},\overline{2}}$$

$$e_{\substack{012100\\1}} = e_{3,11} + e_{4,13} + e_{8,20} + e_{10,23} + e_{18,\overline{25}} + e_{22,\overline{21}} + e_{21,\overline{22}} + e_{25,\overline{18}} + e_{\overline{23},\overline{10}} + e_{\overline{20},\overline{8}} + e_{\overline{13},\overline{4}} + e_{\overline{11},\overline{3}}$$

$$e_{\substack{011110\\1}} = e_{2,9} + e_{5,16} - e_{8,21} + e_{12,26} + e_{15,\overline{28}} - e_{22,\overline{20}} - e_{20,\overline{22}} + e_{28,\overline{15}} + e_{\overline{26},\overline{12}} - e_{\overline{21},\overline{8}} + e_{\overline{16},\overline{5}} + e_{\overline{9},\overline{2}}$$

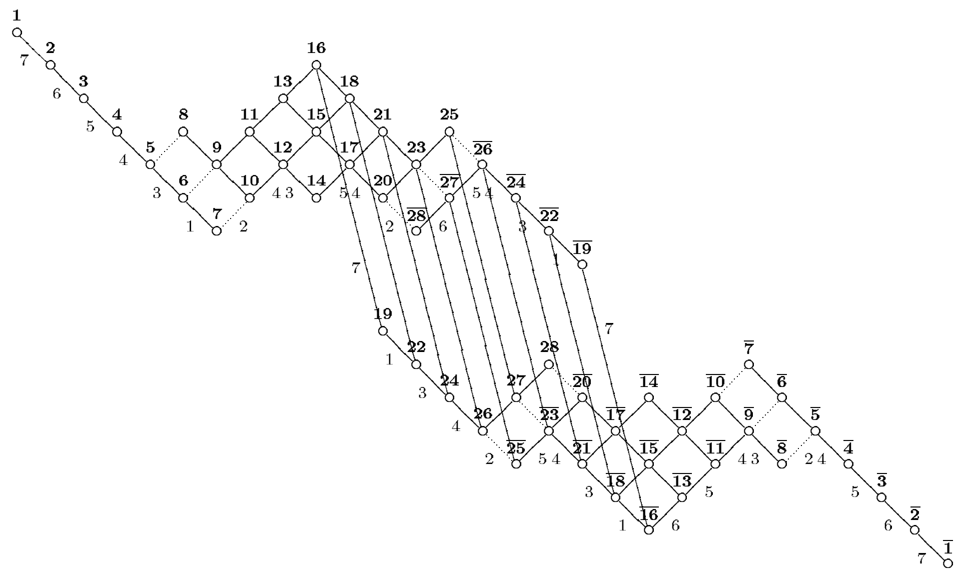
$$e_{\substack{001111\\1}} = e_{1,6} + e_{7,19} + e_{8,22} + e_{14,\overline{27}} + e_{17,\overline{24}} + e_{21,\overline{20}} + e_{20,\overline{21}} + e_{24,\overline{17}} + e_{27,\overline{14}} + e_{\overline{22},\overline{8}} + e_{\overline{19},\overline{7}} + e_{\overline{6},\overline{1}}$$

$$e_{\substack{011111\\0}} = e_{1,7} + e_{6,19} - e_{10,25} - e_{12,28} - e_{15,\overline{26}} - e_{18,\overline{23}} - e_{23,\overline{18}} - e_{26,\overline{15}} - e_{\overline{28},\overline{12}} - e_{\overline{25},\overline{10}} + e_{\overline{19},\overline{6}} + e_{\overline{7},\overline{1}}$$

$$e_{\substack{112100\\1}} = -e_{3,12} - e_{4,15} + e_{7,20} + e_{9,23} + e_{16,\overline{25}} + e_{19,\overline{21}} - e_{21,\overline{19}} - e_{25,\overline{16}} - e_{\overline{23},\overline{9}} - e_{\overline{20},\overline{7}} + e_{\overline{15},\overline{4}} + e_{\overline{12},\overline{3}}$$

$$e_{\substack{111110\\1}} = -e_{2,10} - e_{5,18} - e_{7,21} + e_{11,26} + e_{13,\overline{28}} - e_{19,\overline{20}} + e_{20,\overline{19}} - e_{28,\overline{13}} - e_{\overline{26},\overline{11}} + e_{\overline{21},\overline{7}} + e_{\overline{18},\overline{5}} + e_{\overline{10},\overline{2}}$$
$$e_{\substack{111111\\0}} = -e_{1,8} - e_{6,22} - e_{9,25} - e_{11,28} - e_{13,\overline{26}} - e_{16,\overline{23}} + e_{23,\overline{16}} + e_{26,\overline{13}} + e_{\overline{28},\overline{11}} + e_{\overline{25},\overline{9}} + e_{\overline{22},\overline{6}} + e_{\overline{8},\overline{1}}$$
$$e_{\substack{012110\\1}} = -e_{2,11} + e_{4,16} + e_{8,24} + e_{10,26} - e_{15,\overline{25}} + e_{22,\overline{17}} - e_{17,\overline{22}} + e_{25,\overline{15}} - e_{\overline{26},\overline{10}} - e_{\overline{24},\overline{8}} - e_{\overline{16},\overline{4}} + e_{\overline{11},\overline{2}}$$
$$e_{\substack{011111\\1}} = -e_{1,9} + e_{5,19} - e_{8,25} + e_{12,\overline{27}} + e_{15,\overline{24}} + e_{18,\overline{20}} - e_{20,\overline{18}} - e_{24,\overline{15}} - e_{27,\overline{12}} + e_{\overline{25},\overline{8}} - e_{\overline{19},\overline{5}} + e_{\overline{9},\overline{1}}$$
$$e_{\substack{122100\\1}} = e_{3,14} + e_{4,17} + e_{5,20} + e_{6,23} - e_{16,\overline{22}} - e_{19,\overline{18}} - e_{18,\overline{19}} - e_{22,\overline{16}} + e_{\overline{23},\overline{6}} + e_{\overline{20},\overline{5}} + e_{\overline{17},\overline{4}} + e_{\overline{14},\overline{3}}$$
$$e_{\substack{112110\\1}} = e_{2,12} - e_{4,18} + e_{7,24} + e_{9,26} - e_{13,\overline{25}} + e_{19,\overline{17}} + e_{17,\overline{19}} - e_{25,\overline{13}} + e_{\overline{26},\overline{9}} + e_{\overline{24},\overline{7}} - e_{\overline{18},\overline{4}} + e_{\overline{12},\overline{2}}$$
$$e_{\substack{111111\\1}} = e_{1,10} - e_{5,22} - e_{7,25} + e_{11,\overline{27}} + e_{13,\overline{24}} + e_{16,\overline{20}} + e_{20,\overline{16}} + e_{24,\overline{13}} + e_{27,\overline{11}} - e_{\overline{25},\overline{7}} - e_{\overline{22},\overline{5}} + e_{\overline{10},\overline{1}}$$
$$e_{\substack{012210\\1}} = e_{2,13} + e_{3,16} - e_{8,27} - e_{10,\overline{28}} - e_{12,\overline{25}} - e_{22,\overline{14}} - e_{14,\overline{22}} - e_{25,\overline{12}} - e_{28,\overline{10}} - e_{\overline{27},\overline{8}} + e_{\overline{16},\overline{3}} + e_{\overline{13},\overline{2}}$$
$$e_{\substack{012111\\1}} = e_{1,11} + e_{4,19} + e_{8,28} + e_{10,\overline{27}} - e_{15,\overline{21}} - e_{18,\overline{17}} - e_{17,\overline{18}} - e_{21,\overline{15}} + e_{27,\overline{10}} + e_{\overline{28},\overline{8}} + e_{\overline{19},\overline{4}} + e_{\overline{11},\overline{1}}$$
$$e_{\substack{122110\\1}} = -e_{2,14} + e_{4,21} + e_{5,24} + e_{6,26} + e_{13,\overline{22}} - e_{19,\overline{15}} + e_{15,\overline{19}} - e_{22,\overline{13}} - e_{\overline{26},\overline{6}} - e_{\overline{24},\overline{5}} - e_{\overline{21},\overline{4}} + e_{\overline{14},\overline{2}}$$
$$e_{\substack{112210\\1}} = -e_{2,15} - e_{3,18} - e_{7,27} - e_{9,\overline{28}} - e_{11,\overline{25}} - e_{19,\overline{14}} + e_{14,\overline{19}} + e_{25,\overline{11}} + e_{28,\overline{9}} + e_{\overline{27},\overline{7}} + e_{\overline{18},\overline{3}} + e_{\overline{15},\overline{2}}$$
$$e_{\substack{112111\\1}} = -e_{1,12} - e_{4,22} + e_{7,28} + e_{9,\overline{27}} - e_{13,\overline{21}} - e_{16,\overline{17}} + e_{17,\overline{16}} + e_{21,\overline{13}} - e_{27,\overline{9}} - e_{\overline{28},\overline{7}} + e_{\overline{22},\overline{4}} + e_{\overline{12},\overline{1}}$$
$$e_{\substack{012211\\1}} = -e_{1,13} + e_{3,19} - e_{8,\overline{26}} - e_{10,\overline{24}} - e_{12,\overline{21}} + e_{18,\overline{14}} - e_{14,\overline{18}} + e_{21,\overline{12}} + e_{24,\overline{10}} + e_{26,\overline{8}} - e_{\overline{19},\overline{3}} + e_{\overline{13},\overline{1}}$$
$$e_{\substack{122210\\1}} = e_{2,17} + e_{3,21} - e_{5,27} - e_{6,\overline{28}} + e_{11,\overline{22}} + e_{19,\overline{12}} + e_{12,\overline{19}} + e_{22,\overline{11}} - e_{28,\overline{6}} - e_{\overline{27},\overline{5}} + e_{\overline{21},\overline{3}} + e_{\overline{17},\overline{2}}$$
$$e_{\substack{122111\\1}} = e_{1,14} + e_{4,25} + e_{5,28} + e_{6,\overline{27}} + e_{13,\overline{18}} + e_{16,\overline{15}} + e_{15,\overline{16}} + e_{18,\overline{13}} + e_{27,\overline{6}} + e_{\overline{28},\overline{5}} + e_{\overline{25},\overline{4}} + e_{\overline{14},\overline{1}}$$
$$e_{\substack{112211\\1}} = e_{1,15} - e_{3,22} - e_{7,\overline{26}} - e_{9,\overline{24}} - e_{11,\overline{21}} + e_{16,\overline{14}} + e_{14,\overline{16}} - e_{21,\overline{11}} - e_{24,\overline{9}} - e_{26,\overline{7}} - e_{\overline{22},\overline{3}} + e_{\overline{15},\overline{1}}$$
$$e_{\substack{012221\\1}} = e_{1,16} + e_{2,19} + e_{8,\overline{23}} + e_{10,\overline{20}} + e_{12,\overline{17}} + e_{15,\overline{14}} + e_{14,\overline{15}} + e_{17,\overline{12}} + e_{20,\overline{10}} + e_{23,\overline{8}} + e_{\overline{19},\overline{2}} + e_{\overline{16},\overline{1}}$$
$$e_{\substack{123210\\1}} = -e_{2,20} - e_{3,24} - e_{4,27} + e_{6,\overline{25}} + e_{9,\overline{22}} - e_{19,\overline{10}} + e_{10,\overline{19}} - e_{22,\overline{9}} - e_{25,\overline{6}} + e_{\overline{27},\overline{4}} + e_{\overline{24},\overline{3}} + e_{\overline{20},\overline{2}}$$
$$e_{\substack{122211\\1}} = -e_{1,17} + e_{3,25} - e_{5,\overline{26}} - e_{6,\overline{24}} + e_{11,\overline{18}} - e_{16,\overline{12}} + e_{12,\overline{16}} - e_{18,\overline{11}} + e_{24,\overline{6}} + e_{26,\overline{5}} - e_{\overline{25},\overline{3}} + e_{\overline{17},\overline{1}}$$
$$e_{\substack{112221\\1}} = -e_{1,18} - e_{2,22} + e_{7,\overline{23}} + e_{9,\overline{20}} + e_{11,\overline{17}} + e_{13,\overline{14}} - e_{14,\overline{13}} - e_{17,\overline{11}} - e_{20,\overline{9}} - e_{23,\overline{7}} + e_{\overline{22},\overline{2}} + e_{\overline{18},\overline{1}}$$
$$e_{\substack{123210\\2}} = e_{2,23} + e_{3,26} + e_{4,\overline{28}} + e_{5,\overline{25}} + e_{7,\overline{22}} + e_{19,\overline{8}} + e_{8,\overline{19}} + e_{22,\overline{7}} + e_{25,\overline{5}} + e_{28,\overline{4}} + e_{\overline{26},\overline{3}} + e_{\overline{23},\overline{2}}$$
$$e_{\substack{123211\\1}} = e_{1,20} - e_{3,28} - e_{4,\overline{26}} + e_{6,\overline{21}} + e_{9,\overline{18}} + e_{16,\overline{10}} + e_{10,\overline{16}} + e_{18,\overline{9}} + e_{21,\overline{6}} - e_{26,\overline{4}} - e_{\overline{28},\overline{3}} + e_{\overline{20},\overline{1}}$$
$$e_{\substack{122221\\1}} = e_{1,21} + e_{2,25} + e_{5,\overline{23}} + e_{6,\overline{20}} - e_{11,\overline{15}} - e_{13,\overline{12}} - e_{12,\overline{13}} - e_{15,\overline{11}} + e_{20,\overline{6}} + e_{23,\overline{5}} + e_{\overline{25},\overline{2}} + e_{\overline{21},\overline{1}}$$
$$e_{\substack{123211\\2}} = -e_{1,23} + e_{3,\overline{27}} + e_{4,\overline{24}} + e_{5,\overline{21}} + e_{7,\overline{18}} - e_{16,\overline{8}} + e_{8,\overline{16}} - e_{18,\overline{7}} - e_{21,\overline{5}} - e_{24,\overline{4}} - e_{27,\overline{3}} + e_{\overline{23},\overline{1}}$$
$$e_{\substack{123221\\1}} = -e_{1,24} - e_{2,28} + e_{4,\overline{23}} - e_{6,\overline{17}} - e_{9,\overline{15}} + e_{13,\overline{10}} - e_{10,\overline{13}} + e_{15,\overline{9}} + e_{17,\overline{6}} - e_{23,\overline{4}} + e_{\overline{28},\overline{2}} + e_{\overline{24},\overline{1}}$$
$$e_{\substack{123221\\2}} = e_{1,26} + e_{2,\overline{27}} - e_{4,\overline{20}} - e_{5,\overline{17}} - e_{7,\overline{15}} - e_{13,\overline{8}} - e_{8,\overline{13}} - e_{15,\overline{7}} - e_{17,\overline{5}} - e_{20,\overline{4}} + e_{27,\overline{2}} + e_{\overline{26},\overline{1}}$$
$$e_{\substack{123321\\1}} = e_{1,27} + e_{2,\overline{26}} + e_{3,\overline{23}} + e_{6,\overline{14}} + e_{9,\overline{12}} + e_{11,\overline{10}} + e_{10,\overline{11}} + e_{12,\overline{9}} + e_{14,\overline{6}} + e_{23,\overline{3}} + e_{26,\overline{2}} + e_{\overline{27},\overline{1}}$$
$$e_{\substack{123321\\2}} = -e_{1,\overline{28}} - e_{2,\overline{24}} - e_{3,\overline{20}} + e_{5,\overline{14}} + e_{7,\overline{12}} - e_{11,\overline{8}} + e_{8,\overline{11}} - e_{12,\overline{7}} - e_{14,\overline{5}} + e_{20,\overline{3}} + e_{24,\overline{2}} + e_{28,\overline{1}}$$
$$e_{\substack{124321\\2}} = e_{1,\overline{25}} + e_{2,\overline{21}} + e_{3,\overline{17}} + e_{4,\overline{14}} - e_{7,\overline{10}} - e_{9,\overline{8}} - e_{8,\overline{9}} - e_{10,\overline{7}} + e_{14,\overline{4}} + e_{17,\overline{3}} + e_{21,\overline{2}} + e_{25,\overline{1}}$$
$$e_{\substack{134321\\2}} = -e_{1,\overline{22}} - e_{2,\overline{18}} - e_{3,\overline{15}} - e_{4,\overline{12}} - e_{5,\overline{10}} - e_{6,\overline{8}} + e_{8,\overline{6}} + e_{10,\overline{5}} + e_{12,\overline{4}} + e_{15,\overline{3}} + e_{18,\overline{2}} + e_{22,\overline{1}}$$
$$e_{\substack{234321\\2}} = e_{1,\overline{19}} + e_{2,\overline{16}} + e_{3,\overline{13}} + e_{4,\overline{11}} + e_{5,\overline{9}} + e_{7,\overline{6}} + e_{6,\overline{7}} + e_{9,\overline{5}} + e_{11,\overline{4}} + e_{13,\overline{3}} + e_{16,\overline{2}} + e_{19,\overline{1}}$$

Table 11. The matrix of signs of $V(\varpi_7)$: the A_6-numeration.



	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28
1	0	+	−	+	−	+	−	+	−	+	+	−	−	+	+	+	−	−	0	+	+	0	−	0	+	0	0	0
2	+	0	+	−	+	−	+	−	+	−	−	+	+	−	−	0	+	0	+	−	0	−	0	+	0	−	+	0
3	−	+	0	+	−	+	−	+	−	+	+	−	0	+	0	+	0	−	+	0	+	−	−	+	0	−	0	+
4	+	−	+	0	+	−	+	−	+	−	0	0	+	0	−	+	+	−	+	0	+	−	0	+	−	0	−	+
5	−	+	−	+	0	+	−	+	0	0	+	−	+	0	−	+	0	−	+	+	0	−	+	0	−	+	−	+
6	+	−	+	−	+	0	+	0	+	0	+	0	+	−	0	+	−	0	+	+	−	0	+	−	−	+	−	+
7	−	+	−	+	−	+	0	0	0	+	0	+	0	−	+	0	−	+	0	+	−	+	+	−	−	+	−	+
8	+	−	+	−	+	0	0	0	+	−	+	−	+	0	−	+	0	−	+	0	0	−	0	0	0	0	0	0
9	−	+	−	+	0	+	0	+	0	+	+	0	+	−	0	+	−	0	+	0	−	0	0	−	0	0	0	0
10	+	−	+	−	0	0	+	−	+	0	0	+	0	−	+	0	−	+	0	0	−	+	0	−	0	0	0	0
11	+	−	+	0	+	+	0	+	+	0	0	+	+	+	0	+	0	0	+	−	0	0	−	0	0	−	0	0
12	−	+	−	0	−	0	+	−	0	+	+	0	0	+	+	0	0	+	0	−	0	+	−	0	0	−	0	0
13	−	+	0	+	+	+	0	+	+	0	+	0	0	0	+	+	+	0	+	+	0	0	0	0	−	0	−	0
14	+	−	+	0	0	−	−	0	−	−	+	+	0	0	0	0	+	0	0	−	+	0	−	+	0	−	0	0
15	+	−	0	−	−	0	+	−	0	+	0	+	+	0	0	0	+	+	0	+	0	+	0	0	−	0	−	0
16	+	0	+	+	+	+	0	+	+	0	+	0	+	0	0	0	0	+	+	0	+	0	+	0	+	0	0	−
17	−	+	0	+	0	−	−	0	−	−	0	0	+	+	+	0	0	0	0	+	+	0	0	+	−	0	−	0
18	−	0	−	−	−	0	+	−	0	+	0	+	0	0	+	+	0	0	0	0	+	+	+	0	+	0	0	−
19	0	+	+	+	+	+	0	+	+	0	+	0	+	0	0	+	0	0	0	0	0	+	0	+	0	+	+	+
20	+	−	0	0	+	+	+	0	0	0	−	−	+	−	+	0	+	0	0	0	0	0	+	0	−	+	−	0
21	+	0	+	+	0	−	−	0	−	−	0	0	0	+	0	+	+	+	0	0	0	0	+	+	+	0	0	−
22	0	−	−	−	−	0	+	−	0	+	0	+	0	0	+	0	0	+	+	0	0	0	0	+	0	+	+	+
23	−	0	−	0	+	+	+	0	0	0	−	−	0	−	0	+	0	+	0	+	+	0	0	0	+	+	0	−
24	0	+	+	+	0	−	−	0	−	−	0	0	0	+	0	0	+	0	+	0	+	+	0	0	0	+	+	+
25	+	0	0	−	−	−	−	0	0	0	0	0	−	0	−	+	−	+	0	−	+	0	+	0	0	0	+	−
26	0	−	−	0	+	+	+	0	0	0	−	−	0	−	0	0	0	0	+	+	0	+	+	+	0	0	+	+
27	0	+	0	−	−	−	−	0	0	0	0	0	−	0	−	0	−	0	+	−	0	+	0	+	+	+	0	+
28	0	0	+	+	+	+	+	0	0	0	0	0	0	0	0	−	0	−	+	0	−	+	−	+	−	+	+	0

	$\overline{28}$ $\overline{27}$ $\overline{26}$ $\overline{25}$ $\overline{24}$	$\overline{23}$ $\overline{22}$ $\overline{21}$ $\overline{20}$ $\overline{19}$	$\overline{18}$ $\overline{17}$ $\overline{16}$ $\overline{15}$ $\overline{14}$	$\overline{13}$ $\overline{12}$ $\overline{11}$ $\overline{10}$ $\overline{9}$	$\overline{8}$ $\overline{7}$ $\overline{6}$ $\overline{5}$ $\overline{4}$	$\overline{3}$ $\overline{2}$ $\overline{1}$
1	− + − 0 +	0 − 0 0 +	0 0 0 0 0	0 0 0 0 0	0 0 0 0 0	0 0 0
2	+ 0 0 + 0	− 0 + 0 0	− 0 + 0 0	0 0 0 0 0	0 0 0 0 0	0 0 0
3	0 + 0 + 0	0 0 0 − 0	0 + 0 − 0	+ 0 0 0 0	0 0 0 0 0	0 0 0
4	0 0 + 0 0	+ 0 0 − 0	0 0 0 0 +	0 − + 0 0	0 0 0 0 0	0 0 0
5	0 0 0 0 +	0 0 + 0 0	0 − 0 0 +	0 0 0 − +	0 0 0 0 0	0 0 0
6	0 0 0 0 0	0 + 0 0 0	+ 0 0 − 0	0 + 0 − 0	+ 0 0 0 0	0 0 0
7	0 0 0 0 0	0 0 0 0 +	0 0 + 0 0	− 0 + 0 −	+ 0 0 0 0	0 0 0
8	+ + − + +	− 0 + + 0	0 − 0 0 +	0 0 0 0 0	0 − + 0 0	0 0 0
9	+ + − + 0	− + 0 + 0	+ 0 0 − 0	0 + 0 0 0	0 − 0 + 0	0 0 0
10	+ + − + 0	− 0 0 + +	0 0 + 0 0	− 0 + 0 0	0 0 − + 0	0 0 0
11	+ + 0 + −	0 + − 0 0	+ + 0 − 0	0 0 0 + 0	0 − 0 0 +	0 0 0
12	+ + 0 + −	0 0 − 0 +	0 + + 0 0	− 0 0 0 +	0 0 − 0 +	0 0 0
13	− 0 + 0 −	+ + − 0 0	+ 0 0 0 +	0 − 0 + 0	0 − 0 0 0	+ 0 0
14	+ + 0 + 0	0 − 0 0 +	− 0 + + 0	− 0 0 0 0	+ 0 0 − +	0 0 0
15	− 0 + 0 −	+ 0 − 0 +	0 0 + 0 +	0 0 − 0 +	0 0 − 0 0	+ 0 0
16	0 − − 0 +	0 − 0 + 0	0 − 0 + +	0 − 0 + 0	0 − 0 0 0	0 + 0
17	− 0 + 0 0	+ − 0 0 +	− 0 + 0 0	0 + − 0 0	+ 0 0 − 0	+ 0 0
18	0 − − 0 +	0 0 0 + −	0 − 0 0 +	+ 0 − 0 +	0 0 − 0 0	0 + 0
19	0 0 0 − 0	− 0 + − 0	− + 0 − −	0 + 0 − 0	0 + 0 0 0	0 0 +
20	+ 0 0 0 +	0 − + 0 +	− 0 + 0 0	0 0 0 + −	+ 0 0 0 −	+ 0 0
21	0 − − 0 0	0 + 0 + −	0 0 0 − 0	+ + − 0 0	+ 0 0 − 0	0 + 0
22	0 0 0 − 0	− 0 + − 0	0 + − 0 −	− 0 + 0 −	0 0 + 0 0	0 0 +
23	0 + 0 0 −	0 + 0 0 −	0 + 0 − 0	+ 0 0 + −	+ 0 0 0 −	0 + 0
24	0 0 0 − 0	− 0 0 − 0	+ 0 − + 0	− − + 0 0	− 0 0 + 0	0 0 +
25	0 0 + 0 +	0 − 0 0 +	0 0 0 0 +	0 − + + −	+ 0 0 0 0	− + 0
26	0 0 0 + 0	0 0 − 0 0	+ − − + 0	− 0 0 − +	− 0 0 0 +	0 0 +
27	0 0 0 0 0	+ 0 + 0 0	− 0 + 0 −	0 + − − +	− 0 0 0 0	+ 0 +
28	0 0 0 0 0	0 0 0 + 0	0 + 0 − +	+ − + + −	+ 0 0 0 0	0 + +

	1 2 3 4 5	6 7 8 9 10	11 12 13 14 15	16 17 18 19 20	21 22 23 24 25	26 27 28
$\overline{28}$	− + 0 0 0	0 0 + + +	+ + − + −	0 − 0 0 +	0 0 0 0 0	0 0 0
$\overline{27}$	+ 0 + 0 0	0 0 + + +	+ + 0 + 0	− 0 − 0 0	− 0 + 0 0	0 0 0
$\overline{26}$	− 0 0 + 0	0 0 − − −	0 0 + 0 +	− + − 0 0	− 0 0 0 +	0 0 0
$\overline{25}$	0 + + 0 0	0 0 + + +	+ + 0 + 0	0 0 0 − 0	0 − 0 − 0	+ 0 0
$\overline{24}$	+ 0 0 0 +	0 0 + 0 0	− − − 0 −	+ 0 + 0 +	0 0 − 0 +	0 0 0
$\overline{23}$	0 − 0 + 0	0 0 − − −	0 0 + 0 +	0 + 0 − 0	0 − 0 − 0	0 + 0
$\overline{22}$	− 0 0 0 0	+ 0 0 + 0	+ 0 + − 0	− − 0 0 −	+ 0 + 0 −	0 0 0
$\overline{21}$	0 + 0 0 +	0 0 + 0 0	− − − 0 −	0 0 0 + +	0 + 0 0 0	− + 0
$\overline{20}$	0 0 − − 0	0 0 + + +	0 0 0 0 0	+ 0 + − 0	+ − 0 − 0	0 0 +
$\overline{19}$	+ 0 0 0 0	0 + 0 0 +	0 + 0 + +	0 + − 0 +	− 0 − 0 +	0 0 0
$\overline{18}$	0 − 0 0 0	+ 0 0 + 0	+ 0 + − 0	0 − 0 − −	0 0 0 + 0	+ − 0
$\overline{17}$	0 0 + 0 −	0 0 − 0 0	+ + 0 0 0	− 0 − + 0	0 + + 0 0	− 0 +
$\overline{16}$	0 + 0 0 0	0 + 0 0 +	0 + 0 + +	0 + 0 0 +	0 − 0 − 0	− + 0
$\overline{15}$	0 0 − 0 0	− 0 0 − 0	− 0 0 + 0	+ 0 0 − 0	− 0 − + 0	+ 0 −
$\overline{14}$	0 0 0 + +	0 0 + 0 0	0 0 + 0 +	+ 0 + − 0	0 − 0 0 +	0 − +
$\overline{13}$	0 0 + 0 0	0 − 0 0 −	0 − 0 − 0	0 0 + 0 0	+ − + − 0	− 0 +
$\overline{12}$	0 0 0 − 0	+ 0 0 + 0	0 0 − 0 0	− + 0 + 0	+ 0 0 − −	0 + −
$\overline{11}$	0 0 0 + 0	0 + 0 0 +	0 0 0 0 −	0 − − 0 0	− + 0 + +	0 − +
$\overline{10}$	0 0 0 0 −	− 0 0 0 0	+ 0 + 0 0	+ 0 0 − +	0 0 + 0 +	− − +
$\overline{9}$	0 0 0 0 +	0 − 0 0 0	0 + 0 0 +	0 0 + 0 −	0 − − 0 −	+ + −
$\overline{8}$	0 0 0 0 0	+ + 0 0 0	0 0 0 + 0	0 + 0 0 +	+ 0 + − +	− − +
$\overline{7}$	0 0 0 0 0	0 0 − − 0	− 0 − 0 0	− 0 0 + 0	0 0 0 0 0	0 0 0
$\overline{6}$	0 0 0 0 0	0 0 + 0 −	0 − 0 0 −	0 0 − 0 0	0 + 0 0 0	0 0 0
$\overline{5}$	0 0 0 0 0	0 0 0 + +	0 0 0 − 0	0 − 0 0 0	− 0 0 + 0	0 0 0
$\overline{4}$	0 0 0 0 0	0 0 0 0 0	+ + 0 + 0	0 0 0 0 −	0 0 − 0 0	+ 0 0
$\overline{3}$	0 0 0 0 0	0 0 0 0 0	0 0 + 0 +	0 + 0 0 +	0 0 0 0 −	0 + 0
$\overline{2}$	0 0 0 0 0	0 0 0 0 0	0 0 0 0 0	+ 0 + 0 0	+ 0 + 0 +	0 0 +
$\overline{1}$	0 0 0 0 0	0 0 0 0 0	0 0 0 0 0	0 0 0 + 0	0 + 0 + 0	+ + +

	$\overline{28}$	$\overline{27}$	$\overline{26}$	$\overline{25}$	$\overline{24}$	$\overline{23}$	$\overline{22}$	$\overline{21}$	$\overline{20}$	$\overline{19}$	$\overline{18}$	$\overline{17}$	$\overline{16}$	$\overline{15}$	$\overline{14}$	$\overline{13}$	$\overline{12}$	$\overline{11}$	$\overline{10}$	$\overline{9}$	$\overline{8}$	$\overline{7}$	$\overline{6}$	$\overline{5}$	$\overline{4}$	$\overline{3}$	$\overline{2}$	$\overline{1}$
$\overline{28}$	0	+	−	+	+	−	−	+	0	+	−	0	+	0	0	0	0	0	0	0	0	+	−	+	−	+	0	0
$\overline{27}$	+	0	+	+	−	0	+	0	−	−	0	+	0	−	0	+	0	0	0	0	0	+	−	+	−	0	+	0
$\overline{26}$	−	+	0	0	+	+	−	0	−	+	0	0	0	0	+	0	−	+	0	0	0	+	−	+	0	−	+	0
$\overline{25}$	+	+	0	0	0	+	0	−	+	0	+	−	−	+	0	−	0	0	0	0	0	−	+	−	+	0	0	+
$\overline{24}$	+	−	+	0	0	0	+	+	0	−	0	−	0	0	+	0	0	0	−	+	0	+	−	0	+	−	+	0
$\overline{23}$	−	0	+	+	0	0	0	+	+	0	−	0	+	0	−	0	+	−	0	0	0	−	+	−	0	+	0	+
$\overline{22}$	−	+	−	0	+	0	0	0	0	+	+	0	0	−	0	0	+	0	−	0	+	+	0	−	+	−	+	0
$\overline{21}$	+	0	0	−	+	+	0	0	0	0	+	+	−	0	−	0	0	0	+	−	0	−	+	0	−	+	0	+
$\overline{20}$	0	−	−	+	0	+	0	0	0	0	0	+	0	−	+	+	−	+	0	0	0	+	−	+	0	0	+	+
$\overline{19}$	+	−	+	0	−	0	+	0	0	0	0	0	+	0	0	−	0	+	0	−	+	0	+	−	+	−	+	0
$\overline{18}$	−	0	0	+	0	−	+	+	0	0	0	0	+	+	0	0	−	0	+	0	−	−	0	+	−	+	0	+
$\overline{17}$	0	+	0	−	−	0	0	+	+	0	0	0	0	+	+	−	0	0	−	+	0	+	−	0	+	0	+	+
$\overline{16}$	+	0	0	−	0	+	0	−	0	+	+	0	0	0	0	+	0	−	0	+	−	0	−	+	−	+	0	+
$\overline{15}$	0	−	0	+	0	0	−	0	−	0	+	+	0	0	0	+	+	0	−	0	+	+	0	−	+	0	+	+
$\overline{14}$	0	0	+	0	+	−	0	−	+	0	0	+	0	0	0	0	+	−	+	−	0	−	+	0	0	+	+	+
$\overline{13}$	0	+	0	−	0	0	0	0	+	−	0	−	+	+	0	0	0	+	0	−	+	0	+	−	+	0	+	+
$\overline{12}$	0	0	−	0	0	+	+	0	−	0	−	0	0	+	+	0	0	+	+	0	−	−	0	+	0	+	+	+
$\overline{11}$	0	0	+	0	0	−	0	0	+	+	0	0	−	0	−	+	+	0	0	+	−	0	−	+	0	+	+	+
$\overline{10}$	0	0	0	0	−	0	−	+	0	0	+	−	0	−	+	0	+	0	0	+	+	+	0	0	+	+	+	+
$\overline{9}$	0	0	0	0	+	0	0	−	0	−	0	+	+	0	−	−	0	+	+	0	+	0	+	0	+	+	+	+
$\overline{8}$	0	0	0	0	0	0	+	0	0	+	−	0	−	+	0	+	−	−	+	+	0	0	0	+	+	+	+	+
$\overline{7}$	+	+	+	−	+	−	+	−	+	0	−	+	0	+	−	0	−	0	+	0	0	0	+	+	+	+	+	+
$\overline{6}$	−	−	−	+	−	+	0	+	−	+	0	−	−	0	+	+	0	−	0	+	0	+	0	+	+	+	+	+
$\overline{5}$	+	+	+	−	0	−	−	0	+	−	+	0	+	−	0	−	+	+	0	0	+	+	+	0	+	+	+	+
$\overline{4}$	−	−	0	+	+	0	+	−	0	+	−	+	−	+	0	+	0	0	+	+	+	+	+	+	0	+	+	+
$\overline{3}$	+	0	−	0	−	+	−	+	0	−	+	0	+	0	+	0	+	+	+	+	+	+	+	+	+	0	+	+
$\overline{2}$	0	+	+	0	+	0	+	0	+	+	0	+	0	+	+	+	+	+	+	+	+	+	+	+	+	+	0	+
$\overline{1}$	0	0	0	+	0	+	0	+	+	0	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	0

TABLE 12. Root elements: the A_6-numeration.

$$e_{\substack{100000\\0}} = e_{6,7} + e_{9,10} + e_{11,12} + e_{13,15} + e_{16,18} + e_{19,22} + e_{\overline{22},\overline{19}} + e_{\overline{18},\overline{16}} + e_{\overline{15},\overline{13}} + e_{\overline{12},\overline{11}} + e_{\overline{10},\overline{9}} + e_{\overline{7},\overline{6}}$$

$$e_{\substack{000000\\1}} = e_{5,8} + e_{6,9} + e_{7,10} + e_{20,\overline{28}} + e_{23,\overline{27}} + e_{26,\overline{25}} + e_{25,\overline{26}} + e_{27,\overline{23}} + e_{28,\overline{20}} + e_{\overline{10},\overline{7}} + e_{\overline{9},\overline{6}} + e_{\overline{8},\overline{5}}$$

$$e_{\substack{010000\\0}} = e_{5,6} + e_{8,9} + e_{12,14} + e_{15,17} + e_{18,21} + e_{22,24} + e_{\overline{24},\overline{22}} + e_{\overline{21},\overline{18}} + e_{\overline{17},\overline{15}} + e_{\overline{14},\overline{12}} + e_{\overline{9},\overline{8}} + e_{\overline{6},\overline{5}}$$

$$e_{\substack{001000\\0}} = e_{4,5} + e_{9,11} + e_{10,12} + e_{17,20} + e_{21,23} + e_{24,26} + e_{\overline{26},\overline{24}} + e_{\overline{23},\overline{21}} + e_{\overline{20},\overline{17}} + e_{\overline{12},\overline{10}} + e_{\overline{11},\overline{9}} + e_{\overline{5},\overline{4}}$$

$$e_{\substack{000100\\0}} = e_{3,4} + e_{11,13} + e_{12,15} + e_{14,17} + e_{23,25} + e_{26,27} + e_{\overline{27},\overline{26}} + e_{\overline{25},\overline{23}} + e_{\overline{17},\overline{14}} + e_{\overline{15},\overline{12}} + e_{\overline{13},\overline{11}} + e_{\overline{4},\overline{3}}$$

$$e_{\substack{000010\\0}} = e_{2,3} + e_{13,16} + e_{15,18} + e_{17,21} + e_{20,23} + e_{27,28} + e_{\overline{28},\overline{27}} + e_{\overline{23},\overline{20}} + e_{\overline{21},\overline{17}} + e_{\overline{18},\overline{15}} + e_{\overline{16},\overline{13}} + e_{\overline{3},\overline{2}}$$

$$e_{\substack{000001\\0}} = e_{1,2} + e_{16,19} + e_{18,22} + e_{21,24} + e_{23,26} + e_{25,27} + e_{\overline{27},\overline{25}} + e_{\overline{26},\overline{23}} + e_{\overline{24},\overline{21}} + e_{\overline{22},\overline{18}} + e_{\overline{19},\overline{16}} + e_{\overline{2},\overline{1}}$$

$$e_{\substack{110000\\0}} = -e_{5,7} - e_{8,10} + e_{11,14} + e_{13,17} + e_{16,21} + e_{19,24} - e_{\overline{24},\overline{19}} - e_{\overline{21},\overline{16}} - e_{\overline{17},\overline{13}} - e_{\overline{14},\overline{11}} + e_{\overline{10},\overline{8}} + e_{\overline{7},\overline{5}}$$

$$e_{\substack{001000\\1}} = -e_{4,8} + e_{6,11} + e_{7,12} - e_{17,\overline{28}} - e_{21,\overline{27}} - e_{24,\overline{25}} + e_{25,\overline{24}} + e_{27,\overline{21}} + e_{28,\overline{17}} - e_{\overline{12},\overline{7}} - e_{\overline{11},\overline{6}} + e_{\overline{8},\overline{4}}$$

$$e_{\substack{011000\\0}} = -e_{4,6} + e_{8,11} - e_{10,14} + e_{15,20} + e_{18,23} + e_{22,26} - e_{\overline{26},\overline{22}} - e_{\overline{23},\overline{18}} - e_{\overline{20},\overline{15}} + e_{\overline{14},\overline{10}} - e_{\overline{11},\overline{8}} + e_{\overline{6},\overline{4}}$$

$$e_{\substack{001100\\0}} = -e_{3,5} + e_{9,13} + e_{10,15} - e_{14,20} + e_{21,25} + e_{24,27} - e_{\overline{27},\overline{24}} - e_{\overline{25},\overline{21}} + e_{\overline{20},\overline{14}} - e_{\overline{15},\overline{10}} - e_{\overline{13},\overline{9}} + e_{\overline{5},\overline{3}}$$

$$e_{\substack{000110\\0}} = -e_{2,4} + e_{11,16} + e_{12,18} + e_{14,21} - e_{20,25} + e_{26,28} - e_{\overline{28},\overline{26}} + e_{\overline{25},\overline{20}} - e_{\overline{21},\overline{14}} - e_{\overline{18},\overline{12}} - e_{\overline{16},\overline{11}} + e_{\overline{4},\overline{2}}$$

$$e_{\substack{000011\\0}} = -e_{1,3} + e_{13,19} + e_{15,22} + e_{17,24} + e_{20,26} - e_{25,28} + e_{\overline{28},\overline{25}} - e_{\overline{26},\overline{20}} - e_{\overline{24},\overline{17}} - e_{\overline{22},\overline{15}} - e_{\overline{19},\overline{13}} + e_{\overline{3},\overline{1}}$$

$$e_{\substack{111000\\0}} = e_{4,7} - e_{8,12} - e_{9,14} + e_{13,20} + e_{16,23} + e_{19,26} + e_{\overline{26},\overline{19}} + e_{\overline{23},\overline{16}} + e_{\overline{20},\overline{13}} - e_{\overline{14},\overline{9}} - e_{\overline{12},\overline{8}} + e_{\overline{7},\overline{4}}$$

$$e_{\substack{011000\\1}} = e_{4,9} + e_{5,11} - e_{7,14} - e_{15,\overline{28}} - e_{18,\overline{27}} - e_{22,\overline{25}} - e_{25,\overline{22}} - e_{27,\overline{18}} - e_{28,\overline{15}} - e_{\overline{14},\overline{7}} + e_{\overline{11},\overline{5}} + e_{\overline{9},\overline{4}}$$

$$e_{\substack{001100\\1}} = e_{3,8} + e_{6,13} + e_{7,15} + e_{14,\overline{28}} - e_{21,\overline{26}} - e_{24,\overline{23}} - e_{23,\overline{24}} - e_{26,\overline{21}} + e_{28,\overline{14}} + e_{\overline{15},\overline{7}} + e_{\overline{13},\overline{6}} + e_{\overline{8},\overline{3}}$$

$$e_{\substack{011100\\0}} = e_{3,6} + e_{8,13} - e_{10,17} - e_{12,20} + e_{18,25} + e_{22,27} + e_{\overline{27},\overline{22}} + e_{\overline{25},\overline{18}} - e_{\overline{20},\overline{12}} - e_{\overline{17},\overline{10}} + e_{\overline{13},\overline{8}} + e_{\overline{6},\overline{3}}$$

$$e_{\substack{001110\\0}} = e_{2,5} + e_{9,16} + e_{10,18} - e_{14,23} - e_{17,25} + e_{24,28} + e_{\overline{28},\overline{24}} - e_{\overline{25},\overline{17}} - e_{\overline{23},\overline{14}} + e_{\overline{18},\overline{10}} + e_{\overline{16},\overline{9}} + e_{\overline{5},\overline{2}}$$

$$e_{\substack{000111\\0}} = e_{1,4} + e_{11,19} + e_{12,22} + e_{14,24} - e_{20,27} - e_{23,28} - e_{\overline{28},\overline{23}} - e_{\overline{27},\overline{20}} + e_{\overline{24},\overline{14}} + e_{\overline{22},\overline{12}} + e_{\overline{19},\overline{11}} + e_{\overline{4},\overline{1}}$$

$$e_{\substack{111000\\1}} = -e_{4,10} - e_{5,12} - e_{6,14} - e_{13,\overline{28}} - e_{16,\overline{27}} - e_{19,\overline{25}} + e_{25,\overline{19}} + e_{27,\overline{16}} + e_{28,\overline{13}} + e_{\overline{14},\overline{6}} + e_{\overline{12},\overline{5}} + e_{\overline{10},\overline{4}}$$

$$e_{\substack{111100\\0}} = -e_{3,7} - e_{8,15} - e_{9,17} - e_{11,20} + e_{16,25} + e_{19,27} - e_{\overline{27},\overline{19}} - e_{\overline{25},\overline{16}} + e_{\overline{20},\overline{11}} + e_{\overline{17},\overline{9}} + e_{\overline{15},\overline{8}} + e_{\overline{7},\overline{3}}$$

$$e_{\substack{011100\\1}} = -e_{3,9} + e_{5,13} - e_{7,17} + e_{12,\overline{28}} - e_{18,\overline{26}} - e_{22,\overline{23}} + e_{23,\overline{22}} + e_{26,\overline{18}} - e_{28,\overline{12}} + e_{\overline{17},\overline{7}} - e_{\overline{13},\overline{5}} + e_{\overline{9},\overline{3}}$$

$$e_{\substack{001110\\1}} = -e_{2,8} + e_{6,16} + e_{7,18} + e_{14,\overline{27}} + e_{17,\overline{26}} - e_{24,\overline{20}} + e_{20,\overline{24}} - e_{26,\overline{17}} - e_{27,\overline{14}} - e_{\overline{18},\overline{7}} - e_{\overline{16},\overline{6}} + e_{\overline{8},\overline{2}}$$

$$e_{\substack{011110\\0}} = -e_{2,6} + e_{8,16} - e_{10,21} - e_{12,23} - e_{15,25} + e_{22,28} - e_{\overline{28},\overline{22}} + e_{\overline{25},\overline{15}} + e_{\overline{23},\overline{12}} + e_{\overline{21},\overline{10}} - e_{\overline{16},\overline{8}} + e_{\overline{6},\overline{2}}$$

$$e_{\substack{001111\\0}} = -e_{1,5} + e_{9,19} + e_{10,22} - e_{14,26} - e_{17,27} - e_{21,28} + e_{\overline{28},\overline{21}} + e_{\overline{27},\overline{17}} + e_{\overline{26},\overline{14}} - e_{\overline{22},\overline{10}} - e_{\overline{19},\overline{9}} + e_{\overline{5},\overline{1}}$$

$$e_{\substack{111100\\1}} = e_{3,10} - e_{5,15} - e_{6,17} + e_{11,\overline{28}} - e_{16,\overline{26}} - e_{19,\overline{23}} - e_{23,\overline{19}} - e_{26,\overline{16}} + e_{28,\overline{11}} - e_{\overline{17},\overline{6}} - e_{\overline{15},\overline{5}} + e_{\overline{10},\overline{3}}$$

$$e_{\substack{111110\\0}} = e_{2,7} - e_{8,18} - e_{9,21} - e_{11,23} - e_{13,25} + e_{19,28} + e_{\overline{28},\overline{19}} - e_{\overline{25},\overline{13}} - e_{\overline{23},\overline{11}} - e_{\overline{21},\overline{9}} - e_{\overline{18},\overline{8}} + e_{\overline{7},\overline{2}}$$

$$e_{\substack{012100\\1}} = e_{3,11} + e_{4,13} + e_{7,20} + e_{10,\overline{28}} + e_{18,\overline{24}} + e_{22,\overline{21}} + e_{21,\overline{22}} + e_{24,\overline{18}} + e_{28,\overline{10}} + e_{\overline{20},\overline{7}} + e_{\overline{13},\overline{4}} + e_{\overline{11},\overline{3}}$$

$$e_{\substack{011110\\1}} = e_{2,9} + e_{5,16} - e_{7,21} + e_{12,\overline{27}} + e_{15,\overline{26}} - e_{22,\overline{20}} - e_{20,\overline{22}} + e_{26,\overline{15}} + e_{27,\overline{12}} - e_{\overline{21},\overline{7}} + e_{\overline{16},\overline{5}} + e_{\overline{9},\overline{2}}$$

$$e_{\substack{001111\\1}} = e_{1,8} + e_{6,19} + e_{7,22} + e_{14,\overline{25}} + e_{17,\overline{23}} + e_{21,\overline{20}} + e_{20,\overline{21}} + e_{23,\overline{17}} + e_{25,\overline{14}} + e_{\overline{22},\overline{7}} + e_{\overline{19},\overline{6}} + e_{\overline{8},\overline{1}}$$

$$e_{\substack{011111\\0}} = e_{1,6} + e_{8,19} - e_{10,24} - e_{12,26} - e_{15,27} - e_{18,28} - e_{\overline{28},\overline{18}} - e_{\overline{27},\overline{15}} - e_{\overline{26},\overline{12}} - e_{\overline{24},\overline{10}} + e_{\overline{19},\overline{8}} + e_{\overline{6},\overline{1}}$$

$$e_{\substack{112100\\1}} = -e_{3,12} - e_{4,15} + e_{6,20} + e_{9,\overline{28}} + e_{16,\overline{24}} + e_{19,\overline{21}} - e_{21,\overline{19}} - e_{24,\overline{16}} - e_{28,\overline{9}} - e_{\overline{20},\overline{6}} + e_{\overline{15},\overline{4}} + e_{\overline{12},\overline{3}}$$

$$e_{\substack{111110\\1}} = -e_{2,10} - e_{5,18} - e_{6,21} + e_{11,\overline{27}} + e_{13,\overline{26}} - e_{19,\overline{20}} + e_{20,\overline{19}} - e_{26,\overline{13}} - e_{27,\overline{11}} + e_{\overline{21},\overline{6}} + e_{\overline{18},\overline{5}} + e_{\overline{10},\overline{2}}$$

$$e_{\substack{111111\\0}} = -e_{1,7} - e_{8,22} - e_{9,24} - e_{11,26} - e_{13,27} - e_{16,28} + e_{\overline{28},\overline{16}} + e_{\overline{27},\overline{13}} + e_{\overline{26},\overline{11}} + e_{\overline{24},\overline{9}} + e_{\overline{22},\overline{8}} + e_{\overline{7},\overline{1}}$$

$$e_{\substack{012110\\1}} = -e_{2,11} + e_{4,16} + e_{7,23} + e_{10,\overline{27}} - e_{15,\overline{24}} + e_{22,\overline{17}} - e_{17,\overline{22}} + e_{24,\overline{15}} - e_{27,\overline{10}} - e_{\overline{23},\overline{7}} - e_{\overline{16},\overline{4}} + e_{\overline{11},\overline{2}}$$

$$e_{\substack{011111\\1}} = -e_{1,9} + e_{5,19} - e_{7,24} + e_{12,\overline{25}} + e_{15,\overline{23}} + e_{18,\overline{20}} - e_{20,\overline{18}} - e_{23,\overline{15}} - e_{25,\overline{12}} + e_{\overline{24},\overline{7}} - e_{\overline{19},\overline{5}} + e_{\overline{9},\overline{1}}$$

$$e_{\substack{122100\\1}} = e_{3,14} + e_{4,17} + e_{5,20} + e_{8,\overline{28}} - e_{16,\overline{22}} - e_{19,\overline{18}} - e_{18,\overline{19}} - e_{22,\overline{16}} + e_{28,\overline{8}} + e_{\overline{20},\overline{5}} + e_{\overline{17},\overline{4}} + e_{\overline{14},\overline{3}}$$

$$e_{\substack{112110\\1}} = e_{2,12} - e_{4,18} + e_{6,23} + e_{9,\overline{27}} - e_{13,\overline{24}} + e_{19,\overline{17}} + e_{17,\overline{19}} - e_{24,\overline{13}} + e_{27,\overline{9}} + e_{\overline{23},\overline{6}} - e_{\overline{18},\overline{4}} + e_{\overline{12},\overline{2}}$$

$$e_{\substack{111111\\1}} = e_{1,10} - e_{5,22} - e_{6,24} + e_{11,\overline{25}} + e_{13,\overline{23}} + e_{16,\overline{20}} + e_{20,\overline{16}} + e_{23,\overline{13}} + e_{25,\overline{11}} - e_{\overline{24},\overline{6}} - e_{\overline{22},\overline{5}} + e_{\overline{10},\overline{1}}$$

$$e_{\substack{012210\\1}} = e_{2,13} + e_{3,16} - e_{7,25} - e_{10,\overline{26}} - e_{12,\overline{24}} - e_{22,\overline{14}} - e_{14,\overline{22}} - e_{24,\overline{12}} - e_{26,\overline{10}} - e_{\overline{25},\overline{7}} + e_{\overline{16},\overline{3}} + e_{\overline{13},\overline{2}}$$

$$e_{\substack{012111\\1}} = e_{1,11} + e_{4,19} + e_{7,26} + e_{10,\overline{25}} - e_{15,\overline{21}} - e_{18,\overline{17}} - e_{17,\overline{18}} - e_{21,\overline{15}} + e_{25,\overline{10}} + e_{\overline{26},\overline{7}} + e_{\overline{19},\overline{4}} + e_{\overline{11},\overline{1}}$$

$$e_{\substack{122110\\1}} = -e_{2,14} + e_{4,21} + e_{5,23} + e_{8,\overline{27}} + e_{13,\overline{22}} - e_{19,\overline{15}} + e_{15,\overline{19}} - e_{22,\overline{13}} - e_{27,\overline{8}} - e_{\overline{23},\overline{5}} - e_{\overline{21},\overline{4}} + e_{\overline{14},\overline{2}}$$

$$e_{\substack{112210\\1}} = -e_{2,15} - e_{3,18} - e_{6,25} - e_{9,\overline{26}} - e_{11,\overline{24}} - e_{19,\overline{14}} + e_{14,\overline{19}} + e_{24,\overline{11}} + e_{26,\overline{9}} + e_{\overline{25},\overline{6}} + e_{\overline{18},\overline{3}} + e_{\overline{15},\overline{2}}$$

$$e_{\substack{112111\\1}} = -e_{1,12} - e_{4,22} + e_{6,26} + e_{9,\overline{25}} - e_{13,\overline{21}} - e_{16,\overline{17}} + e_{17,\overline{16}} + e_{21,\overline{13}} - e_{25,\overline{9}} - e_{\overline{26},\overline{6}} + e_{\overline{22},\overline{4}} + e_{\overline{12},\overline{1}}$$

$$e_{\substack{012211\\1}} = -e_{1,13} + e_{3,19} - e_{7,27} - e_{10,\overline{23}} - e_{12,\overline{21}} + e_{18,\overline{14}} - e_{14,\overline{18}} + e_{21,\overline{12}} + e_{23,\overline{10}} + e_{\overline{27},\overline{7}} - e_{\overline{19},\overline{3}} + e_{\overline{13},\overline{1}}$$

$$e_{\substack{122210\\1}} = e_{2,17} + e_{3,21} - e_{5,25} - e_{8,\overline{26}} + e_{11,\overline{22}} + e_{19,\overline{12}} + e_{12,\overline{19}} + e_{22,\overline{11}} - e_{26,\overline{8}} - e_{\overline{25},\overline{5}} + e_{\overline{21},\overline{3}} + e_{\overline{17},\overline{2}}$$

$$e_{\substack{122111\\1}} = e_{1,14} + e_{4,24} + e_{5,26} + e_{8,\overline{25}} + e_{13,\overline{18}} + e_{16,\overline{15}} + e_{15,\overline{16}} + e_{18,\overline{13}} + e_{25,\overline{8}} + e_{\overline{26},\overline{5}} + e_{\overline{24},\overline{4}} + e_{\overline{14},\overline{1}}$$

$$e_{\substack{112211\\1}} = e_{1,15} - e_{3,22} - e_{6,27} - e_{9,\overline{23}} - e_{11,\overline{21}} + e_{16,\overline{14}} + e_{14,\overline{16}} - e_{21,\overline{11}} - e_{23,\overline{9}} - e_{\overline{27},\overline{6}} - e_{\overline{22},\overline{3}} + e_{\overline{15},\overline{1}}$$

$$e_{\substack{012221\\1}} = e_{1,16} + e_{2,19} + e_{7,28} + e_{10,\overline{20}} + e_{12,\overline{17}} + e_{15,\overline{14}} + e_{14,\overline{15}} + e_{17,\overline{12}} + e_{20,\overline{10}} + e_{\overline{28},\overline{7}} + e_{\overline{19},\overline{2}} + e_{\overline{16},\overline{1}}$$

$$e_{\substack{123210\\1}} = -e_{2,20} - e_{3,23} - e_{4,25} + e_{8,\overline{24}} + e_{9,\overline{22}} - e_{19,\overline{10}} + e_{10,\overline{19}} - e_{22,\overline{9}} - e_{24,\overline{8}} + e_{\overline{25},\overline{4}} + e_{\overline{23},\overline{3}} + e_{\overline{20},\overline{2}}$$

$$e_{\substack{122211\\1}} = -e_{1,17} + e_{3,24} - e_{5,27} - e_{8,\overline{23}} + e_{11,\overline{18}} - e_{16,\overline{12}} + e_{12,\overline{16}} - e_{18,\overline{11}} + e_{23,\overline{8}} + e_{\overline{27},\overline{5}} - e_{\overline{24},\overline{3}} + e_{\overline{17},\overline{1}}$$

$$e_{\substack{112221\\1}} = -e_{1,18} - e_{2,22} + e_{6,28} + e_{9,\overline{20}} + e_{11,\overline{17}} + e_{13,\overline{14}} - e_{14,\overline{13}} - e_{17,\overline{11}} - e_{20,\overline{9}} - e_{\overline{28},\overline{6}} + e_{\overline{22},\overline{2}} + e_{\overline{18},\overline{1}}$$

$$e_{\substack{123210\\2}} = e_{2,\overline{28}} + e_{3,\overline{27}} + e_{4,\overline{26}} + e_{5,\overline{24}} + e_{6,\overline{22}} + e_{19,\overline{7}} + e_{7,\overline{19}} + e_{22,\overline{6}} + e_{24,\overline{5}} + e_{26,\overline{4}} + e_{27,\overline{3}} + e_{28,\overline{2}}$$

$$e_{\substack{123211\\1}} = e_{1,20} - e_{3,26} - e_{4,27} + e_{8,\overline{21}} + e_{9,\overline{18}} + e_{16,\overline{10}} + e_{10,\overline{16}} + e_{18,\overline{9}} + e_{21,\overline{8}} - e_{\overline{27},\overline{4}} - e_{\overline{26},\overline{3}} + e_{\overline{20},\overline{1}}$$

$$e_{\substack{122221\\1}} = e_{1,21} + e_{2,24} + e_{5,28} + e_{8,\overline{20}} - e_{11,\overline{15}} - e_{13,\overline{12}} - e_{12,\overline{13}} - e_{15,\overline{11}} + e_{20,\overline{8}} + e_{\overline{28},\overline{5}} + e_{\overline{24},\overline{2}} + e_{\overline{21},\overline{1}}$$

$$e_{\substack{123211\\2}} = -e_{1,\overline{28}} + e_{3,\overline{25}} + e_{4,\overline{23}} + e_{5,\overline{21}} + e_{6,\overline{18}} - e_{16,\overline{7}} + e_{7,\overline{16}} - e_{18,\overline{6}} - e_{21,\overline{5}} - e_{23,\overline{4}} - e_{25,\overline{3}} + e_{28,\overline{1}}$$

$$e_{\substack{123221\\1}} = -e_{1,23} - e_{2,26} + e_{4,28} - e_{8,\overline{17}} - e_{9,\overline{15}} + e_{13,\overline{10}} - e_{10,\overline{13}} + e_{15,\overline{9}} + e_{17,\overline{8}} - e_{\overline{28},\overline{4}} + e_{\overline{26},\overline{2}} + e_{\overline{23},\overline{1}}$$

$$e_{\substack{123221\\2}} = e_{1,\overline{27}} + e_{2,\overline{25}} - e_{4,\overline{20}} - e_{5,\overline{17}} - e_{6,\overline{15}} - e_{13,\overline{7}} - e_{7,\overline{13}} - e_{15,\overline{6}} - e_{17,\overline{5}} - e_{20,\overline{4}} + e_{25,\overline{2}} + e_{27,\overline{1}}$$

$$e_{\substack{123321\\1}} = e_{1,25} + e_{2,27} + e_{3,28} + e_{8,\overline{14}} + e_{9,\overline{12}} + e_{11,\overline{10}} + e_{10,\overline{11}} + e_{12,\overline{9}} + e_{14,\overline{8}} + e_{\overline{28},\overline{3}} + e_{\overline{27},\overline{2}} + e_{\overline{25},\overline{1}}$$

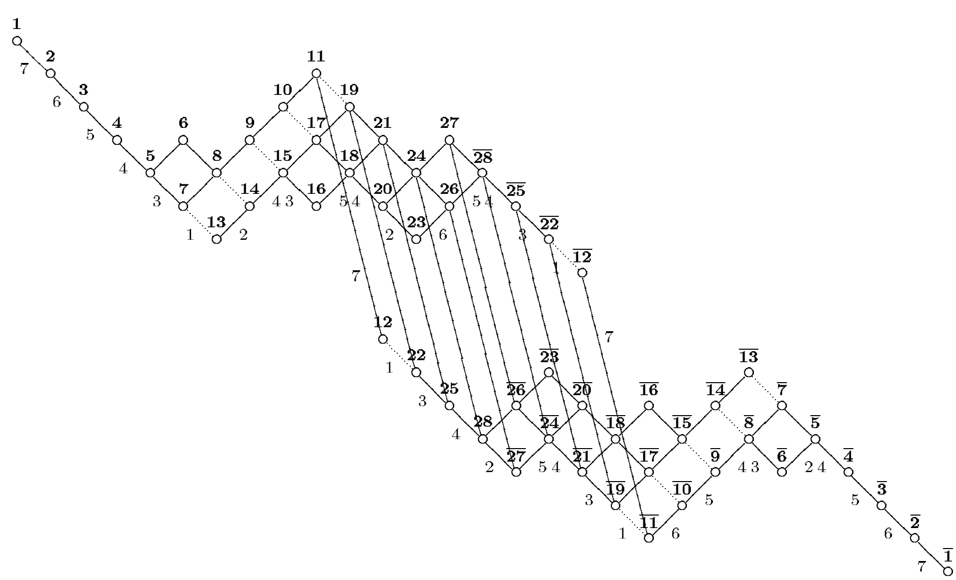
$$e_{\substack{123321\\2}} = -e_{1,\overline{26}} - e_{2,\overline{23}} - e_{3,\overline{20}} + e_{5,\overline{14}} + e_{6,\overline{12}} - e_{11,\overline{7}} + e_{7,\overline{11}} - e_{12,\overline{6}} - e_{14,\overline{5}} + e_{20,\overline{3}} + e_{23,\overline{2}} + e_{26,\overline{1}}$$

$$e_{\substack{124321\\2}} = e_{1,\overline{24}} + e_{2,\overline{21}} + e_{3,\overline{17}} + e_{4,\overline{14}} - e_{6,\overline{10}} - e_{9,\overline{7}} - e_{7,\overline{9}} - e_{10,\overline{6}} + e_{14,\overline{4}} + e_{17,\overline{3}} + e_{21,\overline{2}} + e_{24,\overline{1}}$$

$$e_{\substack{134321\\2}} = -e_{1,\overline{22}} - e_{2,\overline{18}} - e_{3,\overline{15}} - e_{4,\overline{12}} - e_{5,\overline{10}} - e_{8,\overline{7}} + e_{7,\overline{8}} + e_{10,\overline{5}} + e_{12,\overline{4}} + e_{15,\overline{3}} + e_{18,\overline{2}} + e_{22,\overline{1}}$$

$$e_{\substack{234321\\2}} = e_{1,\overline{19}} + e_{2,\overline{16}} + e_{3,\overline{13}} + e_{4,\overline{11}} + e_{5,\overline{9}} + e_{6,\overline{8}} + e_{8,\overline{6}} + e_{9,\overline{5}} + e_{11,\overline{4}} + e_{13,\overline{3}} + e_{16,\overline{2}} + e_{19,\overline{1}}$$

TABLE 13. The matrix of signs of $V(\varpi_7)$: the D_6-numeration.



	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28
1	0	+	−	+	−	+	+	−	+	−	+	0	−	+	−	+	+	−	−	+	+	0	−	−	0	+	+	0
2	+	0	+	−	+	−	−	+	−	+	0	+	+	−	+	−	−	+	0	−	0	−	+	0	+	0	0	−
3	−	+	0	+	−	+	+	−	+	0	+	+	−	+	−	+	0	0	−	0	+	−	0	−	+	+	0	−
4	+	−	+	0	+	−	−	+	0	+	+	+	+	−	0	0	−	+	−	0	+	−	0	0	+	0	−	0
5	−	+	−	+	0	+	+	0	+	+	+	+	−	0	−	0	−	0	−	+	0	−	0	+	0	0	−	+
6	+	−	+	−	+	0	0	+	+	+	+	+	0	−	−	0	−	0	−	0	0	−	+	0	0	+	0	0
7	+	−	+	−	+	0	0	+	+	+	+	+	+	0	0	−	0	−	0	+	−	0	0	+	−	0	−	+
8	−	+	−	+	0	+	+	0	+	+	+	+	0	+	0	−	0	−	0	0	−	0	+	0	−	+	0	0
9	+	−	+	0	+	+	+	+	0	+	+	+	0	0	+	+	0	0	0	−	0	0	+	−	0	+	0	−
10	−	+	0	+	+	+	+	+	+	0	+	+	0	0	0	0	+	+	0	+	0	0	−	0	0	0	−	0
11	+	0	+	+	+	+	+	+	+	+	0	+	0	0	0	0	0	0	+	0	+	0	0	+	0	−	+	0
12	0	+	+	+	+	+	+	+	+	+	+	0	0	0	0	0	0	0	0	0	0	+	0	0	+	0	0	+
13	−	+	−	+	−	0	+	0	0	0	0	0	0	+	+	−	+	−	+	+	−	+	0	+	−	0	−	+
14	+	−	+	−	0	−	0	+	0	0	0	0	+	0	+	−	+	−	+	0	−	+	+	0	−	+	0	0
15	−	+	−	0	−	−	0	0	+	0	0	0	+	+	0	+	+	0	+	−	0	+	+	−	0	+	0	−
16	+	−	+	0	0	0	−	−	+	0	0	0	−	−	+	0	0	+	0	−	+	0	+	−	+	+	0	−
17	+	−	0	−	−	−	0	0	0	+	0	0	+	+	+	0	0	+	+	+	0	+	−	0	0	0	−	0
18	−	+	0	+	0	0	−	−	0	+	0	0	−	−	0	+	+	0	0	+	+	0	−	0	+	0	−	0
19	−	0	−	−	−	−	0	0	0	0	+	0	+	+	+	0	+	0	0	0	+	+	0	+	0	−	+	0
20	+	−	0	0	+	0	+	0	−	+	0	0	+	0	−	−	+	+	0	0	0	0	+	+	0	0	−	+
21	+	0	+	+	0	0	−	−	0	0	+	0	−	−	0	+	0	+	+	0	0	0	0	+	+	−	+	0
22	0	−	−	−	−	−	0	0	0	0	0	+	+	+	+	0	+	0	+	0	0	0	0	0	+	0	0	+
23	−	+	0	0	0	+	0	+	+	−	0	0	0	+	+	+	−	−	0	+	0	0	0	0	0	+	0	0
24	−	0	−	0	+	0	+	0	−	0	+	0	+	0	−	−	0	0	+	+	+	0	0	0	0	+	+	+
25	0	+	+	+	0	0	−	−	0	0	0	+	−	−	0	+	0	+	0	0	+	+	0	0	0	0	0	+
26	+	0	+	0	0	+	0	+	+	0	−	0	0	+	+	+	0	0	−	0	−	0	+	+	0	0	0	0
27	+	0	0	−	−	0	−	0	0	−	+	0	−	0	0	0	−	−	+	−	+	0	0	+	0	0	0	0
28	0	−	−	0	+	0	+	0	−	0	0	+	+	0	−	−	0	0	0	+	0	+	0	+	+	0	0	0

	$\overline{28}$	$\overline{27}$	$\overline{26}$	$\overline{25}$	$\overline{24}$	$\overline{23}$	$\overline{22}$	$\overline{21}$	$\overline{20}$	$\overline{19}$	$\overline{18}$	$\overline{17}$	$\overline{16}$	$\overline{15}$	$\overline{14}$	$\overline{13}$	$\overline{12}$	$\overline{11}$	$\overline{10}$	$\overline{9}$	$\overline{8}$	$\overline{7}$	$\overline{6}$	$\overline{5}$	$\overline{4}$	$\overline{3}$	$\overline{2}$	$\overline{1}$
1	−	0	0	+	0	0	−	0	0	0	0	0	0	0	0	0	+	0	0	0	0	0	0	0	0	0	0	0
2	0	+	+	0	−	0	0	+	0	−	0	0	0	0	0	0	0	+	0	0	0	0	0	0	0	0	0	0
3	0	+	0	0	0	+	0	0	−	0	+	−	0	0	0	0	0	0	+	0	0	0	0	0	0	0	0	0
4	+	0	−	0	+	+	0	0	−	0	0	0	+	−	0	0	0	0	0	+	0	0	0	0	0	0	0	0
5	0	0	−	+	0	+	0	+	0	0	−	0	+	0	−	0	0	0	0	0	+	0	0	0	0	0	0	0
6	−	+	0	+	−	0	0	+	+	0	−	0	+	0	0	−	0	0	0	0	0	+	0	0	0	0	0	0
7	0	0	−	0	0	+	+	0	0	+	0	−	0	+	−	0	0	0	0	0	0	0	+	0	0	0	0	0
8	−	+	0	0	−	0	+	0	+	+	0	−	0	+	0	−	0	0	0	0	0	0	0	+	0	0	0	0
9	0	+	0	−	0	0	+	−	0	+	+	−	0	0	+	−	0	0	0	0	0	0	0	0	+	0	0	0
10	+	0	−	−	+	0	+	−	0	+	0	0	+	−	+	−	0	0	0	0	0	0	0	0	0	+	0	0
11	−	0	0	+	0	−	−	0	+	0	−	+	+	−	+	−	0	0	0	0	0	0	0	0	0	0	+	0
12	0	−	+	0	−	+	0	+	−	−	+	−	−	+	−	+	0	0	0	0	0	0	0	0	0	0	0	+
13	0	0	−	0	0	+	0	0	0	0	0	0	0	0	0	0	+	+	−	+	−	0	+	0	0	0	0	0
14	−	+	0	0	−	0	0	0	+	0	0	0	0	0	0	0	+	+	−	+	0	−	0	+	0	0	0	0
15	0	+	0	−	0	0	0	−	0	0	+	0	0	0	0	0	+	+	−	0	+	−	0	0	+	0	0	0
16	0	+	0	0	0	0	−	0	0	−	0	+	0	0	0	0	+	+	−	0	0	0	+	−	+	0	0	0
17	+	0	−	−	+	0	0	−	0	0	0	0	+	0	0	0	+	+	0	−	+	−	0	0	0	+	0	0
18	+	0	−	0	+	0	−	0	0	−	0	0	0	+	0	0	+	+	0	−	0	0	+	−	0	+	0	0
19	−	0	0	+	0	−	0	0	+	0	−	0	+	0	0	0	−	0	+	−	+	−	0	0	0	0	+	0
20	0	0	−	+	0	0	−	+	0	−	0	0	0	0	+	0	+	+	0	0	−	0	+	0	−	+	0	0
21	−	0	0	0	0	−	+	0	+	0	0	−	0	+	0	0	−	0	+	−	0	0	+	−	0	0	+	0
22	0	−	+	0	−	+	0	+	−	0	+	0	−	0	0	0	0	−	−	+	−	+	0	0	0	0	0	+
23	−	+	0	+	−	0	−	+	0	−	0	0	0	0	0	+	+	+	0	0	0	−	0	+	−	+	0	0
24	0	0	0	−	0	−	+	0	0	0	+	−	0	0	+	0	−	0	+	0	−	0	+	0	−	0	+	0
25	0	−	+	0	−	+	0	0	−	+	0	+	0	−	0	0	0	−	−	+	0	0	−	+	0	0	0	+
26	+	+	0	−	0	0	+	0	−	0	+	−	0	0	0	+	−	0	+	0	0	−	0	+	−	0	+	0
27	+	0	+	+	0	−	−	0	0	0	0	0	+	−	+	0	+	0	0	+	−	0	+	0	0	−	+	0
28	0	+	+	0	0	+	0	−	0	+	−	+	0	0	−	0	0	−	−	0	+	0	−	0	+	0	0	+

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28
$\overline{28}$	−	0	0	+	0	−	0	−	0	+	−	0	0	−	0	0	+	+	−	0	−	0	−	0	0	+	+	0
$\overline{27}$	0	+	+	0	0	+	0	+	+	0	0	−	0	+	+	+	0	0	0	0	0	−	+	0	−	+	0	+
$\overline{26}$	0	+	0	−	−	0	−	0	0	−	0	+	−	0	0	0	−	−	0	−	0	+	0	0	+	0	+	+
$\overline{25}$	+	0	0	0	+	+	0	0	−	−	+	0	0	0	−	0	−	0	+	+	0	0	+	−	0	−	+	0
$\overline{24}$	0	−	0	+	0	−	0	−	0	+	0	−	0	−	0	0	+	+	0	0	0	−	−	0	−	0	0	0
$\overline{23}$	0	0	+	+	+	0	+	0	0	0	−	+	+	0	0	0	0	0	−	0	−	+	0	−	+	0	−	+
$\overline{22}$	−	0	0	0	0	0	+	+	+	+	−	0	0	0	0	−	0	−	0	−	+	0	−	+	0	+	−	0
$\overline{21}$	0	+	0	0	+	+	0	0	−	−	0	+	0	0	−	0	−	0	0	+	0	+	+	0	0	0	0	−
$\overline{20}$	0	0	−	−	0	+	0	+	0	0	+	−	0	+	0	0	0	0	+	0	+	−	0	0	−	−	0	0
$\overline{19}$	0	−	0	0	0	0	+	+	+	+	0	−	0	0	0	−	0	−	0	−	0	0	−	0	+	0	0	+
$\overline{18}$	0	0	+	0	−	−	0	0	+	0	−	+	0	0	+	0	0	0	−	0	0	+	0	+	0	+	0	−
$\overline{17}$	0	0	−	0	0	0	−	−	−	0	+	−	0	0	0	+	0	0	0	0	−	0	0	−	+	−	0	+
$\overline{16}$	0	0	0	+	+	+	0	0	0	+	+	−	0	0	0	0	+	0	+	0	0	−	0	0	0	0	+	0
$\overline{15}$	0	0	0	−	0	0	+	+	0	−	−	+	0	0	0	0	0	+	0	0	+	0	0	0	−	0	−	0
$\overline{14}$	0	0	0	0	−	0	−	0	+	+	+	−	0	0	0	0	0	0	0	+	0	0	0	+	0	0	+	−
$\overline{13}$	0	0	0	0	0	−	0	−	−	−	−	+	0	0	0	0	0	0	0	0	0	0	+	0	0	+	0	0
$\overline{12}$	+	0	0	0	0	0	0	0	0	0	0	0	+	+	+	+	+	+	−	+	−	0	+	−	0	−	+	0
$\overline{11}$	0	+	0	0	0	0	0	0	0	0	0	0	+	+	+	+	+	+	0	+	0	−	+	0	−	0	0	−
$\overline{10}$	0	0	+	0	0	0	0	0	0	0	0	0	−	−	−	−	0	0	+	0	+	−	0	+	−	+	0	−
$\overline{9}$	0	0	0	+	0	0	0	0	0	0	0	0	+	+	0	0	−	−	−	0	−	+	0	0	+	0	+	0
$\overline{8}$	0	0	0	0	+	0	0	0	0	0	0	0	−	0	+	0	+	0	+	−	0	−	0	−	0	0	−	+
$\overline{7}$	0	0	0	0	0	+	0	0	0	0	0	0	0	−	−	0	−	0	−	0	0	+	−	0	0	−	0	0
$\overline{6}$	0	0	0	0	0	0	+	0	0	0	0	0	+	0	0	+	0	+	0	+	+	0	0	+	−	0	+	−
$\overline{5}$	0	0	0	0	0	0	0	+	0	0	0	0	0	+	0	−	0	−	0	0	−	0	+	0	+	+	0	0
$\overline{4}$	0	0	0	0	0	0	0	0	+	0	0	0	0	0	+	+	0	0	0	−	0	0	−	−	0	−	0	+
$\overline{3}$	0	0	0	0	0	0	0	0	0	+	0	0	0	0	0	0	+	+	0	+	0	0	+	0	0	0	−	0
$\overline{2}$	0	0	0	0	0	0	0	0	0	0	+	0	0	0	0	0	0	0	+	0	+	0	0	+	0	+	+	0
$\overline{1}$	0	0	0	0	0	0	0	0	0	0	0	+	0	0	0	0	0	0	0	0	0	+	0	0	+	0	0	+

	$\overline{28}$	$\overline{27}$	$\overline{26}$	$\overline{25}$	$\overline{24}$	$\overline{23}$	$\overline{22}$	$\overline{21}$	$\overline{20}$	$\overline{19}$	$\overline{18}$	$\overline{17}$	$\overline{16}$	$\overline{15}$	$\overline{14}$	$\overline{13}$	$\overline{12}$	$\overline{11}$	$\overline{10}$	$\overline{9}$	$\overline{8}$	$\overline{7}$	$\overline{6}$	$\overline{5}$	$\overline{4}$	$\overline{3}$	$\overline{2}$	$\overline{1}$
$\overline{28}$	0	0	0	+	+	0	−	0	−	0	0	0	+	−	0	+	+	0	0	+	0	−	0	+	0	−	+	0
$\overline{27}$	0	0	0	0	+	0	0	−	+	+	−	+	0	0	0	−	0	−	−	0	0	+	0	−	+	0	0	+
$\overline{26}$	0	0	0	0	+	+	0	+	0	−	0	0	−	+	−	0	0	+	0	−	+	0	−	0	0	+	0	+
$\overline{25}$	+	0	0	0	0	0	+	+	0	0	−	0	+	0	−	+	−	0	0	0	+	−	0	0	+	−	+	0
$\overline{24}$	+	+	+	0	0	0	0	+	+	−	0	0	−	+	0	−	0	+	0	−	0	+	0	−	0	+	0	+
$\overline{23}$	0	0	+	0	0	0	0	0	+	0	+	−	+	−	+	0	0	0	+	+	−	0	+	0	0	0	+	+
$\overline{22}$	−	0	0	+	0	0	0	0	0	+	0	−	0	+	−	+	+	0	0	0	0	0	+	−	+	−	+	0
$\overline{21}$	0	−	+	+	+	0	0	0	0	+	+	0	−	0	+	−	0	−	0	0	−	+	0	0	−	+	0	+
$\overline{20}$	−	+	0	0	+	+	0	0	0	0	+	−	+	−	0	+	0	0	+	+	0	−	0	+	0	0	+	+
$\overline{19}$	0	+	−	0	−	0	+	+	0	0	0	+	0	−	+	−	0	+	0	0	0	0	−	+	−	+	0	+
$\overline{18}$	0	−	0	−	0	+	0	+	+	0	0	+	+	0	−	+	0	0	−	0	+	−	0	0	+	0	+	+
$\overline{17}$	0	+	0	0	0	−	−	0	−	+	+	0	0	+	−	+	0	0	+	0	0	0	+	−	+	0	+	+
$\overline{16}$	+	0	−	+	−	+	0	−	+	0	+	0	0	+	+	−	0	0	0	−	−	+	0	0	0	+	+	+
$\overline{15}$	−	0	+	0	+	−	+	0	−	−	0	+	+	0	+	−	0	0	0	+	0	0	−	+	0	+	+	+
$\overline{14}$	0	0	−	−	0	+	−	+	0	+	−	−	+	+	0	+	0	0	0	0	+	0	+	0	+	+	+	+
$\overline{13}$	+	−	0	+	−	0	+	−	+	−	+	+	−	−	+	0	0	0	0	0	0	+	0	+	+	+	+	+
$\overline{12}$	+	0	0	−	0	0	+	0	0	0	0	0	0	0	0	0	0	+	−	+	−	+	+	−	+	−	+	0
$\overline{11}$	0	−	+	0	+	0	0	−	0	+	0	0	0	0	0	0	+	0	+	−	+	−	−	+	−	+	0	+
$\overline{10}$	0	−	0	0	0	+	0	0	+	0	−	+	0	0	0	0	−	+	0	+	−	+	+	−	+	0	+	+
$\overline{9}$	+	0	−	0	−	+	0	0	+	0	0	0	−	+	0	0	+	−	+	0	+	−	−	+	0	+	+	+
$\overline{8}$	0	0	+	+	0	−	0	−	0	0	+	0	−	0	+	0	−	+	−	+	0	+	+	0	+	+	+	+
$\overline{7}$	−	+	0	−	+	0	0	+	−	0	−	0	+	0	0	+	+	−	+	−	+	0	0	+	+	+	+	+
$\overline{6}$	0	0	−	0	0	+	+	0	0	−	0	+	0	−	+	0	+	−	+	−	+	0	0	+	+	+	+	+
$\overline{5}$	+	−	0	0	−	0	−	0	+	+	0	−	0	+	0	+	−	+	−	+	0	+	+	0	+	+	+	+
$\overline{4}$	0	+	0	+	0	0	+	−	0	−	+	+	0	0	+	+	+	−	+	0	+	+	+	+	0	+	+	+
$\overline{3}$	−	0	+	−	+	0	−	+	0	+	0	0	+	+	+	+	−	+	0	+	+	+	+	+	+	0	+	+
$\overline{2}$	+	0	0	+	0	+	+	0	+	0	+	+	+	+	+	+	+	0	+	+	+	+	+	+	+	+	0	+
$\overline{1}$	0	+	+	0	+	+	0	+	+	+	+	+	+	+	+	+	0	+	+	+	+	+	+	+	+	+	+	0

Table 14. Root elements: the D_6-numeration.

$$e_{\substack{100000\\0}} = e_{7,13} + e_{8,14} + e_{9,15} + e_{10,17} + e_{11,19} + e_{12,22} + e_{\overline{22},\overline{12}} + e_{\overline{19},\overline{11}} + e_{\overline{17},\overline{10}} + e_{\overline{15},\overline{9}} + e_{\overline{14},\overline{8}} + e_{\overline{13},\overline{7}}$$

$$e_{\substack{000000\\1}} = e_{5,6} + e_{7,8} + e_{13,14} + e_{20,23} + e_{24,26} + e_{28,\overline{27}} + e_{27,\overline{28}} + e_{\overline{26},\overline{24}} + e_{\overline{23},\overline{20}} + e_{\overline{14},\overline{13}} + e_{\overline{8},\overline{7}} + e_{\overline{6},\overline{5}}$$

$$e_{\substack{010000\\0}} = e_{5,7} + e_{6,8} + e_{15,16} + e_{17,18} + e_{19,21} + e_{22,25} + e_{\overline{25},\overline{22}} + e_{\overline{21},\overline{19}} + e_{\overline{18},\overline{17}} + e_{\overline{16},\overline{15}} + e_{\overline{8},\overline{6}} + e_{\overline{7},\overline{5}}$$

$$e_{\substack{001000\\0}} = e_{4,5} + e_{8,9} + e_{14,15} + e_{18,20} + e_{21,24} + e_{25,28} + e_{\overline{28},\overline{25}} + e_{\overline{24},\overline{21}} + e_{\overline{20},\overline{18}} + e_{\overline{15},\overline{14}} + e_{\overline{9},\overline{8}} + e_{\overline{5},\overline{4}}$$

$$e_{\substack{000100\\0}} = e_{3,4} + e_{9,10} + e_{15,17} + e_{16,18} + e_{24,27} + e_{28,\overline{26}} + e_{26,\overline{28}} + e_{\overline{27},\overline{24}} + e_{\overline{18},\overline{16}} + e_{\overline{17},\overline{15}} + e_{\overline{10},\overline{9}} + e_{\overline{4},\overline{3}}$$

$$e_{\substack{000010\\0}} = e_{2,3} + e_{10,11} + e_{17,19} + e_{18,21} + e_{20,24} + e_{\overline{26},\overline{23}} + e_{23,26} + e_{\overline{24},\overline{20}} + e_{\overline{21},\overline{18}} + e_{\overline{19},\overline{17}} + e_{\overline{11},\overline{10}} + e_{\overline{3},\overline{2}}$$

$$e_{\substack{000001\\0}} = e_{1,2} + e_{11,12} + e_{19,22} + e_{21,25} + e_{24,28} + e_{27,\overline{26}} + e_{26,\overline{27}} + e_{\overline{28},\overline{24}} + e_{\overline{25},\overline{21}} + e_{\overline{22},\overline{19}} + e_{\overline{12},\overline{11}} + e_{\overline{2},\overline{1}}$$

$$e_{\substack{110000\\0}} = -e_{5,13} - e_{6,14} + e_{9,16} + e_{10,18} + e_{11,21} + e_{12,25} - e_{\overline{25},\overline{12}} - e_{\overline{21},\overline{11}} - e_{\overline{18},\overline{10}} - e_{\overline{16},\overline{9}} + e_{\overline{14},\overline{6}} + e_{\overline{13},\overline{5}}$$

$$e_{\substack{001000\\1}} = -e_{4,6} + e_{7,9} + e_{13,15} - e_{18,23} - e_{21,26} - e_{25,\overline{27}} + e_{27,\overline{25}} + e_{\overline{26},\overline{21}} + e_{\overline{23},\overline{18}} - e_{\overline{15},\overline{13}} - e_{\overline{9},\overline{7}} + e_{\overline{6},\overline{4}}$$

$$e_{\substack{011000\\0}} = -e_{4,7} + e_{6,9} - e_{14,16} + e_{17,20} + e_{19,24} + e_{22,28} - e_{\overline{28},\overline{22}} - e_{\overline{24},\overline{19}} - e_{\overline{20},\overline{17}} + e_{\overline{16},\overline{14}} - e_{\overline{9},\overline{6}} + e_{\overline{7},\overline{4}}$$

$$e_{\substack{001100\\0}} = -e_{3,5} + e_{8,10} + e_{14,17} - e_{16,20} + e_{21,27} + e_{25,\overline{26}} - e_{26,\overline{25}} - e_{\overline{27},\overline{21}} + e_{\overline{20},\overline{16}} - e_{\overline{17},\overline{14}} - e_{\overline{10},\overline{8}} + e_{\overline{5},\overline{3}}$$

$$e_{\substack{000110\\0}} = -e_{2,4} + e_{9,11} + e_{15,19} + e_{16,21} - e_{20,27} + e_{28,\overline{23}} - e_{23,\overline{28}} + e_{\overline{27},\overline{20}} - e_{\overline{21},\overline{16}} - e_{\overline{19},\overline{15}} - e_{\overline{11},\overline{9}} + e_{\overline{4},\overline{2}}$$

$$e_{\substack{000011\\0}} = -e_{1,3} + e_{10,12} + e_{17,22} + e_{18,25} + e_{20,28} - e_{27,\overline{23}} + e_{23,\overline{27}} - e_{\overline{28},\overline{20}} - e_{\overline{25},\overline{18}} - e_{\overline{22},\overline{17}} - e_{\overline{12},\overline{10}} + e_{\overline{3},\overline{1}}$$

$$e_{\substack{111000\\0}} = e_{4,13} - e_{6,15} - e_{8,16} + e_{10,20} + e_{11,24} + e_{12,28} + e_{\overline{28},\overline{12}} + e_{\overline{24},\overline{11}} + e_{\overline{20},\overline{10}} - e_{\overline{16},\overline{8}} - e_{\overline{15},\overline{6}} + e_{\overline{13},\overline{4}}$$

$$e_{\substack{011000\\1}} = e_{4,8} + e_{5,9} - e_{13,16} - e_{17,23} - e_{19,26} - e_{22,\overline{27}} - e_{27,\overline{22}} - e_{\overline{26},\overline{19}} - e_{\overline{23},\overline{17}} - e_{\overline{16},\overline{13}} + e_{\overline{9},\overline{5}} + e_{\overline{8},\overline{4}}$$

$$e_{\substack{001100\\1}} = e_{3,6} + e_{7,10} + e_{13,17} + e_{16,23} - e_{21,\overline{28}} - e_{25,\overline{24}} - e_{24,\overline{25}} - e_{28,\overline{21}} + e_{\overline{23},\overline{16}} + e_{\overline{17},\overline{13}} + e_{\overline{10},\overline{7}} + e_{\overline{6},\overline{3}}$$

$$e_{\substack{011100\\0}} = e_{3,7} + e_{6,10} - e_{14,18} - e_{15,20} + e_{19,27} + e_{22,\overline{26}} + e_{26,\overline{22}} + e_{\overline{27},\overline{19}} - e_{\overline{20},\overline{15}} - e_{\overline{18},\overline{14}} + e_{\overline{10},\overline{6}} + e_{\overline{7},\overline{3}}$$

$$e_{\substack{001110\\0}} = e_{2,5} + e_{8,11} + e_{14,19} - e_{16,24} - e_{18,27} + e_{25,\overline{23}} + e_{23,\overline{25}} - e_{\overline{27},\overline{18}} - e_{\overline{24},\overline{16}} + e_{\overline{19},\overline{14}} + e_{\overline{11},\overline{8}} + e_{\overline{5},\overline{2}}$$

$$e_{\substack{000111\\0}} = e_{1,4} + e_{9,12} + e_{15,22} + e_{16,25} - e_{20,\overline{26}} - e_{24,\overline{23}} - e_{23,\overline{24}} - e_{26,\overline{20}} + e_{\overline{25},\overline{16}} + e_{\overline{22},\overline{15}} + e_{\overline{12},\overline{9}} + e_{\overline{4},\overline{1}}$$

$$e_{\substack{111000\\1}} = -e_{4,14} - e_{5,15} - e_{7,16} - e_{10,23} - e_{11,26} - e_{12,\overline{27}} + e_{27,\overline{12}} + e_{\overline{26},\overline{11}} + e_{\overline{23},\overline{10}} + e_{\overline{16},\overline{7}} + e_{\overline{15},\overline{5}} + e_{\overline{14},\overline{4}}$$

$$e_{\substack{111100\\0}} = -e_{3,13} - e_{6,17} - e_{8,18} - e_{9,20} + e_{11,27} + e_{12,\overline{26}} - e_{26,\overline{12}} - e_{\overline{27},\overline{11}} + e_{\overline{20},\overline{9}} + e_{\overline{18},\overline{8}} + e_{\overline{17},\overline{6}} + e_{\overline{13},\overline{3}}$$

$$e_{\substack{011100\\1}} = -e_{3,8} + e_{5,10} - e_{13,18} + e_{15,23} - e_{19,\overline{28}} - e_{22,\overline{24}} + e_{24,\overline{22}} + e_{28,\overline{19}} - e_{\overline{23},\overline{15}} + e_{\overline{18},\overline{13}} - e_{\overline{10},\overline{5}} + e_{\overline{8},\overline{3}}$$

$$e_{\substack{001110\\1}} = -e_{2,6} + e_{7,11} + e_{13,19} + e_{16,26} + e_{18,\overline{28}} - e_{25,\overline{20}} + e_{20,\overline{25}} - e_{28,\overline{18}} - e_{\overline{26},\overline{16}} - e_{\overline{19},\overline{13}} - e_{\overline{11},\overline{7}} + e_{\overline{6},\overline{2}}$$

$$e_{\substack{011110\\0}} = -e_{2,7} + e_{6,11} - e_{14,21} - e_{15,24} - e_{17,27} + e_{22,\overline{23}} - e_{23,\overline{22}} + e_{\overline{27},\overline{17}} + e_{\overline{24},\overline{15}} + e_{\overline{21},\overline{14}} - e_{\overline{11},\overline{6}} + e_{\overline{7},\overline{2}}$$

$$e_{\substack{001111\\0}} = -e_{1,5} + e_{8,12} + e_{14,22} - e_{16,28} - e_{18,\overline{26}} - e_{21,\overline{23}} + e_{23,\overline{21}} + e_{26,\overline{18}} + e_{\overline{28},\overline{16}} - e_{\overline{22},\overline{14}} - e_{\overline{12},\overline{8}} + e_{\overline{5},\overline{1}}$$

$$e_{\substack{111100\\1}} = e_{3,14} - e_{5,17} - e_{7,18} + e_{9,23} - e_{11,\overline{28}} - e_{12,\overline{24}} - e_{24,\overline{12}} - e_{28,\overline{11}} + e_{\overline{23},\overline{9}} - e_{\overline{18},\overline{7}} - e_{\overline{17},\overline{5}} + e_{\overline{14},\overline{3}}$$

$$e_{\substack{111110\\0}} = e_{2,13} - e_{6,19} - e_{8,21} - e_{9,24} - e_{10,27} + e_{12,\overline{23}} + e_{23,\overline{12}} - e_{\overline{27},\overline{10}} - e_{\overline{24},\overline{9}} - e_{\overline{21},\overline{8}} - e_{\overline{19},\overline{6}} + e_{\overline{13},\overline{2}}$$

$$e_{\substack{012100\\1}} = e_{3,9} + e_{4,10} + e_{13,20} + e_{14,23} + e_{19,\overline{25}} + e_{22,\overline{21}} + e_{21,\overline{22}} + e_{25,\overline{19}} + e_{\overline{23},\overline{14}} + e_{\overline{20},\overline{13}} + e_{\overline{10},\overline{4}} + e_{\overline{9},\overline{3}}$$

$$e_{\substack{011110\\1}} = e_{2,8} + e_{5,11} - e_{13,21} + e_{15,26} + e_{17,\overline{28}} - e_{22,\overline{20}} - e_{20,\overline{22}} + e_{28,\overline{17}} + e_{\overline{26},\overline{15}} - e_{\overline{21},\overline{13}} + e_{\overline{11},\overline{5}} + e_{\overline{8},\overline{2}}$$

$$e_{\substack{001111\\1}} = e_{1,6} + e_{7,12} + e_{13,22} + e_{16,\overline{27}} + e_{18,\overline{24}} + e_{21,\overline{20}} + e_{20,\overline{21}} + e_{24,\overline{18}} + e_{27,\overline{16}} + e_{\overline{22},\overline{13}} + e_{\overline{12},\overline{7}} + e_{\overline{6},\overline{1}}$$

$$e_{\substack{011111\\0}} = e_{1,7} + e_{6,12} - e_{14,25} - e_{15,28} - e_{17,\overline{26}} - e_{19,\overline{23}} - e_{23,\overline{19}} - e_{26,\overline{17}} - e_{\overline{28},\overline{15}} - e_{\overline{25},\overline{14}} + e_{\overline{12},\overline{6}} + e_{\overline{7},\overline{1}}$$

$$e_{\substack{112100\\1}} = -e_{3,15} - e_{4,17} + e_{7,20} + e_{8,23} + e_{11,\overline{25}} + e_{12,\overline{21}} - e_{21,\overline{12}} - e_{25,\overline{11}} - e_{\overline{23},\overline{8}} - e_{\overline{20},\overline{7}} + e_{\overline{17},\overline{4}} + e_{\overline{15},\overline{3}}$$

$$e_{\substack{111110\\1}} = -e_{2,14} - e_{5,19} - e_{7,21} + e_{9,26} + e_{10,\overline{28}} - e_{12,\overline{20}} + e_{20,\overline{12}} - e_{28,\overline{10}} - e_{\overline{26},\overline{9}} + e_{\overline{21},\overline{7}} + e_{\overline{19},\overline{5}} + e_{\overline{14},\overline{2}}$$

$$e_{\substack{111111\\0}} = -e_{1,13} - e_{6,22} - e_{8,25} - e_{9,28} - e_{10,\overline{26}} - e_{11,\overline{23}} + e_{23,\overline{11}} + e_{26,\overline{10}} + e_{\overline{28},\overline{9}} + e_{\overline{25},\overline{8}} + e_{\overline{22},\overline{6}} + e_{\overline{13},\overline{1}}$$

$$e_{\substack{012110\\1}} = -e_{2,9} + e_{4,11} + e_{13,24} + e_{14,26} - e_{17,\overline{25}} + e_{22,\overline{18}} - e_{18,\overline{22}} + e_{25,\overline{17}} - e_{\overline{26},\overline{14}} - e_{\overline{24},\overline{13}} - e_{\overline{11},\overline{4}} + e_{\overline{9},\overline{2}}$$

$$e_{\substack{011111\\1}} = -e_{1,8} + e_{5,12} - e_{13,25} + e_{15,\overline{27}} + e_{17,\overline{24}} + e_{19,\overline{20}} - e_{20,\overline{19}} - e_{24,\overline{17}} - e_{27,\overline{15}} + e_{\overline{25},\overline{13}} - e_{\overline{12},\overline{5}} + e_{\overline{8},\overline{1}}$$

$$e_{\substack{122100\\1}} = e_{3,16} + e_{4,18} + e_{5,20} + e_{6,23} - e_{11,\overline{22}} - e_{12,\overline{19}} - e_{19,\overline{12}} - e_{22,\overline{11}} + e_{\overline{23},\overline{6}} + e_{\overline{20},\overline{5}} + e_{\overline{18},\overline{4}} + e_{\overline{16},\overline{3}}$$

$$e_{\substack{112110\\1}} = e_{2,15} - e_{4,19} + e_{7,24} + e_{8,26} - e_{10,\overline{25}} + e_{12,\overline{18}} + e_{18,\overline{12}} - e_{25,\overline{10}} + e_{\overline{26},\overline{8}} + e_{\overline{24},\overline{7}} - e_{\overline{19},\overline{4}} + e_{\overline{15},\overline{2}}$$

$$e_{\substack{111111\\1}} = e_{1,14} - e_{5,22} - e_{7,25} + e_{9,\overline{27}} + e_{10,\overline{24}} + e_{11,\overline{20}} + e_{20,\overline{11}} + e_{24,\overline{10}} + e_{27,\overline{9}} - e_{\overline{25},\overline{7}} - e_{\overline{22},\overline{5}} + e_{\overline{14},\overline{1}}$$

$$e_{\substack{012210\\1}} = e_{2,10} + e_{3,11} - e_{13,27} - e_{14,\overline{28}} - e_{15,\overline{25}} - e_{22,\overline{16}} - e_{16,\overline{22}} - e_{25,\overline{15}} - e_{28,\overline{14}} - e_{\overline{27},\overline{13}} + e_{\overline{11},\overline{3}} + e_{\overline{10},\overline{2}}$$

$$e_{\substack{012111\\1}} = e_{1,9} + e_{4,12} + e_{13,28} + e_{14,\overline{27}} - e_{17,\overline{21}} - e_{19,\overline{18}} - e_{18,\overline{19}} - e_{21,\overline{17}} + e_{27,\overline{14}} + e_{\overline{28},\overline{13}} + e_{\overline{12},\overline{4}} + e_{\overline{9},\overline{1}}$$

$$e_{\substack{122110\\1}} = -e_{2,16} + e_{4,21} + e_{5,24} + e_{6,26} + e_{10,\overline{22}} - e_{12,\overline{17}} + e_{17,\overline{12}} - e_{22,\overline{10}} - e_{\overline{26},\overline{6}} - e_{\overline{24},\overline{5}} - e_{\overline{21},\overline{4}} + e_{\overline{16},\overline{2}}$$

$$e_{\substack{112210\\1}} = -e_{2,17} - e_{3,19} - e_{7,27} - e_{8,\overline{28}} - e_{9,\overline{25}} - e_{12,\overline{16}} + e_{16,\overline{12}} + e_{25,\overline{9}} + e_{28,\overline{8}} + e_{\overline{27},\overline{7}} + e_{\overline{19},\overline{3}} + e_{\overline{17},\overline{2}}$$

$$e_{\substack{112111\\1}} = -e_{1,15} - e_{4,22} + e_{7,28} + e_{8,\overline{27}} - e_{10,\overline{21}} - e_{11,\overline{18}} + e_{18,\overline{11}} + e_{21,\overline{10}} - e_{27,\overline{8}} - e_{\overline{28},\overline{7}} + e_{\overline{22},\overline{4}} + e_{\overline{15},\overline{1}}$$

$$e_{\substack{012211\\1}} = -e_{1,10} + e_{3,12} - e_{13,\overline{26}} - e_{14,\overline{24}} - e_{15,\overline{21}} + e_{19,\overline{16}} - e_{16,\overline{19}} + e_{21,\overline{15}} + e_{24,\overline{14}} + e_{26,\overline{13}} - e_{\overline{12},\overline{3}} + e_{\overline{10},\overline{1}}$$

$$e_{\substack{122210\\1}} = e_{2,18} + e_{3,21} - e_{5,27} - e_{6,\overline{28}} + e_{9,\overline{22}} + e_{12,\overline{15}} + e_{15,\overline{12}} + e_{22,\overline{9}} - e_{28,\overline{6}} - e_{\overline{27},\overline{5}} + e_{\overline{21},\overline{3}} + e_{\overline{18},\overline{2}}$$

$$e_{\substack{122111\\1}} = e_{1,16} + e_{4,25} + e_{5,28} + e_{6,\overline{27}} + e_{10,\overline{19}} + e_{11,\overline{17}} + e_{17,\overline{11}} + e_{19,\overline{10}} + e_{27,\overline{6}} + e_{\overline{28},\overline{5}} + e_{\overline{25},\overline{4}} + e_{\overline{16},\overline{1}}$$

$$e_{\substack{112211\\1}} = e_{1,17} - e_{3,22} - e_{7,\overline{26}} - e_{8,\overline{24}} - e_{9,\overline{21}} + e_{11,\overline{16}} + e_{16,\overline{11}} - e_{21,\overline{9}} - e_{24,\overline{8}} - e_{26,\overline{7}} - e_{\overline{22},\overline{3}} + e_{\overline{17},\overline{1}}$$

$$e_{\substack{012221\\1}} = e_{1,11} + e_{2,12} + e_{13,\overline{23}} + e_{14,\overline{20}} + e_{15,\overline{18}} + e_{17,\overline{16}} + e_{16,\overline{17}} + e_{18,\overline{15}} + e_{20,\overline{14}} + e_{23,\overline{13}} + e_{\overline{12},\overline{2}} + e_{\overline{11},\overline{1}}$$

$$e_{\substack{123210\\1}} = -e_{2,20} - e_{3,24} - e_{4,27} + e_{6,\overline{25}} + e_{8,\overline{22}} - e_{12,\overline{14}} + e_{14,\overline{12}} - e_{22,\overline{8}} - e_{25,\overline{6}} + e_{\overline{27},\overline{4}} + e_{\overline{24},\overline{3}} + e_{\overline{20},\overline{2}}$$

$$e_{\substack{122211\\1}} = -e_{1,18} + e_{3,25} - e_{5,\overline{26}} - e_{6,\overline{24}} + e_{9,\overline{19}} - e_{11,\overline{15}} + e_{15,\overline{11}} - e_{19,\overline{9}} + e_{24,\overline{6}} + e_{26,\overline{5}} - e_{\overline{25},\overline{3}} + e_{\overline{18},\overline{1}}$$

$$e_{\substack{112221\\1}} = -e_{1,19} - e_{2,22} + e_{7,\overline{23}} + e_{8,\overline{20}} + e_{9,\overline{18}} + e_{10,\overline{16}} - e_{16,\overline{10}} - e_{18,\overline{9}} - e_{20,\overline{8}} - e_{23,\overline{7}} + e_{\overline{22},\overline{2}} + e_{\overline{19},\overline{1}}$$

$$e_{\substack{123210\\2}} = e_{2,23} + e_{3,26} + e_{4,\overline{28}} + e_{5,\overline{25}} + e_{7,\overline{22}} + e_{12,\overline{13}} + e_{13,\overline{12}} + e_{22,\overline{7}} + e_{25,\overline{5}} + e_{28,\overline{4}} + e_{\overline{26},\overline{3}} + e_{\overline{23},\overline{2}}$$

$$e_{\substack{123211\\1}} = e_{1,20} - e_{3,28} - e_{4,\overline{26}} + e_{6,\overline{21}} + e_{8,\overline{19}} + e_{11,\overline{14}} + e_{14,\overline{11}} + e_{19,\overline{8}} + e_{21,\overline{6}} - e_{26,\overline{4}} - e_{\overline{28},\overline{3}} + e_{\overline{20},\overline{1}}$$

$$e_{\substack{122221\\1}} = e_{1,21} + e_{2,25} + e_{5,\overline{23}} + e_{6,\overline{20}} - e_{9,\overline{17}} - e_{10,\overline{15}} - e_{15,\overline{10}} - e_{17,\overline{9}} + e_{20,\overline{6}} + e_{23,\overline{5}} + e_{\overline{25},\overline{2}} + e_{\overline{21},\overline{1}}$$

$$e_{\substack{123211\\2}} = -e_{1,23} + e_{3,\overline{27}} + e_{4,\overline{24}} + e_{5,\overline{21}} + e_{7,\overline{19}} - e_{11,\overline{13}} + e_{13,\overline{11}} - e_{19,\overline{7}} - e_{21,\overline{5}} - e_{24,\overline{4}} - e_{27,\overline{3}} + e_{\overline{23},\overline{1}}$$

$$e_{\substack{123221\\1}} = -e_{1,24} - e_{2,28} + e_{4,\overline{23}} - e_{6,\overline{18}} - e_{8,\overline{17}} + e_{10,\overline{14}} - e_{14,\overline{10}} + e_{17,\overline{8}} + e_{18,\overline{6}} - e_{23,\overline{4}} + e_{\overline{28},\overline{2}} + e_{\overline{24},\overline{1}}$$

$$e_{\substack{123221\\2}} = e_{1,26} + e_{2,\overline{27}} - e_{4,\overline{20}} - e_{5,\overline{18}} - e_{7,\overline{17}} - e_{10,\overline{13}} - e_{13,\overline{10}} - e_{17,\overline{7}} - e_{18,\overline{5}} - e_{20,\overline{4}} + e_{27,\overline{2}} + e_{\overline{26},\overline{1}}$$

$$e_{\substack{123321\\1}} = e_{1,27} + e_{2,\overline{26}} + e_{3,\overline{23}} + e_{6,\overline{16}} + e_{8,\overline{15}} + e_{9,\overline{14}} + e_{14,\overline{9}} + e_{15,\overline{8}} + e_{16,\overline{6}} + e_{23,\overline{3}} + e_{26,\overline{2}} + e_{\overline{27},\overline{1}}$$

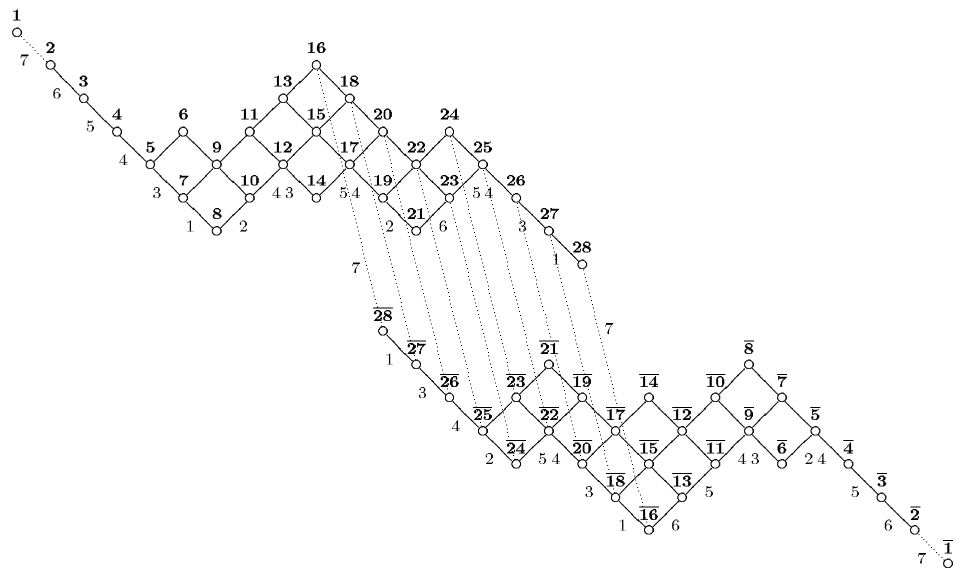
$$e_{\substack{123321\\2}} = -e_{1,\overline{28}} - e_{2,\overline{24}} - e_{3,\overline{20}} + e_{5,\overline{16}} + e_{7,\overline{15}} - e_{9,\overline{13}} + e_{13,\overline{9}} - e_{15,\overline{7}} - e_{16,\overline{5}} + e_{20,\overline{3}} + e_{24,\overline{2}} + e_{28,\overline{1}}$$

$$e_{\substack{124321\\2}} = e_{1,\overline{25}} + e_{2,\overline{21}} + e_{3,\overline{18}} + e_{4,\overline{16}} - e_{7,\overline{14}} - e_{8,\overline{13}} - e_{13,\overline{8}} - e_{14,\overline{7}} + e_{16,\overline{4}} + e_{18,\overline{3}} + e_{21,\overline{2}} + e_{25,\overline{1}}$$

$$e_{\substack{134321\\2}} = -e_{1,\overline{22}} - e_{2,\overline{19}} - e_{3,\overline{17}} - e_{4,\overline{15}} - e_{5,\overline{14}} - e_{6,\overline{13}} + e_{13,\overline{6}} + e_{14,\overline{5}} + e_{15,\overline{4}} + e_{17,\overline{3}} + e_{19,\overline{2}} + e_{22,\overline{1}}$$

$$e_{\substack{234321\\2}} = e_{1,\overline{12}} + e_{2,\overline{11}} + e_{3,\overline{10}} + e_{4,\overline{9}} + e_{5,\overline{8}} + e_{7,\overline{6}} + e_{6,\overline{7}} + e_{8,\overline{5}} + e_{9,\overline{4}} + e_{10,\overline{3}} + e_{11,\overline{2}} + e_{12,\overline{1}}$$

TABLE 15. The matrix of signs of $V(\varpi_7)$: the E_6-numeration.



	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28
1	0	+	−	+	−	+	+	−	−	+	+	−	−	+	+	+	−	−	+	+	−	−	+	+	−	+	−	+
2	+	0	+	−	+	−	−	+	+	−	−	+	+	−	−	0	+	0	−	0	+	0	0	0	0	0	0	0
3	−	+	0	+	−	+	+	−	−	+	+	−	0	+	0	+	0	−	0	+	0	−	+	0	0	0	0	0
4	+	−	+	0	+	−	−	+	+	−	0	0	+	0	−	+	+	−	0	+	0	0	0	−	+	0	0	0
5	−	+	−	+	0	+	+	−	0	0	+	−	+	0	−	+	0	−	+	0	0	+	0	−	0	+	0	0
6	+	−	+	−	+	0	0	0	+	−	+	−	+	0	−	+	0	−	0	0	+	0	+	0	−	+	0	0
7	+	−	+	−	+	0	0	+	+	0	+	0	+	−	0	+	−	0	+	−	0	+	0	−	0	0	+	0
8	−	+	−	+	−	0	+	0	0	+	0	+	0	−	+	0	−	+	+	−	0	+	0	−	0	0	0	+
9	−	+	−	+	0	+	+	0	0	+	+	0	+	−	0	+	−	0	0	−	+	0	+	0	−	0	+	0
10	+	−	+	−	0	−	0	+	+	0	0	+	0	−	+	0	−	+	0	−	+	0	+	0	−	0	0	+
11	+	−	+	0	+	+	+	0	+	0	0	+	+	+	0	+	0	0	−	0	+	−	+	0	0	−	+	0
12	−	+	−	0	−	−	0	+	0	+	+	0	0	+	+	0	0	+	−	0	+	−	+	0	0	−	0	+
13	−	+	0	+	+	+	+	0	+	0	+	0	0	0	+	+	+	0	+	0	−	0	0	−	+	−	+	0
14	+	−	+	0	0	0	−	−	−	−	+	+	0	0	0	0	+	0	−	+	+	−	+	0	0	0	−	+
15	+	−	0	−	−	−	0	+	0	+	0	+	+	0	0	0	+	+	+	0	−	0	0	−	+	−	0	+
16	+	0	+	+	+	+	+	0	+	0	+	0	+	0	0	0	0	+	0	+	0	+	−	+	−	+	−	0
17	−	+	0	+	0	0	−	−	−	−	0	0	+	+	+	0	0	0	+	+	−	0	0	−	+	0	−	+
18	−	0	−	−	−	−	0	+	0	+	0	+	0	0	+	+	0	0	0	+	0	+	−	+	−	+	0	−
19	+	−	0	0	+	0	+	+	0	0	−	−	+	−	+	0	+	0	0	0	+	+	0	−	0	+	−	+
20	+	0	+	+	0	0	−	−	−	−	0	0	0	+	0	+	+	+	0	0	0	+	−	+	−	0	+	−
21	−	+	0	0	0	+	0	0	+	+	+	+	−	+	−	0	−	0	+	0	0	0	+	0	−	+	−	+
22	−	0	−	0	+	0	+	+	0	0	−	−	0	−	0	+	0	+	+	+	0	0	+	+	0	−	+	−
23	+	0	+	0	0	+	0	0	+	+	+	+	0	+	0	−	0	−	0	−	+	+	0	0	+	−	+	−
24	+	0	0	−	−	0	−	−	0	0	0	0	−	0	−	+	−	+	−	+	0	+	0	0	+	+	−	+
25	−	0	0	+	0	−	0	0	−	−	0	0	+	0	+	−	+	−	0	−	−	0	+	+	0	+	−	+
26	+	0	0	0	+	+	0	0	0	0	−	−	−	0	−	+	0	+	+	0	+	−	−	+	+	0	+	−
27	−	0	0	0	0	0	+	0	+	0	+	0	+	−	0	−	−	0	−	+	−	+	+	−	−	+	0	+
28	+	0	0	0	0	0	0	+	0	+	0	+	0	+	+	0	+	−	+	−	+	−	−	+	+	−	+	0

	$\overline{28}$	$\overline{27}$	$\overline{26}$	$\overline{25}$	$\overline{24}$	$\overline{23}$	$\overline{22}$	$\overline{21}$	$\overline{20}$	$\overline{19}$	$\overline{18}$	$\overline{17}$	$\overline{16}$	$\overline{15}$	$\overline{14}$	$\overline{13}$	$\overline{12}$	$\overline{11}$	$\overline{10}$	$\overline{9}$	$\overline{8}$	$\overline{7}$	$\overline{6}$	$\overline{5}$	$\overline{4}$	$\overline{3}$	$\overline{2}$	$\overline{1}$
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	+	−	+	−	+	+	−	0	+	0	−	0	+	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	+	−	+	−	+	0	0	+	0	−	0	+	0	−	0	+	0	0	0	0	0	0	0	0	0	0	0	0
4	+	−	+	0	0	−	+	+	0	−	0	0	0	0	+	0	−	+	0	0	0	0	0	0	0	0	0	0
5	+	−	0	+	0	−	0	+	+	0	0	−	0	0	+	0	0	0	−	+	0	0	0	0	0	0	0	0
6	+	−	0	0	+	0	−	0	+	+	0	−	0	0	+	0	0	0	0	0	−	+	0	0	0	0	0	0
7	+	0	−	+	0	−	0	+	0	0	+	0	0	−	0	0	+	0	−	0	0	0	+	0	0	0	0	0
8	0	+	−	+	0	−	0	+	0	0	0	0	+	0	0	−	0	+	0	−	0	0	+	0	0	0	0	0
9	+	0	−	0	+	0	−	0	0	+	+	0	0	−	0	0	+	0	0	0	−	0	0	+	0	0	0	0
10	0	+	−	0	+	0	−	0	0	+	0	0	+	0	0	−	0	+	0	0	0	−	0	+	0	0	0	0
11	+	0	0	−	+	0	0	0	−	0	+	+	0	−	0	0	0	0	+	0	−	0	0	0	+	0	0	0
12	0	+	0	−	+	0	0	0	−	0	0	+	+	0	0	−	0	0	0	+	0	−	0	0	+	0	0	0
13	+	0	0	0	0	−	+	0	−	0	+	0	0	0	+	0	−	0	+	0	−	0	0	0	0	+	0	0
14	0	0	+	−	+	0	0	0	0	0	−	0	+	+	0	−	0	0	0	0	0	0	+	−	+	0	0	0
15	0	+	0	0	0	−	+	0	−	0	0	0	+	0	+	0	0	−	0	+	0	−	0	0	0	+	0	0
16	+	0	0	0	0	0	0	−	0	+	0	−	0	+	+	0	−	0	+	0	−	0	0	0	0	0	+	0
17	0	0	+	0	0	−	+	0	0	0	−	0	+	0	0	0	+	−	0	0	0	0	+	−	0	+	0	0
18	0	+	0	0	0	0	0	−	0	+	0	−	0	0	+	+	0	−	0	+	0	−	0	0	0	0	+	0
19	0	0	0	+	0	−	0	0	+	0	−	0	+	0	0	0	0	0	+	−	0	0	+	0	−	+	0	0
20	0	0	+	0	0	0	0	−	0	+	0	0	0	−	0	+	+	−	0	0	0	0	+	−	0	0	+	0
21	0	0	0	0	+	0	−	0	+	0	−	0	+	0	0	0	0	0	0	0	+	−	0	+	−	+	0	0
22	0	0	0	+	0	0	0	−	0	0	0	+	0	−	0	+	0	0	+	−	0	0	+	0	−	0	+	0
23	0	0	0	0	+	0	0	0	0	−	0	+	0	−	0	+	0	0	0	0	+	−	0	+	−	0	+	0
24	0	0	0	0	0	+	0	−	0	0	0	0	0	0	+	0	−	+	+	−	0	0	+	0	0	−	+	0
25	0	0	0	0	0	0	+	0	0	−	0	0	0	0	+	0	−	+	0	0	+	−	0	+	0	−	+	0
26	0	0	0	0	0	0	0	0	+	0	0	−	0	0	+	0	0	0	−	+	+	−	0	0	+	−	+	0
27	0	0	0	0	0	0	0	0	0	0	+	0	0	−	0	0	+	0	−	0	+	0	+	−	+	−	+	0
28	0	0	0	0	0	0	0	0	0	0	0	0	+	0	0	−	0	+	0	−	0	+	+	−	+	−	+	0

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28
$\overline{28}$	0	+	+	+	+	+	+	0	+	0	+	0	+	0	0	+	0	0	0	0	0	0	0	0	0	0	0	0
$\overline{27}$	0	−	−	−	−	−	0	+	0	+	0	+	0	0	+	0	0	+	0	0	0	0	0	0	0	0	0	0
$\overline{26}$	0	+	+	+	0	0	−	−	−	−	0	0	0	+	0	0	+	0	0	+	0	0	0	0	0	0	0	0
$\overline{25}$	0	−	−	0	+	0	+	+	0	0	−	−	0	−	0	0	0	0	+	0	0	+	0	0	0	0	0	0
$\overline{24}$	0	+	+	0	0	+	0	0	+	+	+	+	0	+	0	0	0	0	0	0	+	0	+	0	0	0	0	0
$\overline{23}$	0	+	0	−	−	0	−	−	0	0	0	0	−	0	−	0	−	0	−	0	0	0	0	+	0	0	0	0
$\overline{22}$	0	−	0	+	0	−	0	0	−	−	0	0	+	0	+	0	+	0	0	0	−	0	0	0	+	0	0	0
$\overline{21}$	0	0	+	+	+	0	+	+	0	0	0	0	0	0	0	−	0	−	0	−	0	−	0	−	0	0	0	0
$\overline{20}$	0	+	0	0	+	+	0	0	0	0	−	−	−	0	−	0	0	0	+	0	+	0	0	0	0	+	0	0
$\overline{19}$	0	0	−	−	0	+	0	0	+	+	0	0	0	0	0	+	0	+	0	+	0	0	−	0	−	0	0	0
$\overline{18}$	0	−	0	0	0	0	+	0	+	0	+	0	+	−	0	0	−	0	−	0	−	0	0	0	0	0	+	0
$\overline{17}$	0	0	+	0	−	−	0	0	0	0	+	+	0	0	0	−	0	−	0	0	0	+	+	0	0	−	0	0
$\overline{16}$	0	+	0	0	0	0	0	+	0	+	0	+	0	+	+	0	+	0	+	0	+	0	0	0	0	0	0	+
$\overline{15}$	0	0	−	0	0	0	−	0	−	0	−	0	0	+	0	+	0	0	0	−	0	−	−	0	0	0	−	0
$\overline{14}$	0	0	0	+	+	+	0	0	0	0	0	0	+	0	+	+	0	+	0	0	0	0	0	+	+	+	0	0
$\overline{13}$	0	0	+	0	0	0	0	−	0	−	0	−	0	−	0	0	0	+	0	+	0	+	+	0	0	0	0	−
$\overline{12}$	0	0	0	−	0	0	+	0	+	0	0	0	−	0	0	−	+	0	0	+	0	0	0	−	−	0	+	0
$\overline{11}$	0	0	0	+	0	0	0	+	0	+	0	0	0	0	−	0	−	−	0	−	0	0	0	+	+	0	0	+
$\overline{10}$	0	0	0	0	−	0	−	0	0	0	+	0	+	0	0	+	0	0	+	0	0	+	0	+	0	−	−	0
$\overline{9}$	0	0	0	0	+	0	0	−	0	0	0	+	0	0	+	0	0	+	−	0	0	−	0	−	0	+	0	−
$\overline{8}$	0	0	0	0	0	−	0	0	−	0	−	0	−	0	0	−	0	0	0	0	+	0	+	0	+	+	+	0
$\overline{7}$	0	0	0	0	0	+	0	0	0	−	0	−	0	0	−	0	0	−	0	0	−	0	−	0	−	−	0	+
$\overline{6}$	0	0	0	0	0	0	+	+	0	0	0	0	0	+	0	0	+	0	+	+	0	+	0	+	0	0	+	+
$\overline{5}$	0	0	0	0	0	0	0	0	+	+	0	0	0	−	0	0	−	0	0	−	+	0	+	0	+	0	−	−
$\overline{4}$	0	0	0	0	0	0	0	0	0	0	+	+	0	+	0	0	0	0	−	0	−	−	−	0	0	+	+	+
$\overline{3}$	0	0	0	0	0	0	0	0	0	0	0	0	+	0	+	0	+	0	+	0	+	0	0	−	−	−	−	−
$\overline{2}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	+	0	+	0	+	0	+	+	+	+	+	+	+
$\overline{1}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

	$\overline{28}$ $\overline{27}$ $\overline{26}$ $\overline{25}$ $\overline{24}$	$\overline{23}$ $\overline{22}$ $\overline{21}$ $\overline{20}$ $\overline{19}$	$\overline{18}$ $\overline{17}$ $\overline{16}$ $\overline{15}$ $\overline{14}$	$\overline{13}$ $\overline{12}$ $\overline{11}$ $\overline{10}$ $\overline{9}$	$\overline{8}$ $\overline{7}$ $\overline{6}$ $\overline{5}$ $\overline{4}$	$\overline{3}$ $\overline{2}$ $\overline{1}$
$\overline{28}$	0 + + + −	+ − + + −	− + 0 − −	0 + 0 − 0	+ 0 0 0 0	0 0 +
$\overline{27}$	+ 0 + + −	+ − + + −	0 + − 0 −	− 0 + 0 −	0 + 0 0 0	0 0 +
$\overline{26}$	+ + 0 + −	+ − + 0 −	+ 0 − + 0	− − + 0 0	0 0 − + 0	0 0 +
$\overline{25}$	+ + + 0 +	+ 0 + − 0	+ − − + 0	− 0 0 − +	0 0 − 0 +	0 0 +
$\overline{24}$	− − − + 0	0 + 0 − +	+ − − + 0	− 0 0 0 0	− + 0 − +	0 0 +
$\overline{23}$	+ + + + 0	0 + + + 0	− 0 + 0 −	0 + − − +	0 0 − 0 0	+ 0 +
$\overline{22}$	− − − 0 +	+ 0 0 + +	− 0 + 0 −	0 + − 0 0	− + 0 − 0	+ 0 +
$\overline{21}$	+ + + + 0	+ 0 0 0 +	0 + 0 − +	+ − + + −	0 0 + 0 0	0 + +
$\overline{20}$	+ + 0 − −	+ + 0 0 0	+ + − 0 −	0 0 0 + −	− + 0 0 −	+ 0 +
$\overline{19}$	− − − 0 +	0 + + 0 0	0 + 0 − +	+ − + 0 0	+ − 0 + 0	0 + +
$\overline{18}$	− 0 + + +	− − 0 + 0	0 0 + + 0	0 − 0 + 0	− 0 − + −	+ 0 +
$\overline{17}$	+ + 0 − −	0 0 + + +	0 0 0 + +	− 0 0 − +	+ − 0 0 +	0 + +
$\overline{16}$	0 − − − −	+ + 0 − 0	+ 0 0 0 0	+ 0 − 0 +	0 − − + −	+ 0 +
$\overline{15}$	− 0 + + +	0 0 − 0 −	+ + 0 0 0	+ + 0 − 0	+ 0 + − +	0 + +
$\overline{14}$	− − 0 0 0	− − + − +	0 + 0 0 0	0 + − + −	− + 0 0 0	+ + +
$\overline{13}$	0 − − − −	0 0 + 0 +	0 − + + 0	0 0 + 0 −	0 + + − +	0 + +
$\overline{12}$	+ 0 − 0 0	+ + − 0 −	− 0 0 + +	0 0 + + 0	− 0 − + 0	+ + +
$\overline{11}$	0 + + 0 0	− − + 0 +	0 0 − 0 −	+ + 0 0 +	0 − − + 0	+ + +
$\overline{10}$	− 0 0 − 0	− 0 + + 0	+ − 0 − +	0 + 0 0 +	+ 0 + 0 +	+ + +
$\overline{9}$	0 − 0 + 0	+ 0 − − 0	0 + + 0 −	− 0 + + 0	0 + + 0 +	+ + +
$\overline{8}$	+ 0 0 0 −	0 − 0 − +	− + 0 + −	0 − 0 + 0	0 + 0 + +	+ + +
$\overline{7}$	0 + 0 0 +	0 + 0 + −	0 − − 0 +	+ 0 − 0 +	+ 0 0 + +	+ + +
$\overline{6}$	0 0 − − 0	− 0 + 0 0	− 0 − + 0	+ − − + +	0 0 0 + +	+ + +
$\overline{5}$	0 0 + 0 −	0 − 0 0 +	+ 0 + − 0	− + + 0 0	+ + + 0 +	+ + +
$\overline{4}$	0 0 0 + +	0 0 0 − 0	− + − + 0	+ 0 0 + +	+ + + + 0	+ + +
$\overline{3}$	0 0 0 0 0	+ + 0 + 0	+ 0 + 0 +	0 + + + +	+ + + + +	0 + +
$\overline{2}$	0 0 0 0 0	0 0 + 0 +	0 + 0 + +	+ + + + +	+ + + + +	+ 0 +
$\overline{1}$	+ + + + +	+ + + + +	+ + + + +	+ + + + +	+ + + + +	+ + 0

Table 16. Root elements: the E_6-numeration.

$$e_{\substack{100000\\0}} = e_{7,8} + e_{9,10} + e_{11,12} + e_{13,15} + e_{16,18} + e_{\overline{28},\overline{27}} + e_{27,28} + e_{\overline{18},\overline{16}} + e_{\overline{15},\overline{13}} + e_{\overline{12},\overline{11}} + e_{\overline{10},\overline{9}} + e_{\overline{8},\overline{7}}$$

$$e_{\substack{000000\\1}} = e_{5,6} + e_{7,9} + e_{8,10} + e_{19,21} + e_{22,23} + e_{\overline{25},\overline{24}} + e_{24,25} + e_{\overline{23},\overline{22}} + e_{\overline{21},\overline{19}} + e_{\overline{10},\overline{8}} + e_{\overline{9},\overline{7}} + e_{\overline{6},\overline{5}}$$

$$e_{\substack{010000\\0}} = e_{5,7} + e_{6,9} + e_{12,14} + e_{15,17} + e_{18,20} + e_{\overline{27},\overline{26}} + e_{26,27} + e_{\overline{20},\overline{18}} + e_{\overline{17},\overline{15}} + e_{\overline{14},\overline{12}} + e_{\overline{9},\overline{6}} + e_{\overline{7},\overline{5}}$$

$$e_{\substack{001000\\0}} = e_{4,5} + e_{9,11} + e_{10,12} + e_{17,19} + e_{20,22} + e_{\overline{26},\overline{25}} + e_{25,26} + e_{\overline{22},\overline{20}} + e_{\overline{19},\overline{17}} + e_{\overline{12},\overline{10}} + e_{\overline{11},\overline{9}} + e_{\overline{5},\overline{4}}$$

$$e_{\substack{000100\\0}} = e_{3,4} + e_{11,13} + e_{12,15} + e_{14,17} + e_{22,24} + e_{\overline{25},\overline{23}} + e_{23,25} + e_{\overline{24},\overline{22}} + e_{\overline{17},\overline{14}} + e_{\overline{15},\overline{12}} + e_{\overline{13},\overline{11}} + e_{\overline{4},\overline{3}}$$

$$e_{\substack{000010\\0}} = e_{2,3} + e_{13,16} + e_{15,18} + e_{17,20} + e_{19,22} + e_{\overline{23},\overline{21}} + e_{21,23} + e_{\overline{22},\overline{19}} + e_{\overline{20},\overline{17}} + e_{\overline{18},\overline{15}} + e_{\overline{16},\overline{13}} + e_{\overline{3},\overline{2}}$$

$$e_{\substack{000001\\0}} = e_{1,2} + e_{16,\overline{28}} + e_{18,\overline{27}} + e_{20,\overline{26}} + e_{22,\overline{25}} + e_{24,\overline{23}} + e_{23,\overline{24}} + e_{25,\overline{22}} + e_{26,\overline{20}} + e_{27,\overline{18}} + e_{28,\overline{16}} + e_{\overline{2},\overline{1}}$$

$$e_{\substack{110000\\0}} = -e_{5,8} - e_{6,10} + e_{11,14} + e_{13,17} + e_{16,20} + e_{\overline{28},\overline{26}} - e_{26,28} - e_{\overline{20},\overline{16}} - e_{\overline{17},\overline{13}} - e_{\overline{14},\overline{11}} + e_{\overline{10},\overline{6}} + e_{\overline{8},\overline{5}}$$

$$e_{\substack{001000\\1}} = -e_{4,6} + e_{7,11} + e_{8,12} - e_{17,21} - e_{20,23} - e_{\overline{26},\overline{24}} + e_{24,26} + e_{\overline{23},\overline{20}} + e_{\overline{21},\overline{17}} - e_{\overline{12},\overline{8}} - e_{\overline{11},\overline{7}} + e_{\overline{6},\overline{4}}$$

$$e_{\substack{011000\\0}} = -e_{4,7} + e_{6,11} - e_{10,14} + e_{15,19} + e_{18,22} + e_{\overline{27},\overline{25}} - e_{25,27} - e_{\overline{22},\overline{18}} - e_{\overline{19},\overline{15}} + e_{\overline{14},\overline{10}} - e_{\overline{11},\overline{6}} + e_{\overline{7},\overline{4}}$$

$$e_{\substack{001100\\0}} = -e_{3,5} + e_{9,13} + e_{10,15} - e_{14,19} + e_{20,24} + e_{\overline{26},\overline{23}} - e_{23,26} - e_{\overline{24},\overline{20}} + e_{\overline{19},\overline{14}} - e_{\overline{15},\overline{10}} - e_{\overline{13},\overline{9}} + e_{\overline{5},\overline{3}}$$

$$e_{\substack{000110\\0}} = -e_{2,4} + e_{11,16} + e_{12,18} + e_{14,20} - e_{19,24} + e_{\overline{25},\overline{21}} - e_{21,25} + e_{\overline{24},\overline{19}} - e_{\overline{20},\overline{14}} - e_{\overline{18},\overline{12}} - e_{\overline{16},\overline{11}} + e_{\overline{4},\overline{2}}$$

$$e_{\substack{000011\\0}} = -e_{1,3} + e_{13,\overline{28}} + e_{15,\overline{27}} + e_{17,\overline{26}} + e_{19,\overline{25}} - e_{24,\overline{21}} + e_{21,\overline{24}} - e_{25,\overline{19}} - e_{26,\overline{17}} - e_{27,\overline{15}} - e_{28,\overline{13}} + e_{\overline{3},\overline{1}}$$

$$e_{\substack{111000\\0}} = e_{4,8} - e_{6,12} - e_{9,14} + e_{13,19} + e_{16,22} + e_{\overline{28},\overline{25}} + e_{25,28} + e_{\overline{22},\overline{16}} + e_{\overline{19},\overline{13}} - e_{\overline{14},\overline{9}} - e_{\overline{12},\overline{6}} + e_{\overline{8},\overline{4}}$$

$$e_{\substack{011000\\1}} = e_{4,9} + e_{5,11} - e_{8,14} - e_{15,21} - e_{18,23} - e_{\overline{27},\overline{24}} - e_{24,27} - e_{\overline{23},\overline{18}} - e_{\overline{21},\overline{15}} - e_{\overline{14},\overline{8}} + e_{\overline{11},\overline{5}} + e_{\overline{9},\overline{4}}$$

$$e_{\substack{001100\\1}} = e_{3,6} + e_{7,13} + e_{8,15} + e_{14,21} - e_{20,25} - e_{\overline{26},\overline{22}} - e_{22,26} - e_{\overline{25},\overline{20}} + e_{\overline{21},\overline{14}} + e_{\overline{15},\overline{8}} + e_{\overline{13},\overline{7}} + e_{\overline{6},\overline{3}}$$

$$e_{\substack{011100\\0}} = e_{3,7} + e_{6,13} - e_{10,17} - e_{12,19} + e_{18,24} + e_{\overline{27},\overline{23}} + e_{23,27} + e_{\overline{24},\overline{18}} - e_{\overline{19},\overline{12}} - e_{\overline{17},\overline{10}} + e_{\overline{13},\overline{6}} + e_{\overline{7},\overline{3}}$$
$$e_{\substack{001110\\0}} = e_{2,5} + e_{9,16} + e_{10,18} - e_{14,22} - e_{17,24} + e_{\overline{26},\overline{21}} + e_{21,26} - e_{\overline{24},\overline{17}} - e_{\overline{22},\overline{14}} + e_{\overline{18},\overline{10}} + e_{\overline{16},\overline{9}} + e_{\overline{5},\overline{2}}$$
$$e_{\substack{000111\\0}} = e_{1,4} + e_{11,\overline{28}} + e_{12,\overline{27}} + e_{14,\overline{26}} - e_{19,\overline{23}} - e_{22,\overline{21}} - e_{21,\overline{22}} - e_{23,\overline{19}} + e_{26,\overline{14}} + e_{27,\overline{12}} + e_{28,\overline{11}} + e_{\overline{4},\overline{1}}$$
$$e_{\substack{111000\\1}} = -e_{4,10} - e_{5,12} - e_{7,14} - e_{13,21} - e_{16,23} - e_{\overline{28},\overline{24}} + e_{24,28} + e_{\overline{23},\overline{16}} + e_{\overline{21},\overline{13}} + e_{\overline{14},\overline{7}} + e_{\overline{12},\overline{5}} + e_{\overline{10},\overline{4}}$$
$$e_{\substack{111100\\0}} = -e_{3,8} - e_{6,15} - e_{9,17} - e_{11,19} + e_{16,24} + e_{\overline{28},\overline{23}} - e_{23,28} - e_{\overline{24},\overline{16}} + e_{\overline{19},\overline{11}} + e_{\overline{17},\overline{9}} + e_{\overline{15},\overline{6}} + e_{\overline{8},\overline{3}}$$
$$e_{\substack{011100\\1}} = -e_{3,9} + e_{5,13} - e_{8,17} + e_{12,21} - e_{18,25} - e_{\overline{27},\overline{22}} + e_{22,27} + e_{\overline{25},\overline{18}} - e_{\overline{21},\overline{12}} + e_{\overline{17},\overline{8}} - e_{\overline{13},\overline{5}} + e_{\overline{9},\overline{3}}$$
$$e_{\substack{001110\\1}} = -e_{2,6} + e_{7,16} + e_{8,18} + e_{14,23} + e_{17,25} - e_{\overline{26},\overline{19}} + e_{19,26} - e_{\overline{25},\overline{17}} - e_{\overline{23},\overline{14}} - e_{\overline{18},\overline{8}} - e_{\overline{16},\overline{7}} + e_{\overline{6},\overline{2}}$$
$$e_{\substack{011110\\0}} = -e_{2,7} + e_{6,16} - e_{10,20} - e_{12,22} - e_{15,24} + e_{\overline{27},\overline{21}} - e_{21,27} + e_{\overline{24},\overline{15}} + e_{\overline{22},\overline{12}} + e_{\overline{20},\overline{10}} - e_{\overline{16},\overline{6}} + e_{\overline{7},\overline{2}}$$
$$e_{\substack{001111\\0}} = -e_{1,5} + e_{9,\overline{28}} + e_{10,\overline{27}} - e_{14,\overline{25}} - e_{17,\overline{23}} - e_{20,\overline{21}} + e_{21,\overline{20}} + e_{23,\overline{17}} + e_{25,\overline{14}} - e_{27,\overline{10}} - e_{28,\overline{9}} + e_{\overline{5},\overline{1}}$$
$$e_{\substack{111100\\1}} = e_{3,10} - e_{5,15} - e_{7,17} + e_{11,21} - e_{16,25} - e_{\overline{28},\overline{22}} - e_{22,28} - e_{\overline{25},\overline{16}} + e_{\overline{21},\overline{11}} - e_{\overline{17},\overline{7}} - e_{\overline{15},\overline{5}} + e_{\overline{10},\overline{3}}$$
$$e_{\substack{111110\\0}} = e_{2,8} - e_{6,18} - e_{9,20} - e_{11,22} - e_{13,24} + e_{\overline{28},\overline{21}} + e_{21,28} - e_{\overline{24},\overline{13}} - e_{\overline{22},\overline{11}} - e_{\overline{20},\overline{9}} - e_{\overline{18},\overline{6}} + e_{\overline{8},\overline{2}}$$
$$e_{\substack{012100\\1}} = e_{3,11} + e_{4,13} + e_{8,19} + e_{10,21} + e_{18,26} + e_{\overline{27},\overline{20}} + e_{20,27} + e_{\overline{26},\overline{18}} + e_{\overline{21},\overline{10}} + e_{\overline{19},\overline{8}} + e_{\overline{13},\overline{4}} + e_{\overline{11},\overline{3}}$$
$$e_{\substack{011110\\1}} = e_{2,9} + e_{5,16} - e_{8,20} + e_{12,23} + e_{15,25} - e_{\overline{27},\overline{19}} - e_{19,27} + e_{\overline{25},\overline{15}} + e_{\overline{23},\overline{12}} - e_{\overline{20},\overline{8}} + e_{\overline{16},\overline{5}} + e_{\overline{9},\overline{2}}$$
$$e_{\substack{001111\\1}} = e_{1,6} + e_{7,\overline{28}} + e_{8,\overline{27}} + e_{14,\overline{24}} + e_{17,\overline{22}} + e_{20,\overline{19}} + e_{19,\overline{20}} + e_{22,\overline{17}} + e_{24,\overline{14}} + e_{27,\overline{8}} + e_{28,\overline{7}} + e_{\overline{6},\overline{1}}$$
$$e_{\substack{011111\\0}} = e_{1,7} + e_{6,\overline{28}} - e_{10,\overline{26}} - e_{12,\overline{25}} - e_{15,\overline{23}} - e_{18,\overline{21}} - e_{21,\overline{18}} - e_{23,\overline{15}} - e_{25,\overline{12}} - e_{26,\overline{10}} + e_{28,\overline{6}} + e_{\overline{7},\overline{1}}$$
$$e_{\substack{112100\\1}} = -e_{3,12} - e_{4,15} + e_{7,19} + e_{9,21} + e_{16,26} + e_{\overline{28},\overline{20}} - e_{20,28} - e_{\overline{26},\overline{16}} - e_{\overline{21},\overline{9}} - e_{\overline{19},\overline{7}} + e_{\overline{15},\overline{4}} + e_{\overline{12},\overline{3}}$$
$$e_{\substack{111110\\1}} = -e_{2,10} - e_{5,18} - e_{7,20} + e_{11,23} + e_{13,25} - e_{\overline{28},\overline{19}} + e_{19,28} - e_{\overline{25},\overline{13}} - e_{\overline{23},\overline{11}} + e_{\overline{20},\overline{7}} + e_{\overline{18},\overline{5}} + e_{\overline{10},\overline{2}}$$
$$e_{\substack{111111\\0}} = -e_{1,8} - e_{6,\overline{27}} - e_{9,\overline{26}} - e_{11,\overline{25}} - e_{13,\overline{23}} - e_{16,\overline{21}} + e_{21,\overline{16}} + e_{23,\overline{13}} + e_{25,\overline{11}} + e_{26,\overline{9}} + e_{27,\overline{6}} + e_{\overline{8},\overline{1}}$$
$$e_{\substack{012110\\1}} = -e_{2,11} + e_{4,16} + e_{8,22} + e_{10,23} - e_{15,26} + e_{\overline{27},\overline{17}} - e_{17,27} + e_{\overline{26},\overline{15}} - e_{\overline{23},\overline{10}} - e_{\overline{22},\overline{8}} - e_{\overline{16},\overline{4}} + e_{\overline{11},\overline{2}}$$
$$e_{\substack{011111\\1}} = -e_{1,9} + e_{5,\overline{28}} - e_{8,\overline{26}} + e_{12,\overline{24}} + e_{15,\overline{22}} + e_{18,\overline{19}} - e_{19,\overline{18}} - e_{22,\overline{15}} - e_{24,\overline{12}} + e_{26,\overline{8}} - e_{28,\overline{5}} + e_{\overline{9},\overline{1}}$$
$$e_{\substack{122100\\1}} = e_{3,14} + e_{4,17} + e_{5,19} + e_{6,21} - e_{16,27} - e_{\overline{28},\overline{18}} - e_{18,28} - e_{\overline{27},\overline{16}} + e_{\overline{21},\overline{6}} + e_{\overline{19},\overline{5}} + e_{\overline{17},\overline{4}} + e_{\overline{14},\overline{3}}$$
$$e_{\substack{112110\\1}} = e_{2,12} - e_{4,18} + e_{7,22} + e_{9,23} - e_{13,26} + e_{\overline{28},\overline{17}} + e_{17,28} - e_{\overline{26},\overline{13}} + e_{\overline{23},\overline{9}} + e_{\overline{22},\overline{7}} - e_{\overline{18},\overline{4}} + e_{\overline{12},\overline{2}}$$
$$e_{\substack{111111\\1}} = e_{1,10} - e_{5,\overline{27}} - e_{7,\overline{26}} + e_{11,\overline{24}} + e_{13,\overline{22}} + e_{16,\overline{19}} + e_{19,\overline{16}} + e_{22,\overline{13}} + e_{24,\overline{11}} - e_{26,\overline{7}} - e_{27,\overline{5}} + e_{\overline{10},\overline{1}}$$
$$e_{\substack{012210\\1}} = e_{2,13} + e_{3,16} - e_{8,24} - e_{10,25} - e_{12,26} - e_{\overline{27},\overline{14}} - e_{14,27} - e_{\overline{26},\overline{12}} - e_{\overline{25},\overline{10}} - e_{\overline{24},\overline{8}} + e_{\overline{16},\overline{3}} + e_{\overline{13},\overline{2}}$$
$$e_{\substack{012111\\1}} = e_{1,11} + e_{4,\overline{28}} + e_{8,\overline{25}} + e_{10,\overline{24}} - e_{15,\overline{20}} - e_{18,\overline{17}} - e_{17,\overline{18}} - e_{20,\overline{15}} + e_{24,\overline{10}} + e_{25,\overline{8}} + e_{28,\overline{4}} + e_{\overline{11},\overline{1}}$$
$$e_{\substack{122110\\1}} = -e_{2,14} + e_{4,20} + e_{5,22} + e_{6,23} + e_{13,27} - e_{\overline{28},\overline{15}} + e_{15,28} - e_{\overline{27},\overline{13}} - e_{\overline{23},\overline{6}} - e_{\overline{22},\overline{5}} - e_{\overline{20},\overline{4}} + e_{\overline{14},\overline{2}}$$
$$e_{\substack{112210\\1}} = -e_{2,15} - e_{3,18} - e_{7,24} - e_{9,25} - e_{11,26} - e_{\overline{28},\overline{14}} + e_{14,28} + e_{\overline{26},\overline{11}} + e_{\overline{25},\overline{9}} + e_{\overline{24},\overline{7}} + e_{\overline{18},\overline{3}} + e_{\overline{15},\overline{2}}$$
$$e_{\substack{112111\\1}} = -e_{1,12} - e_{4,\overline{27}} + e_{7,\overline{25}} + e_{9,\overline{24}} - e_{13,\overline{20}} - e_{16,\overline{17}} + e_{17,\overline{16}} + e_{20,\overline{13}} - e_{24,\overline{9}} - e_{25,\overline{7}} + e_{27,\overline{4}} + e_{\overline{12},\overline{1}}$$
$$e_{\substack{012211\\1}} = -e_{1,13} + e_{3,\overline{28}} - e_{8,\overline{23}} - e_{10,\overline{22}} - e_{12,\overline{20}} + e_{18,\overline{14}} - e_{14,\overline{18}} + e_{20,\overline{12}} + e_{22,\overline{10}} + e_{23,\overline{8}} - e_{28,\overline{3}} + e_{\overline{13},\overline{1}}$$
$$e_{\substack{122210\\1}} = e_{2,17} + e_{3,20} - e_{5,24} - e_{6,25} + e_{11,27} + e_{\overline{28},\overline{12}} + e_{12,28} + e_{\overline{27},\overline{11}} - e_{\overline{25},\overline{6}} - e_{\overline{24},\overline{5}} + e_{\overline{20},\overline{3}} + e_{\overline{17},\overline{2}}$$
$$e_{\substack{122111\\1}} = e_{1,14} + e_{4,\overline{26}} + e_{5,\overline{25}} + e_{6,\overline{24}} + e_{13,\overline{18}} + e_{16,\overline{15}} + e_{15,\overline{16}} + e_{18,\overline{13}} + e_{24,\overline{6}} + e_{25,\overline{5}} + e_{26,\overline{4}} + e_{\overline{14},\overline{1}}$$
$$e_{\substack{112211\\1}} = e_{1,15} - e_{3,\overline{27}} - e_{7,\overline{23}} - e_{9,\overline{22}} - e_{11,\overline{20}} + e_{16,\overline{14}} + e_{14,\overline{16}} - e_{20,\overline{11}} - e_{22,\overline{9}} - e_{23,\overline{7}} - e_{27,\overline{3}} + e_{\overline{15},\overline{1}}$$

$$e_{\substack{012221\\1}} = e_{1,16} + e_{2,\overline{28}} + e_{8,\overline{21}} + e_{10,\overline{19}} + e_{12,\overline{17}} + e_{15,\overline{14}} + e_{14,\overline{15}} + e_{17,\overline{12}} + e_{19,\overline{10}} + e_{21,\overline{8}} + e_{28,\overline{2}} + e_{\overline{16},\overline{1}}$$
$$e_{\substack{123210\\1}} = -e_{2,19} - e_{3,22} - e_{4,24} + e_{6,26} + e_{9,27} - e_{\overline{28},\overline{10}} + e_{10,28} - e_{\overline{27},\overline{9}} - e_{\overline{26},\overline{6}} + e_{\overline{24},\overline{4}} + e_{\overline{22},\overline{3}} + e_{\overline{19},\overline{2}}$$
$$e_{\substack{122211\\1}} = -e_{1,17} + e_{3,\overline{26}} - e_{5,\overline{23}} - e_{6,\overline{22}} + e_{11,\overline{18}} - e_{16,\overline{12}} + e_{12,\overline{16}} - e_{18,\overline{11}} + e_{22,\overline{6}} + e_{23,\overline{5}} - e_{26,\overline{3}} + e_{\overline{17},\overline{1}}$$
$$e_{\substack{112221\\1}} = -e_{1,18} - e_{2,\overline{27}} + e_{7,\overline{21}} + e_{9,\overline{19}} + e_{11,\overline{17}} + e_{13,\overline{14}} - e_{14,\overline{13}} - e_{17,\overline{11}} - e_{19,\overline{9}} - e_{21,\overline{7}} + e_{27,\overline{2}} + e_{\overline{18},\overline{1}}$$
$$e_{\substack{123210\\2}} = e_{2,21} + e_{3,23} + e_{4,25} + e_{5,26} + e_{7,27} + e_{\overline{28},\overline{8}} + e_{8,28} + e_{\overline{27},\overline{7}} + e_{\overline{26},\overline{5}} + e_{\overline{25},\overline{4}} + e_{\overline{23},\overline{3}} + e_{\overline{21},\overline{2}}$$
$$e_{\substack{123211\\1}} = e_{1,19} - e_{3,\overline{25}} - e_{4,\overline{23}} + e_{6,\overline{20}} + e_{9,\overline{18}} + e_{16,\overline{10}} + e_{10,\overline{16}} + e_{18,\overline{9}} + e_{20,\overline{6}} - e_{23,\overline{4}} - e_{25,\overline{3}} + e_{\overline{19},\overline{1}}$$
$$e_{\substack{122221\\1}} = e_{1,20} + e_{2,\overline{26}} + e_{5,\overline{21}} + e_{6,\overline{19}} - e_{11,\overline{15}} - e_{13,\overline{12}} - e_{12,\overline{13}} - e_{15,\overline{11}} + e_{19,\overline{6}} + e_{21,\overline{5}} + e_{26,\overline{2}} + e_{\overline{20},\overline{1}}$$
$$e_{\substack{123211\\2}} = -e_{1,21} + e_{3,\overline{24}} + e_{4,\overline{22}} + e_{5,\overline{20}} + e_{7,\overline{18}} - e_{16,\overline{8}} + e_{8,\overline{16}} - e_{18,\overline{7}} - e_{20,\overline{5}} - e_{22,\overline{4}} - e_{24,\overline{3}} + e_{\overline{21},\overline{1}}$$
$$e_{\substack{123221\\1}} = -e_{1,22} - e_{2,\overline{25}} + e_{4,\overline{21}} - e_{6,\overline{17}} - e_{9,\overline{15}} + e_{13,\overline{10}} - e_{10,\overline{13}} + e_{15,\overline{9}} + e_{17,\overline{6}} - e_{21,\overline{4}} + e_{25,\overline{2}} + e_{\overline{22},\overline{1}}$$
$$e_{\substack{123221\\2}} = e_{1,23} + e_{2,\overline{24}} - e_{4,\overline{19}} - e_{5,\overline{17}} - e_{7,\overline{15}} - e_{13,\overline{8}} - e_{8,\overline{13}} - e_{15,\overline{7}} - e_{17,\overline{5}} - e_{19,\overline{4}} + e_{24,\overline{2}} + e_{\overline{23},\overline{1}}$$
$$e_{\substack{123321\\1}} = e_{1,24} + e_{2,\overline{23}} + e_{3,\overline{21}} + e_{6,\overline{14}} + e_{9,\overline{12}} + e_{11,\overline{10}} + e_{10,\overline{11}} + e_{12,\overline{9}} + e_{14,\overline{6}} + e_{21,\overline{3}} + e_{23,\overline{2}} + e_{\overline{24},\overline{1}}$$
$$e_{\substack{123321\\2}} = -e_{1,25} - e_{2,\overline{22}} - e_{3,\overline{19}} + e_{5,\overline{14}} + e_{7,\overline{12}} - e_{11,\overline{8}} + e_{8,\overline{11}} - e_{12,\overline{7}} - e_{14,\overline{5}} + e_{19,\overline{3}} + e_{22,\overline{2}} + e_{\overline{25},\overline{1}}$$
$$e_{\substack{124321\\2}} = e_{1,26} + e_{2,\overline{20}} + e_{3,\overline{17}} + e_{4,\overline{14}} - e_{7,\overline{10}} - e_{9,\overline{8}} - e_{8,\overline{9}} - e_{10,\overline{7}} + e_{14,\overline{4}} + e_{17,\overline{3}} + e_{20,\overline{2}} + e_{\overline{26},\overline{1}}$$
$$e_{\substack{134321\\2}} = -e_{1,27} - e_{2,\overline{18}} - e_{3,\overline{15}} - e_{4,\overline{12}} - e_{5,\overline{10}} - e_{6,\overline{8}} + e_{8,\overline{6}} + e_{10,\overline{5}} + e_{12,\overline{4}} + e_{15,\overline{3}} + e_{18,\overline{2}} + e_{\overline{27},\overline{1}}$$
$$e_{\substack{234321\\2}} = e_{1,28} + e_{2,\overline{16}} + e_{3,\overline{13}} + e_{4,\overline{11}} + e_{5,\overline{9}} + e_{7,\overline{6}} + e_{6,\overline{7}} + e_{9,\overline{5}} + e_{11,\overline{4}} + e_{13,\overline{3}} + e_{16,\overline{2}} + e_{\overline{28},\overline{1}}$$

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