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ON OVERGROUPS OF $\mathrm{E}(\mathrm{E}_6, R)$ AND $\mathrm{E}(\mathrm{E}_7, R)$ IN THEIR MINIMAL REPRESENTATIONS

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Overgroups of elementary Chevalley groups of types E_6 and E_7 in their minimal irreducible representations are dealt with. One of the first steps toward their description, namely, the construction of a series of perfect intermediate groups, is performed. Probably all perfect intermediate groups for groups of type E_6 are described. A conjecture concerning the structure of perfect overgroups of type E_7 is suggested. Bibliography: 23 titles.

1. Introduction

In this paper, we consider the problem of classifying the overgroups of the elementary subgroup $\mathrm{E}(\Phi, R)$ in Chevalley groups of type $\Phi = \mathrm{E}_6, \mathrm{E}_7$ in their minimal irreducible representations (R is a commutative ring). There are many papers devoted to similar problems, but almost all of them consider only classical groups over fields. These are [9–11, 13–17]. Only in recent papers [4–6] has the standard description for overgroups of symplectic and orthogonal elementary groups over a commutative ring been obtained, and Petrov's paper [19] is devoted to a similar description of overgroups for generalized unitary groups over an arbitrary ring with certain conditions on the local stable rank.

The proof of such theorems concerning the standard description for exceptional groups would be of great interest. Let us clarify what this means. We say that the *standard description* of subgroups in $G = \mathrm{GL}(n, R)$ containing $\mathrm{E}(\Phi, R)$ holds if for any such subgroup H there exists a unique ideal $A \trianglelefteq R$ such that

$$\mathrm{E}(\Phi, R)\mathrm{E}(n, R, A) \leq H \leq N_G(\mathrm{E}(\Phi, R)\mathrm{E}(n, R, A)),$$

where $\mathrm{E}(n, R, A) = \mathrm{E}(n, A)^{\mathrm{E}(n,R)}$ is the relative elementary group and N_G denotes a normalizer in G. For exceptional groups of types E_6 and E_7 in their minimal irreducible representations (which have dimensions 27 and 56, respectively), it is not known whether the standard description holds, even if R is a field. In this paper, we prove the existence of the largest ideal A satisfying the condition

$$\mathrm{E}(\Phi, R)\mathrm{E}(n, R, A) \leq H.$$

Moreover, we show that the groups

$$\mathrm{EE}_6(27, R, A) = \mathrm{E}(\mathrm{E}_6, R)\mathrm{E}(27, R, A)$$

and

$$\mathrm{EE}_7(56, R, A) = \mathrm{E}(\mathrm{E}_7, R)\mathrm{E}(56, R, A)$$

are perfect. Here R is an arbitrary commutative ring in which 2 and 3 are invertible.

For groups of type E_7 the standard description seems to be more complicated: we must turn our attention to the group

$$\mathrm{EE}'_7(56, R, A, B) = \mathrm{E}(\mathrm{E}_7, R)\mathrm{E}(56, R, A)\,\mathrm{Ep}(56, R, B),$$

where $A \subseteq B$ are two ideals of R. This is due to the fact that a group of type E_7 can be embedded in a certain symplectic matrix group. Since this embedding is not unique, we must deal with the conjugation of $\mathrm{Ep}(56, R, B)$ by some special diagonal matrix. In the present paper, we make some steps toward a similar description of overgroups of $\mathrm{E}(\mathrm{E}_7, R)$.

This paper is organized as follows: in Sec. 2, we introduce some notation and definitions and in Sec. 3 we state the results in six theorems. The rest of the paper is devoted to their proofs. In Sec. 4, one can find a proof of the first theorem. In Secs. 5, 6, 7, and 8, the second theorem is proved, which proves Theorem 3 as well (see the remark before the statement of Theorem 3). Section 9 is devoted to a proof of Theorem 4. The introduction of the symplectic group in the description of overgroups for $\mathrm{E}(\mathrm{E}_7, R)$ is made in Secs. 10–12. In Sec. 10, we construct the symplectic group we need, whereas in Secs. 11 and 12 we establish Theorems 5 and 6, respectively.

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2. Notation and definitions

Throughout this paper $\Phi = \mathrm{E}_6$ or $\Phi = \mathrm{E}_7$. If statements for $\Phi = \mathrm{E}_6$ and $\Phi = \mathrm{E}_7$ are different, we separate them by the abbreviation "resp." Let P be a lattice lying between the root lattice $Q(\Phi)$ and the weight lattice $P(\Phi)$. For a fixed order on Φ, let Φ^+ be the set of positive roots and Φ^- be the set of negative roots with respect to this order. Let $\Pi = \{\alpha_1, \alpha_2, \dots, \alpha_l\}$ be the corresponding set of fundamental roots ($l = 6$ resp. 7). The numbering of fundamental roots follows that from [1]. Given a root system Φ, a lattice P, and a commutative ring R with 1, one can construct the *Chevalley group* $\mathrm{G} = \mathrm{G}_P(\Phi, R)$, i.e., the group of points over R of the *Chevalley–Demazure affine group scheme*. We always consider a simply-connected Chevalley group, i.e., $P = P(\Phi)$, and we omit P in the notation of the group.

We assume that a Chevalley base is chosen in a complex simple Lie algebra L of type Φ and that the Chevalley algebra L_R is constructed. Therefore, we have a split maximal torus $\mathrm{T}(\Phi, R)$ in G and a fixed parametrization of root unipotent subgroups X_α, $\alpha \in \Phi$, with respect to this torus. For $\alpha \in \Phi$ and $\xi \in R$, we denote by $x_\alpha(\xi)$ the corresponding elementary root unipotent. In the sequel we will frequently make use of Steinberg relations between the elements $x_\alpha(\xi)$, in particular, of the Chevalley commutator formula (see [8]). The group $X_\alpha = \{x_\alpha(\xi), \xi \in R\}$ is called the *elementary root subgroup*; the group $\mathrm{E}(\Phi, R) = \langle X_\alpha, \alpha \in \Phi \rangle$ that is generated by all elementary root subgroups is called the (absolute) *elementary subgroup* of the Chevalley group $\mathrm{G}(\Phi, R)$.

We assume that the group $\mathrm{E}(\Phi, R)$ is a subgroup in $\mathrm{GL}(n, R)$, where $n = 27$ resp. 56, in the usual (minimal) representations. This means that G acts on a *Weyl module* $V = V(\omega)$, where ω is a dominant weight. We assume that the highest weight ω of V is fundamental and is equal to ω_1 resp. ω_7. Then the corresponding modules V are *microweight*; see [2, 3, 23] for more detailed information and further references. By $\Lambda = \Lambda(\omega)$ we denote the set of weights of the module $V = V(\omega)$, taking the multiplicities into account. In fact, in microweight representations all weights have multiplicity one; therefore Λ coincides with the orbit of the highest weight ω under the action of the Weyl group W.

Let us fix an admissible base v^λ, $\lambda \in \Lambda$, for the module V. This means that $x_\alpha(\xi)v^\lambda$ can be expressed as a linear combination of vectors v^μ, $\mu \in \Lambda$, with integer coefficients. For microweight representations, one can normalize an admissible base in such a way that $x_\alpha(\xi)v^\lambda = v^\lambda + c_{\lambda\alpha}\xi v^{\lambda+\alpha}$, where all the action constants $c_{\lambda\alpha}$ are equal to 1 or -1 (this is a "Matsumoto lemma," see [18, 20]). In fact, we always choose a *crystalic* base, in which the constants $c_{\lambda\alpha}$ are equal to 1 for $\alpha \in \pm\Pi$ (an elementary proof of the existence of this base can be found in [22]).

We view a vector $a \in V$, $a = \sum a_\lambda v^\lambda$, as a column of coordinates $a = (a_\lambda)$, $\lambda \in \Lambda$. At the same time, an element b of the contragradient module V^* can be naturally regarded as a row $b = (b_\lambda)$, $\lambda \in \Lambda$. We index rows and columns by weights of the module V, that is, by elements of Λ. Therefore, rows and columns of matrices in $\mathrm{GL}(V, R) = \mathrm{GL}(n, R)$ are indexed by the weights of the representation, that is, by elements $\lambda, \mu, \nu, \rho, \sigma, \tau, \ldots \in \Lambda$. For $x, y \in G$ we denote by $[x, y]$ their commutator $xyx^{-1}y^{-1}$.

The main tool for our calculations is the following lemma, copied from [3].

Lemma 0. *For any $g \in \mathrm{GL}(n, R)$, $\alpha \in \Phi$, and $\xi \in R$, we have*

$$(x_\alpha(\xi)g)_{\rho\sigma} = g_{\rho\sigma} \pm \xi g_{\rho-\alpha,\sigma}, \quad (g x_\alpha(\xi))_{\rho\sigma} = g_{\rho\sigma} \pm \xi g_{\rho,\sigma+\alpha}.$$

3. Main results

The aim of this paper is to prove the following two theorems.

Theorem 1. *Let $A \trianglelefteq R$. Then*

$$\mathrm{E}(n, A)^{\mathrm{E}(\Phi,R)} = \mathrm{E}(n, R, A),$$

where, as usual, $\mathrm{E}(n, R, A) = \mathrm{E}(n, A)^{\mathrm{E}(n,R)}$.

Theorem 2. *Assume that H is a subgroup in $\mathrm{GL}(n, R)$ that contains $\mathrm{E}(\Phi, R)$ and $2, 3 \in R^*$. For $\lambda, \mu \in \Lambda$ and $\lambda \neq \mu$, we set $A_{\lambda\mu} = \{\xi \in R \mid t_{\lambda\mu}(\xi) \in H\}$. If $\rho, \sigma \in \Lambda$, $\rho \neq \sigma$, then $A_{\lambda\mu} = A_{\rho\sigma} = A$, where $A \trianglelefteq R$.*

From this theorems we can immediately deduce the following theorem.

Theorem 3. *Assume that H is a subgroup in $\mathrm{GL}(n, R)$ that contains $\mathrm{E}(\Phi, R)$ and $2, 3 \in R^*$; then there exists a unique largest ideal $A \trianglelefteq R$ such that $\mathrm{E}(n, R, A) \leq H$. Moreover, if $t_{\lambda\mu}(\xi) \in H$ for some $\lambda, \mu \in \Lambda$, $\lambda \neq \mu$, then $\xi \in A$.*

Theorem 3 asserts that for any intermediate subgroup between $\mathrm{E}(\Phi, R)$ and $\mathrm{GL}(n, R)$, there exists an ideal $A \trianglelefteq R$ such that H contains the group $\mathrm{E}(\Phi, R)\mathrm{E}(n, R, A)$. In Theorem 3, we prove that the latter group is perfect.

Theorem 4. *Let R be a commutative ring and $A \trianglelefteq R$. The groups* $\mathrm{EE}_6(27, R, A) = \mathrm{E}(\mathrm{E}_6, R)\mathrm{E}(27, R, A)$ *and* $\mathrm{EE}_7(56, R, A) = \mathrm{E}(\mathrm{E}_7, R)\mathrm{E}(56, R, A)$ *are perfect.*

The following theorem is analogous to Theorem 1 for a symplectic group, which we need in the case $\Phi = \mathrm{E}_7$.

Theorem 5. *Assume that $A \trianglelefteq R$ and $2 \in R^*$; then*

$$\mathrm{Ep}(56, A)^{\mathrm{E}(\mathrm{E}_7,R)} = \mathrm{Ep}(56, R, A),$$

where, as usual, $\mathrm{Ep}(56, R, A) = \mathrm{Ep}(56, A)^{\mathrm{Ep}(56,R)}$.

We construct the above-mentioned symplectic group $\mathrm{Ep}(56, R)$ in Sec. 10.

The last theorem is analogous to Theorem 4.

Theorem 6. *Let R be a commutative ring, $A \trianglelefteq R$, $B \trianglelefteq R$, $A \subseteq B$, and $2 \in R^*$. The group* $\mathrm{EE}'_7(56, R, A, B) = \mathrm{E}(\mathrm{E}_7, R)\mathrm{E}(56, R, A)\,\mathrm{Ep}(56, R, B)$ *is perfect.*

Unfortunately, the author has not yet succeeded in obtaining an analog of Theorem 2 in the symplectic case. Moreover, we did not make use of the fact that there are many different symplectic groups that contain $\mathrm{E}(\mathrm{E}_7, R)$ in the minimal representation (see a short comment in Sec. 1).

4. Proof of Theorem 1

Proof. It is obvious that the left-hand group is contained in the right-hand one. To prove the opposite inclusion, note that for $n \geq 3$ the group $\mathrm{E}(n, R, A)$ is generated by the elements $z_{\lambda\mu}(\xi, \zeta) = {}^{t_{\mu\lambda}(\zeta)}t_{\lambda\mu}(\xi)$ for all $\xi \in A$, $\zeta \in R$, and λ, $\mu \in \Lambda$, $\lambda \neq \mu$ (this fact was proved by Tits in [21]). Thus it remains to show that $z_{\lambda\mu}(\xi, \zeta)$ belongs to $H = \mathrm{E}(n, A)^{\mathrm{E}(\Phi,R)}$. We do this in Lemmas 1, 2, and 3; in Lemma i we consider the case $d(\lambda, \mu) = i$, $i = 1, 2, 3$. It is obvious that in our representation, 2 resp. 3 is the maximal distance between the weights, and the theorem will be established if we prove the following three lemmas.

From now on $\zeta \in R$ and $\xi \in A$. We shall often use the following direct calculation:

$$z_{\lambda\mu}(\xi, \zeta) = t_{\mu\lambda}(\zeta)t_{\lambda\mu}(\xi)t_{\mu\lambda}(-\zeta) = (e + \zeta e_{\mu\lambda})(e + \xi e_{\lambda\mu})(e - \zeta e_{\mu\lambda}) = e + \xi e_{\lambda\mu} + \xi\zeta(e_{\mu\mu} - e_{\lambda\lambda}) - \xi\zeta^2 e_{\mu\lambda}.$$

Lemma 1. *Assume that $d(\lambda, \mu) = 1$; then ${}^{t_{\mu\lambda}(\zeta)}t_{\rho\sigma}(\xi) \in H$ for any ρ, $\sigma \in \Lambda$. In particular, $z_{\lambda\mu}(\xi, \zeta) \in H$.*

Proof. First consider the case $\rho = \lambda$ and $\sigma = \mu$. Denote $\mu - \lambda$ by $\alpha \in \Phi$ and consider an element

$${}^{x_\alpha(\zeta)}t_{\lambda\mu}(\xi) = x_\alpha(\zeta)t_{\lambda\mu}(\xi)x_\alpha(-\zeta) \in H.$$

It easily follows from Lemma 0 that

$$g = x_\alpha(\zeta)t_{\lambda\mu}(\xi) = e + \xi e_{\lambda\mu} \pm \zeta\xi e_{\mu\mu} \pm \zeta e_{\mu\lambda} + \sum_{i=1}^{s} (-1)^{\varepsilon_i}\zeta e_{\nu_i+\alpha,\nu_i},$$

where $\nu_1, \nu_2, \ldots, \nu_s$ are all distinct weights $\nu \in \Lambda \setminus \{\lambda\}$ such that $\nu + \alpha$ is also a weight. We are not interested in the signs $(-1)^{\varepsilon_i}$ in this expression; we need only know the sign of action "between" the weights λ and μ, which we will indicate by the symbols $\pm$ and $\mp$.

It remains to multiply the expression obtained by $x_\alpha(-\zeta)$ and to look at the matrix elements. It is obvious (see Lemma 0 once more) that this multiplication changes only elements in positions (τ, τ'), where $\tau, \tau', \tau' + \alpha \in \Lambda$. But we have already known all τ' such that τ' and $\tau' + \alpha$ are both weights: these are precisely $\lambda, \nu_1, \ldots, \nu_s$. We want to know for which τ the element $g_{\tau,\tau'+\alpha}$ does not equal 0. It is easy to show that the signs of action "between" the weights ν_i and $\nu_i + \alpha$ are opposite to each other, whence all expressions containing ν_i cancel out. That is why we did not try to know what signs of actions we had there. The remaining expression is

$${}^{x_\alpha(\zeta)}t_{\lambda\mu}(\xi) = gx_\alpha(-\zeta)e \mp \xi\zeta e_{\lambda\lambda} + \xi e_{\lambda\mu} - \xi\zeta^2 e_{\mu\lambda} \pm \xi\zeta e_{\mu\mu}.$$

We see that it coincides with the above expression for $z_{\lambda\mu}(\xi, \zeta)$, except for the signs in the positions (λ, λ) and (μ, μ), which we can easily change by substituting $-\zeta$ for ζ at the outset.

The remaining cases where $\rho \neq \lambda$ or $\sigma \neq \mu$ are even easier. Indeed, $y = {}^{t_{\mu\lambda}(\zeta)}t_{\rho\sigma}(\xi) = [t_{\mu\lambda}(\zeta), t_{\rho\sigma}(\xi)] \cdot t_{\rho\sigma}(\xi)$. If $\rho = \lambda$, but $\sigma \neq \mu$, then $y = t_{\mu\sigma}(\xi\zeta)t_{\rho\sigma}(\xi) \in \mathrm{E}(n, A)$. The case $\sigma = \mu$ and $\rho \neq \lambda$ can be treated in the same way. Finally, if $\sigma \neq \mu$ and $\rho \neq \lambda$, then these transvections commute, and $y = t_{\rho\sigma}(\xi) \in \mathrm{E}(n, A)$. The proof is complete.

Corollary. *Assume that* $d(\lambda,\mu)=1$; *then* $\mathrm{E}(n,A)^{t_{\mu\lambda}(\zeta)} \le H$.

Lemma 2. *Assume that* $d(\lambda,\mu)=2$; *then* $z_{\lambda\mu}(\xi,\zeta)\in H$.

Proof. Choose $\alpha,\beta\in\Phi$ such that $\lambda-\mu=\alpha+\beta$, $\lambda-\alpha=\mu+\beta\in\Lambda$, and $\lambda-\beta=\mu+\alpha\in\Lambda$. We can do this: the pair (λ,μ) can be translated to the pair $(\omega,-\omega)$ resp. $(\omega,\omega-{}^{012222}_{\ \ 1})$ by the action of the Weyl group. Hence we can take, for example, $\alpha={}^{12211}_{\ \ 1}$ resp. $\alpha=\alpha_7$ and $\beta={}^{11221}_{\ \ 1}$ resp. $\beta={}^{012221}_{\ \ 1}$.

Set $\kappa=\lambda-\alpha=\mu+\beta$ and $\nu=\lambda-\beta=\mu+\alpha$. We have

$$\begin{aligned} z_{\lambda\mu}(\xi,\zeta) &= {}^{t_{\mu\lambda}(\zeta)}t_{\lambda\mu}(\xi) = {}^{t_{\mu\lambda}(\zeta)}[t_{\lambda\nu}(\xi),t_{\nu\mu}(1)] = [t_{\lambda\nu}(\xi)t_{\mu\nu}(\xi\zeta),t_{\nu\mu}(1)t_{\nu\lambda}(-\zeta)] \\ &= t_{\lambda\nu}(\xi)t_{\mu\nu}(\xi\zeta)\cdot {}^{t_{\nu\mu}(1)t_{\nu\lambda}(-\zeta)}t_{\mu\nu}(-\xi\zeta)t_{\lambda\nu}(-\xi). \end{aligned}$$

Thus, it remains to show that

$${}^{t_{\nu\mu}(1)t_{\nu\lambda}(-\zeta)}t_{\mu\nu}(-\xi\zeta)t_{\lambda\nu}(-\xi)\in H.$$

Now we restrict our calculations to four weights: λ, ν, μ, and κ. Namely, consider the four-dimensional subspace $W\subseteq V$ generated by v^λ, v^ν, v^μ, v^κ. We can perform our calculations inside W, because they will contain only elementary transvections of the form $t_{\rho\sigma}(\zeta)$ for $\zeta\in R$, $\{\rho,\sigma\}\subset\{\lambda,\nu,\mu,\kappa\}$, and conjugations by the elements $x_\alpha(\zeta)$ and $x_\beta(\zeta)$. After these conjugations, we are kept inside this subspace: indeed, consider an element $g\in\mathrm{GL}(n,R)$ such that the nontriviality of its action is contained in W; then the element ${}^{x_\alpha(\zeta)}g$ also has this property. This immediately follows from the fact that our representation is microweight, whence none of the elements $\lambda+\alpha$, $\nu+\alpha$, $\mu-\alpha$, and $\kappa-\alpha$ is a weight. Therefore, the "actions" between the weights ρ and σ (here $\rho-\sigma=\alpha$) cancel each other in just the same way as was in the proof of Lemma 1. Naturally, this argument can also be applied to β.

We use matrices in $\mathrm{GL}(n,R)$ restricted to W in the base $(v^\lambda,v^\nu,v^\mu,v^\kappa)$. Let $h={}^{t_{\nu\lambda}(-\zeta)}t_{\mu\nu}(-\xi\zeta)t_{\lambda\nu}(-\xi)$, $g={}^{t_{\nu\mu}(1)}h$. We want to prove that $g\in H$. Note that, by the corollary to Lemma 1, we have $h\in H$. Let us denote by $\widetilde g$ and $\widetilde h$ the matrices in $\mathrm{GL}(4,R)$ that correspond to $g|_W$ and $h|_W$ in the above-mentioned base. It is clear that

$$\widetilde h=\begin{pmatrix}1-\xi\zeta & -\xi & 0 & 0\\ \xi\zeta^2 & 1+\xi\zeta & 0 & 0\\ -\xi\zeta^2 & -\xi\zeta & 1 & 0\\ 0&0&0&1\end{pmatrix},\qquad \widetilde g=\begin{pmatrix}1-\xi\zeta & -\xi & \xi & 0\\ 0 & 1 & 0 & 0\\ -\xi\zeta^2 & -\xi\zeta & 1+\xi\zeta & 0\\ 0&0&0&1\end{pmatrix},\qquad \widetilde{x_\alpha(\zeta)}=\begin{pmatrix}1&0&0&\zeta\\0&1&\zeta&0\\0&0&1&0\\0&0&0&1\end{pmatrix}.$$

Thus, the element ${}^{x_\alpha(1)}h\in H$ corresponds to the matrix

$$\begin{pmatrix}1-\xi\zeta & -\xi & \xi & \xi\zeta\\ 0&1&0&0\\ -\xi\zeta^2 & -\xi\zeta & 1+\xi\zeta & \xi\zeta^2\\ 0&0&0&1\end{pmatrix}.$$

Upon multiplication from the left by $t_{\lambda\kappa}(-\xi\zeta)t_{\mu\kappa}(-\xi\zeta^2)\in\mathrm{E}(n,R)$, we obtain an element of H, which is expressed by the matrix $\widetilde g$.

However, the argument above is correct only if the action signs of $x_\alpha(\zeta)$ restricted to W are the same: in this case, $x_\alpha(\zeta)$ has actually the above form. But if the signs are opposite, we should slightly change the argument. Conjugating h by $x_\alpha(\pm1)$, we obtain *almost* the same matrix as above; namely, the nondiagonal elements in the last column will have the opposite sign. It is clear that we need only multiply by the same elementary transvections with the opposite signs in their arguments; then we obtain precisely the matrix $\widetilde g$. This means that in every case we have $g\in H$, and the proof is complete.

Lemma 3. *Assume that* $d(\lambda,\mu)=3$; *then* $z_{\lambda\mu}(\xi,\zeta)\in H$.

Proof. Here $\Phi=\mathrm{E}_7$. Now, as before, we construct a certain configuration of weights for the special case $\lambda=\omega$, $\mu=-\omega$ and "translate" it by an element of the Weyl group into an arbitrary pair of weights with distance 3 between them. Let us set $\alpha=\alpha_7$, $\beta={}^{123221}_{\ \ 2}$, and $\gamma={}^{123321}_{\ \ 1}$. It is clear that $-\omega+\alpha+\beta+\gamma=\omega$ and $\alpha,\beta,\gamma\in\Phi$. Let us denote the weight $\mu+\alpha$ by ν.

A calculation similar to the one at the beginning of the proof of Lemma 2 shows that it suffices to prove the inclusion

$$g = {}^{t_{\nu\mu}(1)t_{\nu\lambda}(-\zeta)}t_{\lambda\nu}(-\xi)t_{\mu\nu}(-\xi\zeta) \in H.$$

We obtain

$$g = t_{\nu\mu}(1) \cdot t_{\nu\lambda}(-\zeta)t_{\lambda\nu}(-\xi)t_{\nu\lambda}(\zeta) \cdot t_{\nu\lambda}(-\zeta)t_{\mu\nu}(-\xi\zeta)t_{\nu\lambda}(\zeta) \cdot t_{\nu\mu}(-1).$$

Using Lemma 2, we get $t_{\nu\lambda}(-\zeta)t_{\lambda\nu}(-\xi)t_{\nu\lambda}(\zeta) \in H$ (note that $d(\lambda,\nu)=2$). Consider the expression

$$t_{\nu\lambda}(-\zeta)t_{\mu\nu}(-\xi\zeta)t_{\nu\lambda}(\zeta) = t_{\mu\nu}(-\xi\zeta) \cdot [t_{\mu\nu}(\xi\zeta), t_{\nu\lambda}(-\zeta)] = t_{\mu\nu}(-\xi\zeta)t_{\mu\lambda}(-\xi\zeta^2) \in \mathrm{E}(n,A).$$

Thus we have $g = {}^{t_{\nu\mu}(1)}h$, where $h \in H$. Now we proceed as in the final part of the proof of the previous lemma: we look at the matrices to compare g with ${}^{x_\alpha(1)}h \in H$. The only weights affected by the elements obtained are λ, μ, and ν. We want to be able to conjugate by $x_\alpha(1)$, and thus we need to take the weight $\kappa = \lambda - \alpha$ into account as well. These four weights provide the subspace we need: any addition or subtraction of the root α does not result in new weights. Therefore we may restrict our calculations to the four-dimensional subspace $W \subseteq V$ generated by the vectors v^λ, v^κ, v^ν, v^μ. We express these restrictions by matrices in $\mathrm{GL}(4,R)$ in the base $(v^\lambda, v^\kappa, v^\nu, v^\mu)$:

$$\widetilde{h} = \begin{pmatrix} 1+\xi\zeta & 0 & -\xi & 0 \\ 0 & 1 & 0 & 0 \\ \xi\zeta^2 & 0 & 1+\xi\zeta & 0 \\ -\xi\zeta^2 & 0 & -\xi\zeta & 1 \end{pmatrix}, \qquad \widetilde{x_\alpha(1)} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\widetilde{{}^{x_\alpha(1)}h} = \begin{pmatrix} 1-\xi\zeta & \xi\zeta & -\xi & \xi \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\xi\zeta^2 & \xi\zeta^2 & -\xi\zeta & 1+\xi\zeta \end{pmatrix}, \qquad \widetilde{g} = \begin{pmatrix} 1-\xi\zeta & 0 & -\xi & \xi \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\xi\zeta^2 & 0 & -\xi\zeta & 1+\xi\zeta \end{pmatrix}.$$

Now we can see that, multiplying ${}^{x_\alpha(1)}h$ by

$$t_{\lambda\kappa}(-\xi\zeta)t_{\mu\kappa}(-\xi\zeta^2) \in \mathrm{E}(n,A)$$

from the left, we obtain g. Unfortunately, $\widetilde{x_\alpha(1)}$ does not always have the form given above. Action signs may be different for the pairs of weights (λ,κ) and (ν,μ). It is obvious that there are only two essentially different cases: these signs are the same (and we dealt with this case above), and these signs are opposite to each other. We can reduce all the cases to these two ones by substituting $x_\alpha(-1)$ for $x_\alpha(1)$. Hence if the signs are different, we may assume that

$$\widetilde{x_\alpha(1)} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

whence

$$\widetilde{{}^{x_\alpha(1)}h} = \begin{pmatrix} 1-\xi\zeta & -\xi\zeta & -\xi & \xi \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\xi\zeta^2 & -\xi\zeta^2 & -\xi\zeta & 1+\xi\zeta \end{pmatrix},$$

and we obtain the final result by multiplying from the left by $t_{\lambda\kappa}(\xi\zeta)t_{\mu\kappa}(\xi\zeta^2) \in \mathrm{E}(n,A)$.

5. Proof of Theorem 2. The beginning

Note that now H is a subgroup in $\mathrm{GL}(n,R)$ containing $\mathrm{E}(\Phi,R)$. The following lemma can be obtained by direct calculations.

Lemma 4. *Assume that λ, $\mu \in \Lambda$, $\lambda \neq \mu$, $\alpha \in \Phi$, and $\lambda - \mu \neq \pm\alpha$.*
(a) *If $\mu - \alpha \notin \Lambda$ and $\lambda + \alpha \in \Lambda$, then $[t_{\lambda\mu}(\xi), x_\alpha(\zeta)] = t_{\lambda+\alpha,\mu}(\pm\xi\zeta)$.*
(b) *If $\lambda - \alpha \notin \Lambda$ and $\mu + \alpha \in \Lambda$, then $[t_{\lambda\mu}(\xi), x_\alpha(\zeta)] = t_{\lambda,\mu+\alpha}(\pm\xi\zeta)$.*

Lemma 5. *Assume that λ, $\mu \in \Lambda$ and $d(\lambda,\mu) = 1$. Then for $\alpha \in \Phi$ such that $\lambda+\alpha \in \Lambda$ and $\lambda + \alpha \neq \mu$, we have $\mu - \alpha \notin \Lambda$.*

Proof. Let us look at the weight diagram. We may assume that $\lambda = \omega$ and $\mu = \omega - \delta$. If $\lambda + \alpha \in \Lambda$, then $\alpha \in \Phi^-$ and the root $-\alpha_1$ resp. $-\alpha_7$ must occur in the decomposition of α into the sum of fundamental roots. But to satisfy $\mu - \alpha \in \Lambda$, the root $-\alpha_1$ resp. $-\alpha_7$ must be between the weights $\mu-\alpha$ and μ, which holds only if $\alpha = \mu - \lambda$.

Now we can prove part of Theorem 2.

Lemma 6. *Assume that $\lambda, \mu, \rho, \sigma \in \Lambda$ and $d(\lambda,\mu) = d(\rho,\sigma) = 1$; then $A_{\lambda\mu} = A_{\rho\sigma} = A_1$ and $A_1 \trianglelefteq R$.*

Proof. It is obvious that $A_{\lambda\mu}$ is an additive subgroup of R. Part (a) of Lemma 4 says in fact that if $\lambda, \mu \in \Lambda$, $\lambda \neq \mu$, $\alpha \in \Phi$, $\lambda - \mu \neq \pm\alpha$, and $\mu - \alpha \notin \Lambda$, then $RA_{\lambda\mu} \subset A_{\lambda+\alpha,\mu}$. Thus, using Lemma 5, we obtain $RA_{\lambda\mu} \subset A_{\rho\sigma}$ (we can move "step by step" along the edges of the weight diagram to replace both weights in the indices by necessary ones). Moreover, $A_{\lambda\mu}$ is an ideal of R. It follows that all such ideals coincide with each other.

The following lemma is a further step toward the theorem.

Lemma 7. *Assume that $\lambda, \mu, \rho, \sigma \in \Lambda$ and $d(\lambda,\mu) = d(\rho,\sigma) = 2$; then $A_{\lambda\mu} = A_{\rho\sigma} = A_2$ and $A_2 \trianglelefteq R$.*

Proof. Let us look at the weight diagram again. We may assume that $\mu = \omega$ and $\lambda = \omega - \begin{smallmatrix}22210\\1\end{smallmatrix}$ resp. $\lambda = \omega - \begin{smallmatrix}012222\\1\end{smallmatrix}$ ($\begin{smallmatrix}22210\\1\end{smallmatrix}$ resp. $\begin{smallmatrix}012222\\1\end{smallmatrix}$ is not a root, and thus $d(\lambda,\mu) = 2$). Hence, for any $\alpha \in \Phi^-$, we have $\mu - \alpha \notin \Lambda$, and we can use part (a) of Lemma 4 with $\alpha \in \Phi^-$ such that $\lambda + \alpha \in \Lambda$. In this way, $\rho = \lambda + \alpha$ can be made into any other weight such that $d(\rho,\mu) = 2$, except for $\rho = -\omega$ in the case $\Phi = \mathrm{E}_6$, therefore obtaining $RA_{\lambda\mu} \subset A_{\rho\mu}$. However, if $\Phi = \mathrm{E}_6$ and $\rho = -\omega$, we proceed as follows: first we obtain $RA_{\lambda\mu} \subset A_{\rho\mu}$, where $\rho = \lambda - \begin{smallmatrix}01221\\1\end{smallmatrix}$, and second we pass to $\rho - \begin{smallmatrix}00001\\0\end{smallmatrix} = -\omega$. Therefore, conjugating by $x_{\alpha_6}(\pm 1)$, we have $RA_{\lambda\mu} \subset A_{\rho\mu} \subset A_{-\omega,\mu}$. The proof is accomplished in the same way as in the previous lemma.

Now we show that $A_1 = A_2$. In the following two lemmas, we prove the necessary inclusions. It is worth mentioning that the inclusion $A_1 \subset A_2$ can easily be obtained by the methods used in the proof of Theorem 2. On the other side, in order to prove the opposite inclusion we rely on the fact that H *contains* $\mathrm{E}(\Phi, R)$ (earlier we used only the fact that H *is normalized* by $\mathrm{E}(\Phi, R)$). Moreover, the restriction $2, 3 \in R^*$ arises.

Lemma 8. $A_1 \subseteq A_2$.

Proof. Take $\xi \in A_1$. Using Lemma 6, we obtain $t_{\lambda\mu}(\xi) \in H$ for any $\mu = \omega$ and $\lambda = \omega - \begin{smallmatrix}12210\\1\end{smallmatrix}$ resp. $\lambda = \omega - \begin{smallmatrix}012221\\1\end{smallmatrix}$. Therefore, using Lemma 4, we get $[t_{\lambda\mu}(\xi), x_{-\alpha_1}(1)] = t_{\lambda-\alpha_1,\mu}(\xi)$ resp. $[t_{\lambda\mu}(\xi), x_{-\alpha_7}(1)] = t_{\lambda-\alpha_7,\mu}(\xi)$. But $d(\lambda-\alpha_1,\mu) = 2$ resp. $d(\lambda - \alpha_7, \mu) = 2$, and thus $\xi \in A_2$.

Lemma 9. *Assume that $2, 3 \in R^*$; then $A_2 \subseteq A_1$.*

Proof. Now we need to know what happens when we commute a transvection $t_{\lambda\mu}(\xi)$ with a root element $x_\alpha(\zeta)$ if $\lambda - \mu \neq \pm\alpha$, but $\lambda + \alpha$, $\mu - \alpha \in \Lambda$ ($\alpha \in \Phi$). Assume that $\xi \in A_2$ and $\zeta \in R$ (in fact we will set $\zeta = 1$ below). We may also assume that $d(\lambda,\mu) = 2$. Using the technique from the proof of Theorem 1, it is easy to obtain

$$\begin{aligned}\big[t_{\lambda\mu}(\xi), x_\alpha(\zeta)\big] &= \Big(e + \xi e_{\lambda\mu} + (-1)^\varepsilon \xi\zeta e_{\lambda,\mu-\alpha} + (-1)^\eta \zeta e_{\lambda+\alpha,\lambda} + (-1)^\varepsilon \zeta e_{\mu,\mu-\alpha}\Big)\\ &\times\Big(e - \xi e_{\lambda\mu} + (-1)^\varepsilon \xi\zeta e_{\lambda,\mu-\alpha} - (-1)^\eta \zeta e_{\lambda+\alpha,\lambda} - (-1)^\varepsilon \zeta e_{\mu,\mu-\alpha}\Big)\\ &= e + (-1)^\varepsilon \xi\zeta e_{\lambda,\mu-\alpha} - (-1)^\eta \xi\zeta e_{\lambda+\alpha,\mu} + (-1)^{\varepsilon+\eta}\xi\zeta^2 e_{\lambda+\alpha,\mu-\alpha} \in H. \qquad (1)\end{aligned}$$

Note that $d(\lambda+\alpha, \mu-\alpha) \geq 2$ (this is readily seen, since we may assume that $\lambda = \omega - \begin{smallmatrix}22210\\1\end{smallmatrix}$ resp. $\lambda = \omega - \begin{smallmatrix}012222\\1\end{smallmatrix}$ and $\alpha = \alpha_1$ resp. $\alpha = \alpha_7$, and look at the weight diagram). Therefore, using Lemma 7, we have $t_{\lambda+\alpha,\mu-\alpha}(\pm\xi\zeta^2) \in H$, and, multiplying by this element, we obtain $e + (-1)^\varepsilon \xi\zeta e_{\lambda,\mu-\alpha} - (-1)^\eta \xi\zeta e_{\lambda+\alpha,\mu} \in H$. This means that the product of two transvections $t_{\lambda,\mu-\alpha}((-1)^\varepsilon\xi\zeta) t_{\lambda+\alpha,\mu}((-1)^{\eta+1}\xi\zeta)$ belongs to H. But we need to find *one* transvection lying in H.

Now consider the case $\lambda - (\mu - \alpha) = (\lambda + \alpha) - \mu = \beta$, where $\beta \in \Phi$ is a fixed root. The idea is to take all transvections $t_{\rho\sigma}(\dots)$ with $\rho - \sigma = \beta$. It is obvious that the number k of such pairs (ρ, σ) is equal to 6 resp. 12. Let us denote them by (ρ_i, σ_i), $1 \leq i \leq k$.

The last part of our argument is somewhat tricky, so we first consider the case $\Phi = \mathrm{E}_6$ and next make necessary corrections to include the case $\Phi = \mathrm{E}_7$.

6. Proof of Lemma 9. The case $\Phi = \mathrm{E}_6$

We claim that for every *pair* of such pairs (ρ_i, σ_i) and (ρ_j, σ_j), the expression $\rho_i - \rho_j = \sigma_i - \sigma_j$ is a root. Indeed, one such pair, say (ρ_i, σ_i), can be translated by an element of the Weyl group to a pair $(\omega, \omega - \alpha_1)$. Then, looking at the remaining five pairs (ρ_j, σ_j) on the weight diagram, we see that $\rho_i - \rho_j = \omega - \rho_j$ is always a root. Hence, if we apply the above calculation and set $\zeta = 1$, we derive that the product of transvections $t_{\rho_i,\sigma_i}(\pm\xi)t_{\rho_j,\sigma_j}(\pm\xi)$ belongs to H for any $1 \le i, j \le 6$, $i \ne j$. At the same time, we said nothing about the signs of $\pm\xi$. Obviously, there are two cases: either these signs coincide or they are opposite to each other. Therefore, we may assume that a certain product $t_{\rho_i,\sigma_i}(\xi)t_{\rho_j,\sigma_j}(\pm\xi)$ belongs to H.

Now we introduce the final idea of the proof. Since a root element x_β is a product of six transvections

$$x_\beta(\xi) = \prod_{i=1}^{6} t_{\rho_i,\sigma_i}(\pm\xi),$$

we have a hope to construct a *single* transvection $t_{\rho_i,\sigma_i}(\pm n\xi)$ (for some $n \in R^*$) from this expression of x_β and our pairwise products. Then we would have $\xi \in A_1$, just as we need. Now we set $\beta = \delta$ in order to make positive all signs in the expression of $x_\beta(\xi)$.

Now we need some facts about the action signs of the root element $x_\alpha(1)$, because without knowing the signs we cannot execute our idea. We will use Theorem 1 of [22], which says that if α is a fundamental or a negative fundamental root, then all signs in the decomposition of $x_\alpha(1)$ into the product of transvections are equal to 1. In our case this implies that $\eta = \varepsilon = 0$ in Eq. (1). Thus, if α is a fundamental or a negative fundamental root, then the product of transvections that is obtained by using $x_\alpha(1)$ has opposite signs at ξ. Recall that we put $\beta = \delta$; now we take

$$\begin{array}{ll}
\rho_1 = \omega, & \rho_2 = \omega - \substack{10000\\0}, \\
\rho_3 = \omega - \substack{11000\\0}, & \rho_4 = \omega - \substack{11100\\0}, \\
\rho_5 = \omega - \substack{11110\\0}, & \rho_6 = \omega - \substack{11111\\0}, \\
\sigma_i = \rho_i - \delta, & 1 \le i \le 6.
\end{array}$$

Now note that in considering the product of transvections

$$x t_{\rho_i,\sigma_i}(\xi)t_{\rho_{i+1},\sigma_{i+1}}(\pm\xi),$$

we used commutation with x_α for $\alpha = \rho_i - \rho_{i+1}$, which is a fundamental root for any i such that $1 \le i \le 5$. Therefore we obtain the products

$$y_1 = t_{\rho_1,\sigma_1}(\xi)t_{\rho_2,\sigma_2}(-\xi) \in H, \qquad y_2 = t_{\rho_2,\sigma_2}(\xi)t_{\rho_3,\sigma_3}(-\xi) \in H,$$
$$y_3 = t_{\rho_3,\sigma_3}(\xi)t_{\rho_4,\sigma_4}(-\xi) \in H, \qquad y_4 = t_{\rho_4,\sigma_4}(\xi)t_{\rho_5,\sigma_5}(-\xi) \in H, \qquad y_5 = t_{\rho_5,\sigma_5}(\xi)t_{\rho_6,\sigma_6}(-\xi) \in H.$$

Consider the product $h = y_1 y_2^2 y_3^3 y_4^4 y_5^5 x_\beta(-\xi) \in H$. It is readily seen that $h = t_{\rho_6,\sigma_6}(-6\xi)$, whence $6\xi \in A_1$ and finally $\xi \in A_1$, which proves the lemma in the case $\Phi = \mathrm{E}_6$.

7. Proof of Lemma 9. The case $\Phi = \mathrm{E}_7$

In this case we cannot state that for *any pair* of pairs (ρ_i, σ_i) and (ρ_j, σ_j) the expression $\rho_i - \rho_j = \sigma_i - \sigma_j$ is a root. Consider the pair $(\rho_i, \sigma_i) = (\omega, \omega - \alpha_7)$: the above-mentioned expression is a root only for ten of the remaining eleven pairs (ρ_j, σ_j). However, in the proof for the case $\Phi = \mathrm{E}_6$ we used not all possible combinations of pairs, but only five of them, to construct the elements y_1, y_2, y_3, y_4, y_5. We shall show that for $\Phi = \mathrm{E}_7$ we need exactly eleven combinations. As above, it can easily be checked that the product of transvections $t_{\rho_i,\sigma_i}(\pm\xi)t_{\rho_j,\sigma_j}(\pm\xi)$ belongs to H if $\rho_i - \rho_j = \sigma_i - \sigma_j$ is a root.

The final idea of the proof, described in the previous section, can be applied to our case in the following way. We again set $\beta = \delta$ to make positive all signs in nondiagonal matrix elements of $x_\beta(\xi)$. Using Theorem 1 of [22], we conclude that if α or $-\alpha$ is a fundamental root, then the action signs of $x_\alpha(1)$ are also equal to 1; therefore we have $\eta = \varepsilon = 0$ in (1). Now we take

$$\begin{array}{llll}
\rho_1 = \omega, & \rho_2 = \omega - \begin{smallmatrix}000001\\0\end{smallmatrix}, & \rho_3 = \omega - \begin{smallmatrix}000011\\0\end{smallmatrix}, & \\
\rho_4 = \omega - \begin{smallmatrix}000111\\0\end{smallmatrix}, & \rho_5 = \omega - \begin{smallmatrix}001111\\0\end{smallmatrix}, & \rho_6 = \omega - \begin{smallmatrix}011111\\0\end{smallmatrix}, & \\
\rho_7 = \omega - \begin{smallmatrix}001111\\1\end{smallmatrix}, & \rho_8 = \omega - \begin{smallmatrix}011111\\1\end{smallmatrix}, & \rho_9 = \omega - \begin{smallmatrix}012111\\1\end{smallmatrix}, & \\
\rho_{10} = \omega - \begin{smallmatrix}012211\\1\end{smallmatrix}, & \rho_{11} = \omega - \begin{smallmatrix}012221\\1\end{smallmatrix}, & \rho_{12} = \omega - \begin{smallmatrix}012222\\1\end{smallmatrix}, & \sigma_i = \rho_i - \delta, \quad 1 \le i \le 6.
\end{array}$$

In composing the product of transvections for successive pairs (ρ_i, σ_i), $(\rho_{i+1}, \sigma_{i+1})$, we obtain

$$t_{\rho_i,\sigma_i}(\xi) t_{\rho_{i+1},\sigma_{i+1}}(\pm\xi);$$

here we used commutation with x_α, where $\alpha = \rho_i - \rho_{i+1}$ is a fundamental root for any i, $1 \le i \le 11$, except for $i = 6$. Moreover,

$$t_{\rho_5,\sigma_5}(\xi) t_{\rho_7,\sigma_7}(\pm\xi)$$

can be obtained by commuting with x_{α_2}. Since we used only fundamental roots, we get $y_i = t_{\rho_i,\sigma_i}(\xi) t_{\rho_{i+1},\sigma_{i+1}}(-\xi) \in H$ for $1 \le i \le 11$ and $i \ne 6$. Denote $t_{\rho_5,\sigma_5}(\xi) t_{\rho_7,\sigma_7}(-\xi)$ by y_6. Now consider the expression

$$h = y_1^1 y_2^2 y_3^3 y_4^4 y_5^{-1} y_6^6 y_7^7 y_8^8 y_9^9 y_{10}^{10} y_{11}^{11} x_\alpha(-\xi) \in H.$$

It is clear that $h = t_{\rho_{12},\sigma_{12}}(-12\xi)$ (here, as in the case $\Phi = \mathrm{E}_6$ we use the factorization of $x_\beta(-\xi)$ into twelve transvections $t_{\rho_i,\sigma_i}(\xi)$), whence $12\xi \in A_1$ and thus $\xi \in A_1$. This completes the proof of Lemma 9.

8. Completion of the proof of Theorem 2

Note that the proof of Theorem 2 for $\Phi = \mathrm{E}_6$ has already been completed (in Sec. 5), since 2 is the maximal distance between the weights of the microweight representation of E_6. In the case $\Phi = \mathrm{E}_7$, we need to do some more work to embrace the case where this distance is 3. Denote $A = A_1 = A_2$.

Lemma 10. *Assume that $\lambda, \mu \in \lambda$ and $d(\lambda, \mu) = 3$; then $RA \subset A_{\lambda\mu}$.*

Proof. The proof is similar to that of Lemma 8. Any pair of weights (λ, μ) with $d(\lambda, \mu) = 3$ can be transformed by element of the Weyl group to the pair $(\omega, -\omega)$. Therefore we may assume that $\lambda = -\omega$ and $\mu = \omega$. Take $\xi \in A$. By the above, $t_{\lambda+\alpha_7,\mu}(\xi) \in H$. Using Lemma 4, we obtain $[t_{\lambda+a_7,\mu}(\xi), x_{-\alpha_7}(\pm\zeta)] = t_{\lambda,\mu}(\xi\zeta)$ with a suitable choice of the sign, whence $\xi\zeta \in A_{\lambda\mu}$.

Lemma 11. *Assume that $\lambda, \mu, \rho, \sigma \in \lambda$ and $d(\lambda, \mu) = d(\rho, \sigma) = 3$; then $A_{\lambda\mu} \subset A_{\rho\sigma}$.*

Proof. Here, as at the beginning of the proof of Lemma 9, we commute the element $t_{\lambda\mu}(\xi)$ with $x_\alpha(\zeta)$ (see formula (1)) for $\xi \in A_{\lambda\mu}$, $\zeta \in R$, and $\alpha \in \Phi$. We shall look at the restrictions of our matrices to the four-dimensional subspace W generated by the base vectors corresponding to the weights μ, $\mu - \alpha$, $\lambda + \alpha$, and λ (in the given order). We assume that α is a fundamental or a negative fundamental root; therefore the nondiagonal matrix entries of $x_\alpha(\zeta)$ have equal signs. Now using formula (1) and the information about the signs, we obtain the matrix of the restriction of $h_1 = [t_{\lambda\mu}(\xi), x_\alpha(\zeta)]$:

$$\widetilde{h_1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\xi\zeta & \xi\zeta^2 & 1 & 0 \\ 0 & \xi\zeta & 0 & 1 \end{pmatrix}. \tag{2}$$

Set $\zeta = 1$, change the sign of ξ, and commute the result with $x_{-\alpha}(\zeta)$. Note that, under our choice of α, all nondiagonal matrix entries of $x_{-\alpha}(\zeta)$ have equal signs. Denote $[[t_{\lambda\mu}(-\xi), x_\alpha(1)], x_{-\alpha}(\zeta)]$ by h_2. An easy computation shows that

$$\widetilde{h_2} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \xi\zeta & 0 & 1 & 0 \\ 2\xi\zeta + \xi\zeta^2 & -\xi\zeta & 0 & 1 \end{pmatrix}.$$

By construction, $h_1, h_2 \in H$, and therefore the product $h = h_1 h_2 t_{\lambda\mu}(-2\xi\zeta - \xi\zeta^2)$ belongs to H and can be written in the form

$$\widetilde{h} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \xi\zeta^2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

In the special case $\zeta = 1$, we obtain $t_{\lambda+\alpha,\mu-\alpha}(\xi) \in H$. It is readily seen that the distance between $\lambda + \alpha$ and $\mu - \alpha$ equals 3. Indeed, two weights with distance 3 between them correspond to centrally symmetric points in the weight diagram; thus, if λ and μ are centrally symmetric, then $\lambda+\alpha$ and $\mu-\alpha$ are also centrally symmetric. Finally, we obtain $\xi \in A_{\lambda+\alpha,\mu-\alpha}$, whence $A_{\lambda\mu} \subset A_{\lambda+\alpha,\mu-\alpha}$.

Now it is easy to complete the proof of the lemma. Indeed, since we can take α to be any fundamental or negative fundamental root, we can move step by step along the edges of the diagram and pass from any pair of weights with distance 3 between them to any other such pair.

Let us denote the set $A_{\lambda\mu}$ for $d(\lambda, \mu) = 3$ by A_3. Note that, in contrast to Lemmas 6 and 7, we have not yet proved that A_3 is an ideal in R. We do not intend to prove this directly; we only show that the subgroup (with respect to addition) A_3 coincides with A. The inclusion $A \subset A_3$ is proved in Lemma 10, and now we prove the opposite inclusion.

Lemma 12. $A_3 \subset A$.

Proof. Assume $\xi \in A_3$. For the sake of clarity, we fix the weights $\lambda = -\omega$ and $\mu = \omega$ and the root $\alpha = \alpha_7$. We use some notation from the proof of the previous lemma: $h_1 = [t_{\lambda\mu}(\xi), x_\alpha(\zeta)]$. Let us put $\zeta = 1$; then, using formula (2) and multiplying h_1 by $t_{\lambda+\alpha,\mu-\alpha}(-\xi)$, we obtain the element $h = e - \xi e_{\lambda+\alpha,\mu} + \xi e_{\lambda,\mu-\alpha}$. Fortunately, in the present case $d(\lambda + \alpha, \mu) = d(\lambda, \mu - \alpha) = 2$, and thus we need not repeat giddy tricks in the spirit of the proof of Lemma 9. It suffices to put $\beta = \alpha_6$ and consider an element $[h, x_\beta(1)] \in H$ (note that we know the explicit form of $x_\beta(1)$, since β is a simple root). An easy computation shows that $[h, x_\beta(1)] = e + \xi e_{\lambda,\mu-\alpha-\beta}$. Using the fact that $d(\lambda, \mu - \alpha - \beta) = 2$, we obtain $\xi \in A$. This completes the proof of the lemma and Theorem 2.

9. Proof of Theorem 4

Here we show that the group that naturally arises in Theorem 3 is perfect. Assume that $n = 27$ resp. $n = 56$ and $E = \mathrm{EE}_6(27, R, A)$ resp. $E = \mathrm{EE}_7(56, R, A)$. Since the group $\mathrm{E}(\Phi, R)$ is perfect (see [22, 7, 12]), it suffices to show that generators of $\mathrm{E}(n, R, A)$ belong to $[E, E]$. Denote $z_{\lambda\mu}(\xi, \zeta)$ by x. Recall that

$$z_{\lambda\mu}(\xi, \zeta) = {}^{t_{\mu\lambda}(\zeta)} t_{\lambda\mu}(\xi), \quad \xi \in A, \zeta \in R.$$

Theorem 1 implies that $x \in \mathrm{E}(n, A)^{\mathrm{E}(\Phi, R)}$, and thus x can be expressed as a product

$$x = \prod_i x_i y_i x_i^{-1} \quad \text{for} \quad x_i \in \mathrm{E}(\Phi, R) \quad \text{and} \quad y_i \in \mathrm{E}(n, A) \subset \mathrm{E}(n, R, A).$$

We have $x = \prod_i [x_i, y_i] y_i$, and for any i the commutator $[x_i, y_i]$ belongs to $[E, E]$. It remains to prove that $\mathrm{E}(n, A) \subset [E, E]$. This can easily be deduced from Lemma 4 in the proof of Theorem 2. Namely, consider $t_{\rho\sigma}(\xi) \in \mathrm{E}(n, A)$ and try to find $\lambda, \mu \in \Lambda$ and $\alpha \in \Phi$ such that the condition of part (a) in Lemma 4 is satisfied, and at the same time $\lambda + \alpha = \rho$ and $\mu = \sigma$. In the case of success, we obtain $t_{\rho\sigma}(\xi) = [t_{\lambda\mu}(\xi), x_\alpha(\pm 1)] \in [\mathrm{E}(n, A), \mathrm{E}(\Phi, R)] \subset [E, E]$.

To do this, we may assume that $\mu = \sigma = \omega$ is the maximal root. If $\rho \neq \omega - \alpha_1$ resp. $\rho \neq \omega - \alpha_7$, then we can take a simple root β for which $\rho + \beta \in \Lambda$, and put $\lambda = \rho + \beta$ and $\alpha = -\beta$. Thus $\lambda + \alpha = \rho$ and $\mu - \alpha \notin \Lambda$, since $\alpha \in \Phi^-$ and μ is the highest weight. It is clear that $\lambda \neq \mu$ and $\lambda - \mu \neq \pm\alpha$ (since only $\lambda - \mu = \alpha$ may occur, and thus $\lambda - \alpha$, λ, $\lambda + \alpha \in \Lambda$, a contradiction).

It remains to consider the case $\sigma = \omega$ and $\rho = \omega - \alpha_1$ resp. $\rho = \omega - \alpha_7$. Now we can put $\alpha = \alpha_3$ resp. $\alpha = \alpha_6$ and $\lambda = \rho - \alpha$, and it is readily seen that all the assumptions of Lemma 4 are satisifed. This completes the proof.

10. Embedding of $\mathrm{E}(\mathrm{E}_7, R)$ into the symplectic group

In this section, we describe an embedding of $\mathrm{E}(\mathrm{E}_7.R)$ into a symplectic group of 56×56 matrices. Recall that we have already embedded $\mathrm{E}(\mathrm{E}_7, R)$ into the general linear group $\mathrm{GL}(56, R)$, so it suffices to construct a symplectic bilinear form φ in the given base. Our representation of E_7 has a symmetric weight diagram: for any weight λ there is a weight $-\lambda$, which is centrally symmetric to λ. We define a symplectic product $\varphi(v^\lambda, v^\mu) = 0$ for $\mu \neq -\lambda$. If $\mu = -\lambda$, decompose $\omega - \lambda$ into the sum of fundamental weights. The number of weights in this sum equals the "distance" from the weight λ to the highest weight ω in the weight diagram. This distance is just the number of edges in a minimal path joining these weights (in contrast to the distance d in the *weight graph*, which we often used before). For a while we denote this distance by $d'(\omega, \lambda)$. In fact, we need only know $d'(\omega, \lambda)$ modulo 2; denote $\varepsilon_\lambda = (-1)^{d'(\omega,\lambda)}$. We call ε_λ the *sign* of the weight λ if this does not cause confusion. It is clear that $\varepsilon_{-\lambda} = -\varepsilon_\lambda$, and thus this naming convention makes some sense. Now we define $\varphi(v_\lambda, v_{-\lambda}) = \varepsilon_\lambda$.

It is obvious that, defining the product on the base vectors, we obtain a symplectic bilinear form on the entire representation space; therefore we have the corresponding symplectic group. Now we recall how the symplectic transvections look like:

$$T_{\lambda\mu}(\xi) = T_{-\mu,-\lambda}(-\varepsilon_\lambda\varepsilon_\mu\xi) = \begin{cases} t_{\lambda\mu}(\xi)t_{-\mu,-\lambda}(-\varepsilon_\lambda\varepsilon_\mu\xi) & \text{for} \quad \mu \neq -\lambda, \\ t_{\lambda,-\lambda}(\xi) & \text{for} \quad \mu = -\lambda. \end{cases}$$

We say that the transvection $T_{\lambda\mu}(\xi)$ is a *short root transvection* if $\mu \neq -\lambda$, and it is a *long root transvection* if $\mu = -\lambda$.

We often use the Chevalley commutator formula for symplectic transvections without special notice. Let us write down the most important cases:

$$\begin{aligned}
[T_{\lambda\mu}(\xi), T_{\mu\sigma}(\zeta)] &= T_{\lambda\sigma}(\xi\zeta) \quad \text{if} \quad \lambda \neq \pm\mu, \quad \mu \neq \pm\sigma, \quad \lambda \neq \pm\sigma, \\
[T_{\lambda\mu}(\xi), T_{\mu,-\lambda}(\zeta)] &= T_{\lambda,-\lambda}(2\xi\zeta) \quad \text{if} \quad \lambda \neq \pm\mu, \\
[T_{\lambda\mu}(\xi), T_{\mu,-\mu}(\zeta)] &= T_{\lambda,-\mu}(\xi\zeta)T_{\lambda,-\lambda}(\varepsilon_\lambda\varepsilon_\mu\xi^2\zeta) \quad \text{if} \quad \lambda \neq \pm\mu, \\
[T_{\lambda\mu}(\xi), T_{\rho\sigma}(\zeta)] &= 1 \quad \text{if} \quad \lambda \neq \mu, \quad \rho \neq \sigma, \quad \mu \neq \rho, \quad \lambda \neq \sigma, \quad \mu \neq -\sigma, \quad \lambda \neq -\rho.
\end{aligned}$$

The other cases of the Chevalley commutator formula can easily be produced from these ones. The elementary symplectic group corresponding to this symplectic form will be denoted by $\mathrm{Ep}(56, R) = \langle T_{\lambda\mu}(\xi),\ \lambda \neq \mu,\ \xi \in R\rangle$, and in addition $\mathrm{Ep}(56, R, A) = \mathrm{Ep}(56, A)^{\mathrm{Ep}(56,R)}$, where $\mathrm{Ep}(56, A) = \langle T_{\lambda\mu}(\xi),\ \lambda \neq \mu,\ \xi \in A\rangle$ for any ideal $A \trianglelefteq R$.

Now we check that the group $\mathrm{E}(\mathrm{E}_7, R)$ is contained in the constructed symplectic group $\mathrm{Ep}(56, R)$. It suffices to show that $x_\alpha(\xi) \in \mathrm{Ep}(56, R)$ for $\xi \in R$ and $\alpha \in \mathrm{E}_7$. In fact, it suffices to verify this inclusion only for *fundamental* roots $\alpha \in \mathrm{E}_7$ (due to the Chevalley commutator formula). We show that for any fundamental root α, the root element $x_\alpha(\xi)$ is a product of six symplectic transvections. Indeed, using the weight diagram we see that there are twelve edges labelled by α, situated in a symmetric way: there are six pairs of such edges, and in every pair the edges are centrally symmetric. Consider such a pair: edges (λ, μ) and $(-\mu, -\lambda)$ with $\mu - \lambda = \alpha$. Take a symplectic transvection $T_{\mu\lambda}(1) = t_{\mu\lambda}(1)t_{-\lambda,-\mu}(1)$ (it is really a transvection, because the weights λ and μ are adjacent, whence $\varepsilon_\mu = -\varepsilon_\lambda$). But we see here precisely two (elementary) transvections that correspond to the given pair of edges in the decomposition of x_α (recall that all action signs of the root element x_α are equal to $+1$). Repeating the same argument for each pair of edges, we obtain six short-root symplectic transvections.

We have $x_\alpha(\xi) \in \mathrm{Ep}(56, R)$, and therefore $\mathrm{E}(\mathrm{E}_7, R) \in \mathrm{Ep}(56, R)$. Now we show that for *any* root $\alpha \in \mathrm{E}_7$, the root element $x_\alpha(\xi)$ is a product of exactly six symplectic transvections. As above, we can divide all pairs of weights (λ, μ) such that $\mu - \lambda = \alpha$ into six pairs of pairs, grouping together the pairs (λ, μ) and $(-\mu, -\lambda)$. But we already know that $x_\alpha(\xi)$ belongs to the symplectic group; therefore it satisfies certain simple equations. It can easily be checked that the action signs in these pairs match each other nicely, just as in the symplectic transvection $T_{\lambda\mu}(\pm 1)$.

11. Proof of Theorem 5

We will try to follow the line of proof of Theorem 1.

Proof. It is obvious that the left-hand side is contained in the right-hand one. To prove the opposite inclusion, we recall that for $n \geq 3$ the group $\mathrm{Ep}(n, R, A)$ is generated by the elements $Z_{\lambda\mu}(\xi, \zeta) = {}^{T_{\mu\lambda}(\zeta)}T_{\lambda\mu}(\xi)$, where $\xi \in A$, $\zeta \in R$, $\lambda, \mu \in \Lambda$, and $\lambda \neq \mu$ (see [21]). Hence, it remains to show that $Z_{\lambda\mu}(\xi, \zeta)$ belongs to $H = \mathrm{Ep}(n, A)^{\mathrm{E}(\Phi,R)}$.

The proof is contained in Lemmas 13, 14, and 15; in Lemma $12+i$ we consider the case $d(\lambda,\mu)=i$, $i=1, 2, 3$. It is obvious that 3 is the maximal distance between the weights in our representation, and thus it remains to prove the lemmas.

Lemma 13. *Assume that $d(\lambda,\mu)=1$; then ${}^{T_{\mu\lambda}(\zeta)}T_{\rho\sigma}(\xi)\in H$ for all ρ, $\sigma\in\Lambda$. In particular, $Z_{\lambda\mu}(\xi,\zeta)\in H$.*

Proof. First consider the case $\rho=\lambda$ and $\sigma=\mu$. Denote $\mu-\lambda$ by α. We have

$$Z_{\lambda\mu}(\xi,\zeta)={}^{T_{\mu\lambda}(\zeta)}T_{\lambda\mu}(\xi).$$

Recall that in the proof of Theorem 1 we succeeded in examining the following expression: ${}^{x_\alpha(\zeta)}t_{\lambda\mu}(\xi)$. Let us see what we can count on now. In Sec. 10 we showed that the root element $x_\alpha(\zeta)$ can be expressed as a product of six symplectic transvections:

$$x_\alpha(\zeta)=\prod_{i=1}^{6}T_{\rho_i\sigma_i}(\pm\zeta).$$

But only one transvection plays a role when we conjugate by x_α:

$${}^{x_\alpha(\zeta)}T_{\lambda\mu}(\xi)={}^{T_{\mu\lambda}(\pm\zeta)}T_{\lambda\mu}(\xi).$$

The reason is the same as in the proof of Theorem 1: we have a microweight representation, and of all transvections T_{ρ_i,σ_i} in the decomposition of $x_\alpha(\zeta)$ we need to consider only the transvections that interact with the weights $\pm\lambda$ and $\pm\mu$. There is exactly one such transvection $T_{\mu\lambda}(\pm\zeta)$, because the others commute with $T_{\lambda\mu}(\xi)$ by the Chevalley commutator formula. The following necessary result is obtained: by changing the sign of ζ in $x_\alpha(\pm\zeta)$ when needed, the above expression can be made coincident with $Z_{\lambda\mu}(\xi,\zeta)$.

Now consider the general case: $\rho\neq\lambda$ or $\sigma\neq\mu$. Here we must study more cases than in the proof of Theorem 1. First suppose $\rho=\lambda$ and $\sigma\neq\mu$. It can easily be checked that

$$\begin{aligned}&{}^{T_{\mu\lambda}(\zeta)}T_{\lambda,-\lambda}(\xi)=T_{\mu,-\lambda}(\xi\zeta)\cdot T_{\mu,-\mu}(\varepsilon_i\varepsilon_j\xi^2\zeta)\cdot T_{\lambda,-\lambda}(\xi)\in H;\quad\text{and}\\&{}^{T_{\mu\lambda}(\zeta)}T_{\lambda\sigma}(\xi)=T_{\mu\sigma}(\xi\zeta)\cdot T_{\rho\sigma}(\xi)\in H\quad\text{if}\quad\sigma\neq-\lambda.\end{aligned}$$

By the same argument we deal with the case where $\sigma=\mu$ and $\rho\neq\lambda$. We must also consider the cases where $\rho=-\mu$ or $\sigma=-\lambda$. For example, suppose $\sigma=-\lambda$; then if $\rho=\lambda$, we have the case discussed above. On the other hand, if $\rho\neq-\lambda$, then $T_{\rho\sigma}(\xi)$ is a short-root transvection; therefore we can rewrite it as $T_{-\sigma,-\rho}(-\varepsilon_\rho\varepsilon_\sigma\xi)=T_{\lambda,-\rho}(-\varepsilon_\sigma\varepsilon_\rho\xi)$, and this case was also discussed. It is obvious that for the case where $\rho=\mu$ the argument is just the same. In all remaining cases, the symplectic transvections $T_{\mu\lambda}(\zeta)$ and $T_{\rho\sigma}(\xi)$ commute.

Corollary. *Assume that $d(\lambda,\mu)=1$; then $Ep(56,A)^{T_{\mu\lambda}(\zeta)}\le H$.*

Lemma 14. *Assume that $d(\lambda,\mu)=2$; then $Z_{\lambda\mu}(\xi,\zeta)\in H$.*

Proof. We need a closer inspection of the proof of analogous Lemma 2. We shall freely use the notation from that proof. Using simple calculations, we reduced the problem to proving the fact that a certain element g belongs to H. In order to do this, we introduced a helpful element $h\in H$ such that $g={}^{t_{\nu\mu}(1)}h$ and in fact proved that $t_{\lambda\kappa}(-\xi\zeta)t_{\mu\kappa}(-\xi\zeta^2)\cdot{}^{x_\alpha(1)}h=g$. Next we noted that all interesting things occur in the subspace generated by the vectors v^λ, v^ν, v^μ, v^κ, and therefore we were dealing with matrices of order 4. In fact, these direct calculations of matrices can be expressed as some transformations of a product of transvections. Hence, after some fussing around, we can obtain the necessary result using only elementary relations between the transvections and the Chevalley commutator formula. In fact, we perform these calculations in the corresponding Steinberg group. We can write down this result as follows:

$$t_{\lambda\kappa}(-\xi\zeta)t_{\mu\kappa}(-\xi\zeta^2)\cdot{}^{t_{\lambda\kappa}(1)t_{\nu\mu}(1)}h={}^{t_{\nu\mu}(1)}h,$$

where $h=t_{\nu\lambda}(-\zeta)t_{\mu\nu}(-\xi\zeta)t_{\lambda\nu}(-\xi)t_{\nu\lambda}(\zeta)$ (we just substituted the expression of g and the restriction of $x_\alpha(1)$ to the chosen four-dimensional subspace).

Now we begin to use symplectic transvections instead of elementary ones. The key observation is that our proof in fact remains almost the same, because the Chevalley commutator formula holds true if we consider only short-root transvections. Let us rewrite the proof of the above relation, using symplectic transvections instead of elementary ones. As a result, we obtain the formula

$$T_{\lambda\kappa}(-\xi\zeta)T_{\mu\kappa}(-\xi\zeta^2)\cdot {}^{T_{\lambda\kappa}(1)T_{\nu\mu}(1)}h = {}^{T_{\nu\mu}(1)}h,$$

where

$$h = T_{\nu\lambda}(-\zeta)T_{\mu\nu}(-\xi\zeta)T_{\lambda\nu}(-\xi)T_{\nu\lambda}(\zeta).$$

Now recall that we got the product $t_{\lambda\kappa}(1)t_{\nu\mu}(1)$ from the decomposition of the root element $x_\alpha(1)$ into the product of elementary transvections. Similarly, the product $T_{\lambda\kappa}(1)T_{\nu\mu}(1)$ is the restriction of $x_\alpha(1)$ to the 8-dimensional subspace generated by the vectors v^λ, v^ν, v^μ, v^κ, $v^{-\kappa}$, $v^{-\mu}$, $v^{-\nu}$, $v^{-\lambda}$, since this root element is the product of six symplectic transvections, and exactly two of them act in the chosen subspace.

All these actions are legal, because the pairwise distances between the weights κ, λ, μ, and ν are less than or equal to 2. Therefore all symplectic transvections that we obtain during calculations correspond to short roots.

As in the proof of Lemma 2, so far we have considered only specific sign actions of $x_\alpha(1)$. In fact, we have four possible cases, i.e., the restriction of $x_\alpha(1)$ to the chosen subspace has the form $T_{\lambda\kappa}(\pm1)T_{\nu\mu}(\pm1)$. As before, two of the cases can be eliminated by choosing $x_\alpha(-1)$ instead of $x_\alpha(1)$, and the remaining case can be reduced to the case considered by changing the signs in the arguments of the additional transvections. The lemma is proved.

Lemma 15. *Assume that $d(\lambda,\mu)=3$; then $Z_{\lambda\mu}(\xi,\zeta)\in H$.*

Proof. We again use the notation from analogous Lemma 3. As in the proof of the previous lemma, we use symplectic transvections instead of elementary ones. Calculations become more complicated, because now long root transvections come into play. Since $d(\lambda,\mu)=3$, we write $-\lambda$ instead of μ, as stated before. As in the proof of Lemma 3, we use some additional weights: $\nu=-\lambda+\alpha$ and $-\nu=\lambda-\alpha$, where α is a root.

We are going to prove that $Z_{\lambda,-\lambda}(\xi,\zeta) = {}^{T_{-\lambda,\lambda}(\zeta)}T_{\lambda,-\lambda}(\xi)\in H$, where $\zeta\in R$ and $\xi\in A$. First note that since 2 is invertible, we can substitute 2ξ for ξ and get $[T_{\lambda\nu}(\xi),T_{\nu,-\lambda}(1)]$ instead of $T_{\lambda,-\lambda}(2\xi)$. Therefore,

$$\begin{aligned}Z_{\lambda,-\lambda}(\xi,\zeta) &= {}^{T_{-\lambda,\lambda}(\zeta)}[T_{\lambda\nu}(\xi),T_{\nu,-\lambda}(1)][[T_{-\lambda,\lambda}(\zeta),T_{\lambda\nu}(\xi)]T_{\lambda\nu}(\xi),[T_{-\lambda,\lambda}(\zeta),T_{\nu,-\lambda}(1)]T_{\nu,-\lambda}(1)]\\ &= [T_{-\lambda,\nu}(\xi\zeta)T_{-\nu,\nu}(-\varepsilon_i\varepsilon_j\xi^2\zeta)T_{\lambda\nu}(\xi),T_{\nu,\lambda}(-\zeta)T_{\nu,-\nu}(\varepsilon_i\varepsilon_j\zeta)T_{\nu,-\lambda}(1)].\end{aligned}$$

It suffices to show that

$$g = {}^{T_{\nu,\lambda}(-\zeta)T_{\nu,-\nu}(\varepsilon_i\varepsilon_j\zeta)T_{\nu,-\lambda}(1)}T_{-\lambda,\nu}(-\xi\zeta)T_{-\nu,\nu}(\varepsilon_i\varepsilon_j\xi^2\zeta)T_{\lambda\nu}(-\xi)\in H.$$

Let $f=T_{\nu,\lambda}(-\zeta)T_{\nu,-\nu}(\varepsilon_i\varepsilon_j\zeta)$. We have

$$\begin{aligned}g &= fT_{\nu,-\lambda}(1)T_{-\lambda,\nu}(-\xi\zeta)T_{-\nu,\nu}(\varepsilon_i\varepsilon_j\xi^2\zeta)T_{\lambda\nu}(-\xi)T_{\nu,-\lambda}(-1)f^{-1}\\ &= fT_{\nu,-\lambda}(1)T_{-\lambda,\nu}(-\xi\zeta)T_{-\nu,\nu}(\varepsilon_i\varepsilon_j\xi^2\zeta)T_{\nu,-\lambda}(-1)T_{\lambda,-\lambda}(2\xi)T_{\lambda\nu}(-\xi)f^{-1}\\ &= fT_{\nu,-\lambda}(1)T_{-\lambda,\nu}(-\xi\zeta)T_{\lambda\nu}(\xi^2\zeta)T_{\lambda,-\lambda}(\xi^2\zeta+2\xi),\\ &T_{\nu,-\lambda}(-1)T_{-\nu,\nu}(\varepsilon_i\varepsilon_j\xi^2\zeta)T_{\lambda\nu}(-\xi)f^{-1} = fT_{\nu,-\lambda}(1)T_{-\lambda,\nu}(-\xi\zeta)T_{\lambda,-\lambda}(-2\xi^2\zeta)T_{\nu,-\lambda}(-1),\\ &T_{\lambda\nu}(\xi^2\zeta)T_{\lambda,-\lambda}(\xi^2\zeta+2\xi)T_{-\nu,\nu}(\varepsilon_i\varepsilon_j\xi^2\zeta)T_{\lambda\nu}(-\xi)f^{-1} = fZ_{-\lambda,\nu}(\xi^2\zeta,1)T_{\lambda\nu}(\xi^2\zeta)T_{\lambda,-\lambda}(-\xi^2\zeta+2\xi),\\ &T_{-\nu,\nu}(\varepsilon_i\varepsilon_j\xi^2\zeta)T_{\lambda\nu}(-\xi)f^{-1} = {}^fh,\end{aligned}$$

where

$$h = Z_{-\lambda,\nu}(\xi^2\zeta,1)T_{\lambda\nu}(\xi^2\zeta)T_{\lambda,-\lambda}(-\xi^2\zeta+2\xi)T_{-\nu,\nu}(\varepsilon_i\varepsilon_j\xi^2\zeta)T_{\lambda\nu}(-\xi).$$

Note that $h\in H$ (the distance between the weights $-\lambda$ and ν is equal to 1), and therefore $Z_{-\lambda,\nu}(\xi^2\zeta,1)\in H$). Moreover, $f=T_{\nu\lambda}(-\zeta)[T_{\nu\lambda}(\varepsilon_i\varepsilon_j\zeta),T_{\lambda,-\nu}(1)]$. Now look at $T_{\nu\lambda}(\dots)$: since $d(\nu,\lambda)=2$, we can find a weight τ such that $d(\nu,\tau)=d(\tau,\lambda)=1$. Thus $T_{\nu\lambda}(x)=[T_{\nu\tau}(x),T_{\tau\lambda}(1)]$. We now have an expression for f that uses only symplectic transvections $T_{\rho\sigma}(\dots)$, where $\rho-\sigma$ is a root. Now we again use our trick, restricting the computations to a four-dimensional subspace. We claim that conjugation by a symplectic transvection from the expression of f is the same as conjugation by a *root element* whose decomposition contains this transvection. That is, if we write $x_{\rho-\sigma}(\pm x)$ instead of $T_{\rho\sigma}(x)$ in the decomposition of f, then the conjugation of h by f remains the same. The rule for choosing a sign in this root element is simple: the decomposition of the root element into symplectic transvections should contain $T_{\rho\sigma}(x)$ but not $T_{\rho\sigma}(-x)$. First let us see what happens to $T_{\nu\lambda}(x)$. We replaced this transvection by $[T_{\nu\tau}(x), T_{\tau\lambda}(1)]$ and then by $[x_{\nu-\tau}(\pm x), x_{\tau-\lambda}(\pm 1)]$. Now we replace each of these root elements by the product of six symplectic transvections and consider the subspace spanned by the weight vectors v^ν, v^τ, v^λ, v^κ, $v^{-\nu}$, $v^{-\tau}$, $v^{-\lambda}$, $v^{-\kappa}$ (here $\kappa = \lambda + (\nu - \tau)$). It is clear that only the symplectic transvections that act inside this subspace affect each other, and the other transvections commute with these ones and with themselves. The argument is the same as in the proof of Theorem 1, where we used this trick repeatedly. We derive that $T_{\nu\lambda}(x)$ is a commutator of two root elements, and thus it belongs to $\mathrm{E}(\mathrm{E}_7, R)$.

Unfortunately, the decomposition of f contains the transvection $T_{\lambda,-\nu}(1)$ that does not belong to $\mathrm{E}(\mathrm{E}_7, R)$. But the final part of our calculations uses the conjugation by f, and thus we can use our trick again. Now we restrict ourselves to the four-dimensional subspace W spanned by the weight vectors v^λ, v^ν, $v^{-\lambda}$, and $v^{-\nu}$. Arguing as above, we can show that conjugation by $T_{\lambda,-\nu}(\dots)$ is the same as conjugation by $x_{\lambda-(-\nu)}(\dots)$ if the object that we conjugate acts inside the subspace W (which is the case, because all other parts of g behave in this way).

We have just proved that the conjugation of h by f can be expressed as a sequence of conjugations by elements of $\mathrm{E}(\mathrm{E}_7, R)$. Thus we remain in the group H, and $g = {}^f h \in H$. The proof is complete.

12. Proof of Theorem 6

Let

$$E = \mathrm{EE}_7'(56, R, A, B) = \mathrm{E}(\mathrm{E}_7, R)\mathrm{E}(56, R, A)\,\mathrm{Ep}(56, R, B).$$

Taking Theorem 4 into account, it remains to show the inclusion $\mathrm{Ep}(56, R, B) \subset [E, E]$. We prove that the generators of $\mathrm{Ep}(56, R, B)$ belong to $[E, E]$. Take $x = Z_{\lambda\mu}(\xi, \zeta) = {}^{T_{\mu\lambda}(\zeta)}T_{\lambda\mu}(\xi)$, where $\xi \in B$ and $\zeta \in R$. Using Theorem 1, we get $x \in \mathrm{Ep}(56, A)^{\mathrm{E}(\mathrm{E}_7, R)}$. Therefore x can be expressed as a product

$$x = \prod_i x_i y_i x_i^{-1}, \qquad \text{where } x_i \in \mathrm{E}(\mathrm{E}_7, R) \quad \text{and} \quad y_i \in \mathrm{E}(56, B) \subset \mathrm{E}(56, R, B).$$

Thus $x = \prod_i [x_i, y_i] y_i$ and for any i the commutator $[x_i, y_i]$ belongs to $[E, E]$. It remains to prove that $\mathrm{E}(56, B) \subset [E, E]$.

Take a short root transvection $T_{\rho\sigma}(\xi) \in \mathrm{E}(56, B)$ (here $\rho \neq \pm\sigma$). We try to find a root $\alpha \in \mathrm{E}_7$ in order to use the Chevalley commutator formula

$$T_{\rho\sigma}(\xi) = [T_{\rho,\rho-\alpha}(1), T_{\rho-\alpha,\sigma}(\xi)],$$

and then we write $x_\alpha(\pm 1)$ instead of $T_{\rho,\rho-\alpha}(1)$:

$$T_{\rho\sigma}(\xi) = [x_\alpha(\pm 1), T_{\rho-\alpha,\sigma}(\xi)] \in [\mathrm{E}(\mathrm{E}_7, R), \mathrm{E}(56, B)] \subset [E, E].$$

In order to use this case of the Chevalley commutator formula, it is necessary that $-\rho \neq \rho - \alpha$ (this is the case, because the distance between the opposite weights is always equal to 3 and α is a root), $\rho \neq \pm\sigma$ (it was one of the initial conditions), and $\rho - \alpha \neq \pm\sigma$.

The second step, the substitution of $x_\alpha(1)$ for $T_{\rho,\rho-\alpha}(1)$, is possible if the remaining transvections in the decomposition of $x_\alpha(\pm 1)$ do not affect the commutation with $T_{\rho-\alpha,\sigma}(\xi)$. Thus, $\sigma - \alpha$ should not be a weight.

We may assume that $\sigma = \omega$ is a highest weight (and then we reduce any other case to this one by the action of the Weyl group). Consider the case $\rho \neq \sigma - \alpha_7$. Let α be a negative fundamental root such that $\rho - \alpha$ is not a weight (such an α exists, because ρ is not a highest weight). It is obvious that $\rho - \alpha \neq \pm\sigma$ and $\sigma - \alpha$ is not a weight, and thus all the conditions for applying the Chevalley commutator formula are fulfilled. On the other hand, if $\rho = \sigma - \alpha_7$, we take $\alpha = \alpha_6$, and it is easy to check necessary conditions.

Now consider a long-root transvection $T_{\rho,-\rho}(\xi)$ ($\xi \in B$). Let us take an arbitrary weight σ such that $\sigma - (-\rho) = \alpha$ is a root. We have $T_{\rho,-\rho}(\xi) = [T_{\rho\sigma}(\xi), T_{\sigma,-\rho}(2^{-1})] = [T_{\rho\sigma}(\xi), x_\alpha(\pm 2^{-1})] \in [\mathrm{E}(56, B), \mathrm{E}(\mathrm{E}_7, R)] \subset [E, E]$ (this can easily be verified, arguing as above). The proof is complete.

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