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k – field, $\operatorname{char} k \neq 2$ $\quad pt = \operatorname{Spec} k$

$$CH^*(\mathbb{P}^n) = CH^*(pt)[t]/t^{n+1}$$
$$[H] \leftrightarrow t$$

$$K_*(\mathbb{P}^n) \simeq K_*(pt)[t]/t^{n+1}$$
$$1-[\mathcal{O}(-1)] \leftrightarrow t$$

$\boxed{\text{Thm}}$ X – smooth; $E \to X$ – vector bundle, rank $E = n+1$

$$\rightsquigarrow K_*(\mathbb{P}(E)) \simeq K_*(X) \oplus K_*(X)t \oplus \dots \oplus K_*(X)t^n$$
$$1-[\mathcal{O}(-1)] \leftrightarrow t$$

$\rightsquigarrow$ Chern classes $K_*(Gr(n,m))$

Some $CH^*(\mathbb{P}(E))$

$\boxed{\text{Def}}$ X – smooth

$$W^{4n}(X) = \bigoplus \mathbb{Z}[E, \varphi : E \xrightarrow{\sim} E^\vee] \Big/ \begin{matrix}[E_1+E_2, \varphi_1 \oplus \varphi_2] = [E_1,\varphi_1] \oplus [E_2,\varphi_2] \\ [E,\varphi] \text{ for } (E,\varphi) - \text{metabolic}\end{matrix}$$

symmetric

here $(-)^\vee = \mathrm{Hom}(-, \mathcal{O}_X)$.

$W^{4n+2}(X) =$ — // — // — // — (φ – skew-symmetric)

$W^{2n+1}(X)$ – uses derived categories

$\boxed{\text{Thm}}$ (Walter)

$$W^*(\mathbb{P}^{2n}) \simeq W^*(pt)$$
$$W^*(\mathbb{P}^{2n+1}) \simeq W^*(pt) \oplus W^{*-(2n+1)}(pt)$$

$\boxed{\text{Thm}}$ (Balmer, Calmès):

$W^*(Gr(n,m))$

- the answer is not canonical (it depends on the flag $V_0 < V_1 < \dots < V_n$
- the answer involves twisted Witt groups

$\boxed{\text{Question}}$ What one should use instead of the projective space?

general context: „oriented cohomology theories"
(Levine, Morel, Panin, Smirnov)

– axiomatic approach to cohomology theory:

„ring cohomology theory" : functor $A^*(-)$ s.t.

$\forall Z \hookrightarrow X$ closed (smooth) $\rightsquigarrow A^*_Z(X)$ ← cohomology supported on Z

- localization
- cup product
- excision
- $\mathbb{A}^1$-homotopy invariance

„orientation":

- is a rule that assigns to every vector bundle $E \to X$ a class (Thom class) $th(E) \in A^*_X E$ such that $-\cup th(E) : A^*(X) \simeq A^*_X(E)$ and ...

$\boxed{\text{Thm}}$ $A^*(-)$-oriented

$$A^*(\mathbb{P}^n) \underset{\text{natural}}{\overset{!!}{\simeq}} A^*(pt)[t]/t^{n+1}$$

$\boxed{\text{Example}}$ K_* $E \to X$

$\rightsquigarrow th(E) = 1 - [E^\vee] + [\Lambda^2 E^\vee] - \dots$ (koszul complex)

$W^*(-)$ are not oriented

„symplectic orientation"

- same as orientation, but use (E,φ) - symplectic

Def $HP^n = Sp_{2n+2}/Sp_2 \times Sp_{2n}$ - quaternionic proj. space

$\boxed{\text{Thm}}$ (Walter, Panin) A^* symplectic oriented $\Rightarrow A^*(HP^n) \simeq A^*(pt)[t]/t^{n+1}$

W^* - symplectically oriented

Def $(E, \lambda : \det E \xrightarrow{\sim} \mathcal{O}_X)$ - special linear bundle

$\rightsquigarrow \Lambda^{n-k} E^\vee \underset{\text{canonical}}{\simeq} \Lambda^k E^\vee$ for special linear

„SL orientation"

- same with (E,λ) — SL-vector bundle

oriented	$\mathbb{P}^n \overset{\mathbb{A}^1}{\sim} GL_{n+1}/GL_1 \times GL_n$
Sp-oriented	$H\mathbb{P}^n = Sp_{2n+2}/Sp_2 \times Sp_{2n}$
SL-oriented	$\mathbb{A}^{n+1} - \{0\} \overset{\mathbb{A}^1}{\sim} SL_{n+1}/SL_1 \times SL_n$?
	$SL_{n+2}/SL_2 \times SL_n$?

$$SL_{n+2}/SL_2\times SL_n$$

$$\Big\downarrow \mathbb{A}^1$$

$$\mathbb{G}_m \longrightarrow SL_{n+2}\Big/\left(\begin{array}{c|c} SL_2 & * \\ \hline 0 & SL_n \end{array}\right) \longrightarrow Gr(2, n+2)$$

$$W^*(\mathbb{A}^{n+1}\setminus\{0\}) \simeq W^*(pt) \oplus W^{*-n}(pt)$$

$$\begin{tikzcd}
\mathcal{O}(-1) \arrow[r, tail] \arrow[dr] & \mathcal{O}^{n+1} \arrow[r] \arrow[d] & \tau_n \arrow[dl] \arrow[dr, dashed] & \\
& \mathbb{A}^{n+1}\setminus\{0\} \arrow[drr] & & \tau'_n \arrow[d] \\
& & & \mathbb{P}^n
\end{tikzcd}$$

has SL structure

pullback

τ'_n – tangent bundle

$$\begin{array}{ccc} th(E,\lambda) \in A^*_X(E) & \to & A^*(E) \\ & & \wr\!\!\downarrow \\ e \in A^*(X) & & \end{array}$$

Euler class

$\boxed{\text{Thm}}$ $W^*(\mathbb{A}^{2n+1}-\{0\}) \simeq W^*(pt)[e]/e^2$

$\boxed{\text{Thm}}$ ① $W^*(SL_{2n+1}/SL_2\times SL_{2n-1}) \simeq W^*(pt)[e]/e^{2n}$

Euler class coming from SL_2

② $W^*(SL_{2n}/SL_2\times SL_{2n-2}) \simeq W^*(pt)[e_1, e_2]/(e_1e_2,\ e_1^{2n-2}+(-1)^n e_2^2)$

from SL_2 | from SL_{2n-2}

$\boxed{\text{Thm}}$ ① $W^*(SL_{2n+1}/SL_2^n) \simeq W^*(pt)[e_1, e_2, \dots, e_n]/(\sigma_i(e_1^2, e_2^2, \dots, e_n^2))$

② $W^*(SL_{2n}/SL_2^n) \simeq W^*(pt)[e_1, e_2, \dots, e_n]/(\sigma_i(e_1^2, e_2^2, \dots, e_n^2), e_1e_2\dots e_n)$

$K=\mathbb{R}$

AG	topology
SL_n	$SL_n(\mathbb{R}) \simeq SO_n(\mathbb{R})$
SL-orientation	orientation
Euler class	Euler class $H^*(-,\mathbb{Z})$
SL_2	$SO_2(\mathbb{R}) \simeq S^2$
SL_{2n}/SL_2^n	SO_{2n}/T
$W^*(SL_{2n}/SL_2^n)$	$H^*(SO_{2n}/T, \mathbb{Z}[1/2])$
	$H^*(G/T, \mathbb{Q}) \simeq \mathbb{Q}[x_1, \dots, x_n]_{W(G)}$