

K -field, $\text{char } K \neq 2$ $pt = \text{Spec } K$

$$CH^*(\mathbb{P}^n) = CH^*(pt)[t] / t^{n+1}$$

$$[H] \mapsto t$$

$$K_*(\mathbb{P}^n) \cong K_*(pt)[t] / t^{n+1}$$

$$1 - [\alpha - 1] \mapsto t$$

Thm X -smooth; $E \rightarrow X$ -vector bundle, $\text{rank } E = n+1$

$$\leadsto K_*(\mathbb{P}(E)) \cong K_*(X) \oplus K_*(X) t \oplus \dots \oplus K_*(X) t^n$$

$$1 - [\alpha - 1] \mapsto t$$

\leadsto Chern classes $K_*(Gr(n, m))$

Some $CH^*(\mathbb{P}(E))$

Def X -smooth

$$W^{2n}(X) = \bigoplus \mathbb{Z} [E, \varphi: E \xrightarrow{\text{symmetric}} E^{\vee}] / [E_1 + E_2, \varphi_1 \oplus \varphi_2] [E_1, \varphi_1] \oplus [E_2, \varphi_2]$$

here $(-)^{\vee} = \text{Hom}(-, \mathcal{O}_X)$.

$$W^{2n+2}(X) = \dots // \dots // \dots // \dots$$

(φ - skew-symmetric)

$W^{2n+2}(X)$ - uses derived categories

Thm (Walter)

$$W^*(\mathbb{P}^{2n}) \cong W^*(pt)$$

$$W^*(\mathbb{P}^{2n+1}) \cong W^*(pt) \oplus W^{*- (2n+1)}(pt)$$

Thm (Balmer, Caenès):

- $W^*(Gr(n, m))$
- the answer is not canonical (it depends on the flag $V_0 \subset V_1 \subset \dots \subset V_n$)
 - the answer involves twisted Witt groups

Question What one should use instead of the projective space?

general context: "oriented cohomology theories"
(Levine, Morel, Panin, Smirnov)

-axiomatic approach to cohomology theory:

"ring cohomology theory" : functor $A^*(-)$ s.t.

$\forall Z \hookrightarrow X$ closed $\xrightarrow{\text{smooth}} A_Z^*(X) \longleftarrow$ cohomology supported on Z

- localization
- cup product
- excision
- A^1 -homotopy invariance

"orientation":

• is a rule that assigns to every vector bundle $E \rightarrow X$ a class (Thom class) $th(E) \in A_X^*(E)$ such that

$\cup th(E) : A^*(X) \simeq A_X^*(E)$ and ...

Thm $A^*(-)$ -oriented / **Example** $K_* \quad E \rightarrow X$

$$A^*(\mathbb{P}^n) \xrightarrow{\text{natural}} A^*(pt)[t] / t^{n+1}$$

$$\rightarrow th(E) = 1 - [E^v] + [\Lambda^2 E^v] - \dots$$

(Koszul complex)

$W^*(-)$ are not oriented

"symplectic orientation"

- same as orientation, but use (E, φ) - symplectic

Def $HP^n = Sp_{2n+2} / Sp_2 \times Sp_{2n}$ - quaternionic proj. space

Thm A^* symplectic oriented $\Rightarrow A^*(HP^n) \simeq A^*(pt)[t] / t^{n+1}$

(Walter, Panin)

W^* - symplectically oriented

Def $(E, \lambda: \det E \xrightarrow{\sim} \mathcal{O}_X)$ - special linear bundle

$\rightarrow \Lambda^{n-k} E^v \simeq \Lambda^k E^v$ for special linear

\uparrow canonical

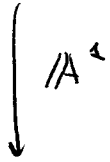
"SL orientation"

- same with (E, λ) - SL-vector bundle

- oriented
- Sp -oriented
- SL -oriented

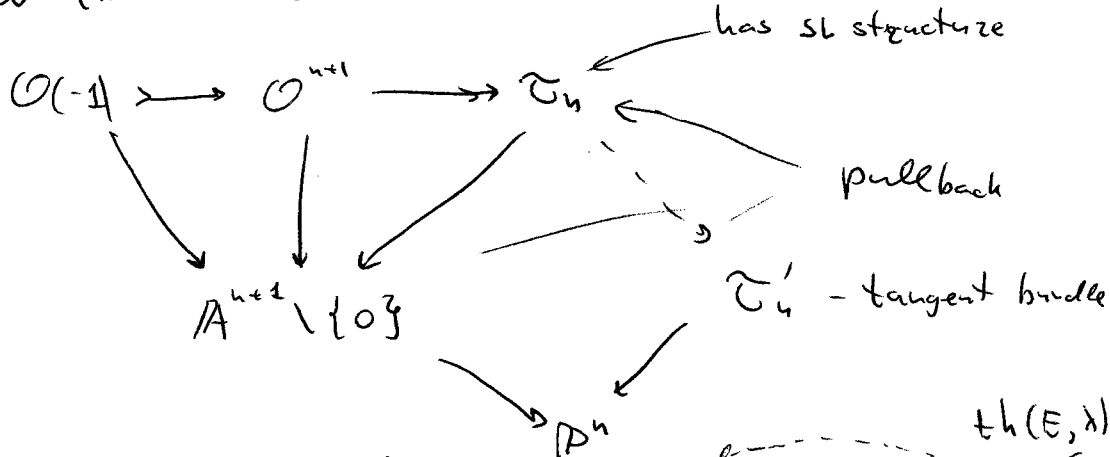
$$\begin{aligned}
 \mathbb{P}^n &\xrightarrow{A^1} GL_{n+1} / GL_1 \times GL_n \\
 HP^n &= Sp_{2n+2} / Sp_2 \times Sp_{2n} \\
 A^{n+1} - \{0\} &\xrightarrow{A^1} SL_{n+1} / SL_1 \times SL_n ? \\
 &SL_{n+2} / SL_2 \times SL_n ?
 \end{aligned}$$

$$SL_{n+2} / SL_2 \times SL_n$$



$$G_m \rightarrow SL_{n+2} / \left(\begin{array}{c|c} SL_2 & * \\ \hline 0 & SL_n \end{array} \right) \rightarrow Gr(2, n+2)$$

$$W^*(\mathbb{A}^{n+1} \setminus \{0\}) \simeq W^*(pt) \oplus W^{*-2n}(pt)$$



Thm $W^*(\mathbb{A}^{2n+1} \setminus \{0\}) \simeq W^*(pt)[e]/e^2$

Thm ① $W^*(SL_{2n+1} / SL_2 \times SL_{2n-1}) \simeq W^*(pt)[e]/e^{2n}$

② $W^*(SL_{2n} / SL_2 \times SL_{2n-2}) \simeq W^*(pt)[e_1, e_2] / (e_1 e_2, e_1^{2n-2} + (-1)^n e_2^2)$

from SL_2
from SL_{2n-2}

Thm ① $W^*(SL_{2n+1} / SL_2^n) \simeq W^*(pt)[e_1, e_2, \dots, e_n] / (\sigma_i(e_1^2, e_2^2, \dots, e_n^2))$

② $W^*(SL_{2n} / SL_2^n) \simeq W^*(pt)[e_1, e_2, \dots, e_n] / (\sigma_i(e_1^2, e_2^2, \dots, e_n^2), e_1 e_2 \dots e_n)$

$K = \mathbb{R}$

AG
 SL_n
 SL -orientation
 Euler class
 SL_2
 SL_{2n}/SL_2^n
 $W^*(SL_{2n}/SL_2^n)$

Topology
 $SL_n(\mathbb{R}) \cong SO_n(\mathbb{R})$
 orientation
 Euler class $H^*(-, \mathbb{Z})$
 $SO_2(\mathbb{R}) \simeq S^2$
 SO_{2n}/\mathbb{T}
 $H^*(SO_{2n}/\mathbb{T}, \mathbb{Z}[1/2])$
 $H^*(G/\mathbb{T}, \mathbb{Q}) \simeq \mathbb{Q}[x_1, \dots, x_n]_{W(G)}$