Contemporary Mathematics
Volume 67, 1987

p- ADIC PERIODS AND p- ADIC ETALE COHOMOLOGY

Jean-Marc Fontaine and William Messing[1]

Introduction

Classically, the relation between the Betti and de Rham cohomology of a projective and smooth variety defined over a number field is expressed in terms of the periods and via Hodge theory. We shall present evidence indicating that, in a precise sense, there is an analogous relation between étale and de Rham cohomology.

Let E be a number field, $\bar{E}$ an algebraic closure of E, and X be a proper smooth variety defined over E. Fix a prime number p and $\mathfrak{p}$ (resp. $\bar{\mathfrak{p}}$) a place of E (resp. $\bar{E}$) lying over p (resp. $\mathfrak{p}$). Let $V = H^m_{et}(X_{\bar{E}}, \mathbf{Q}_p)$ viewed as a representation of the decomposition group associated to $\bar{\mathfrak{p}}$ and $D = H^m_{DR}(X) \otimes E_{\mathfrak{p}}$ endowed with its Hodge filtration. In [31], Tate proved that, if m = 1 and X has good reduction at $\mathfrak{p}$, then setting $C = \hat{\bar{E}}_{\mathfrak{p}}$, $C \otimes V$ admits a Hodge-Tate decomposition; and conjectured that this holds for all m without the good reduction hypothesis. Faltings has recently announced a proof of this conjecture [9].

Changing notation, we are led to consider K a characteristic zero, non-archimedian local field of residue characteristic p and X a proper, smooth variety defined over K. We continue to denote by V (resp. D) the associated p-adic étale (resp. de Rham) cohomology of X. Using a basic result of Tate [31] and Dieudonné theory, Grothendieck proved in [21] that, if X has good reduction, then, for m = 1, V viewed as a representation of Gal ($\bar{K}$/K) determines D endowed with its Hodge filtration and its "Frobenius structure" (coming from crystalline cohomology) and conversely. This mutual determination was not direct or explicit (but rather used the intermediary of the p-divisible group associated to the Albanese variety of X) and Grothendieck raised the problem of finding an explicit recipe for passing between V and D as well as the problem of obtaining similar results for $m \geq 2$. This is his problem of the *mysterious functor.* The case m = 1 was resolved by one of us in [11,14] and a conjectural recipe in the case of m ≥ 2 was also given in [14].

1980 Mathematics Subject Classification. Primary 14F, 14G, 12B.
[1] Partially supported by grants NSF-DMS-82-01120 and NSF-DMS-85-01729.

We summarize here the progress that has been made towards establishing the "crystalline conjecture" of [14] and, in addition, indicate some applications. Intuitively the idea is that, just as p- divisible groups serve as the bridge connecting V and D when m = 1, an appropriate "generalized p- divisible group" should serve to connect them for m ≥ 2 as well. In the following this intuition is not made explicit but rather we use the concepts and techniques which have developed from it.

The paper is divided into three sections. In the first, we state our main results and give both arithmetic and geometric applications. In the second, we introduce the syntomic topology and explain how to calculate crystalline cohomology using it, as well as give an essentially complete proof of the main part of Theorem A (of part I). In the third, we define certain sheaves S_n^r for the syntomic (or syntomic-étale topology) which intuitively can be thought of as an "intelligent version" of $\mathrm{Sym}^k \mu_{p^n}$ and give an outline of how our main result, (theorem B of part I) is proved.[2]

I. Crystalline Representations and the Construction of p-Adic Etale Cohomology

Throughout this paper, p is a fixed prime number, k a perfect field of characteristic p, W its ring of Witt vectors, K the fraction of field W, σ the absolute Frobenius of K ($\sigma(x) = x^p$) as well as (abusively) the induced automorphism of W (resp. K). Let $\overline{K}$ be an algebraic closure of K, G = Gal ($\overline{K}$/K), I the inertia subgroup, $\mathbf{Q_p}(1)$ the one dimensional $\mathbf{Q_p}$-vector space $\mathbf{Q_p} \otimes_{\mathbf{Z_p}} \mathrm{Hom}(\mathbf{Q_p}/\mathbf{Z_p}, \overline{K}^*)$, χ the cylotomic character giving the action of G on $\mathbf{Q_p}(1)$.

1. Crystalline Representations

1.1. A *p-adic representation* is a topological $\mathbf{Q_p}$-vector space equipped with a continuous linear action of G. The *dimension* of the representation is the dimension of the underlying space.

A *filtered module* is a K-vector space D equipped with

i) an exhaustive, separated decreasing filtration indexed by **Z** (i.e. a family of K-subspaces $\mathrm{Fil}^r D$ satisfying $\mathrm{Fil}^r D \supset \mathrm{Fil}^{r+1} D$, $\cup \mathrm{Fil}^r D = D$, $\cap \mathrm{Fil}^r D = 0$),

ii) a Frobenius, i.e. an injective σ-semilinear endomorphism

$$\phi: D \to D \quad .$$

[2]We assume that $p \neq 2$ in the third section. In terms of the "main case" of theorem B, namely when $p > \dim(X)$, this is not a restricition, because for H^1's the result is known.

The *dimension* of the filtered module is the dimension of the underlying K-space. With the obvious definition of morphism, the filtered modules form a category which is additive (but not abelian), $\mathbf{Q}_p$-linear, has direct sums, tensor products, kernels, cokernels and in which the notion of short exact sequence can be defined. If D is a finite dimensional filtered module, then the dual vector space is endowed with a natural structure of filtered module, denoted $D^{\vee}$.

1.2. EXAMPLE : Let X be a proper, smooth variety defined over K and let $\overline{X} = X \otimes \overline{K}$; for each $m \in \mathbf{N}$, $H^m_{ét}(\overline{X}, \mathbf{Q}_p)$ is a finite dimensional *p*- adic representation. We denote the direct sum of these spaces by $H_{et}(X)$ and (for brevity) the m^{th} summand by $H^m_{et}(X)$.

We say that X has *good reduction* provided there exists a proper smooth W-scheme $\mathcal{X}$ such that $\mathcal{X} \otimes K = X$. If $\mathcal{X}_k$ is the special fiber of $\mathcal{X}$, then $K \underset{W}{\otimes} H^m_{cris}(\mathcal{X}_k/W)$ is endowed with a (bijective) Frobenius and is canonically isomorphic to $H^m_{DR}(X/K)$, and thus is endowed, by transport of structure, with a filtration, the Hodge filtration. Thus, we obtain a filtered module, $H^m_{cris}(X)$, which by [19], is independent of the choice of $\mathcal{X}$. As above, we denote the direct sum of these by $H_{cris}(X)$.

1.3. For any commutative ring A, let $A_n = A/p^nA$ Let $\mathcal{O}_{\overline{K}}$ be the ring of integers of $\overline{K}$ and $\tilde{\mathcal{O}}_{\overline{K}}$ be $\mathcal{O}_{\overline{K},1}$. The ring, $W_n(\tilde{\mathcal{O}}_{\overline{K}})$, of Witt vectors of length n with coefficients in $\tilde{\mathcal{O}}_{\overline{K}}$ is endowed with an action of G and a Frobenius. We view $W_n(\tilde{\mathcal{O}}_{\overline{K}})$ as a W_n -algebra via the composite

$$W_n \xrightarrow{\sigma^{-n}} W_n \longrightarrow W_n(\tilde{\mathcal{O}}_{\overline{K}}) .$$

We define a surjective homomorphism of W_n -algebras

$$\theta_n\colon W_n(\tilde{\mathcal{O}}_{\overline{K}}) \longrightarrow \mathcal{O}_{\overline{K},n}$$

by $\theta_n(a_o, \dots, a_{n-1}) = \hat{a}_o^{p^n} + \dots + p^{n-1}\hat{a}_{n-1}^{p}$

(where $\hat{a}_j \in \mathcal{O}_{\overline{K},n}$ is any lifting of a_j). Note θ_n commutes with the action of G.

Denote by $W_n^{DP}(\tilde{\mathcal{O}}_{\overline{K}})$ the *divided power envelope* of the ideal Ker (θ_n) (compatible with the natural divided powers on (p)) and by $J_n(\mathcal{O}_{\overline{K}})$ the corresponding divided power ideal. Thus we have an exact sequence

$$0 \longrightarrow J_n(\mathcal{O}_{\overline{K}}) \longrightarrow W_n^{DP}(\tilde{\mathcal{O}}_{\overline{K}}) \longrightarrow \mathcal{O}_{\overline{K},n} \longrightarrow 0 .$$

By functoriality G acts on this sequence and since $\phi(\mathrm{Ker}(\theta_n)) \subset \mathrm{Ker}(\theta_n) + p\cdot W_n(\tilde{\mathcal{O}}_{\overline{K}})$, it follows that ϕ extends to $W_n^{DP}(\tilde{\mathcal{O}}_{\overline{K}})$.

1.4. For any $n \geq 1$ we have a commutative diagram of W-algebras

$$\begin{array}{ccc} W_{n+1}(\mathcal{O}_{\bar K}) & \xrightarrow{\theta_{n+1}} & \mathcal{O}_{\bar K, n+1} \\ \upsilon_n \big\downarrow & & \big\downarrow \\ W_n(\mathcal{O}_{\bar K}) & \xrightarrow[\theta_n]{} & \mathcal{O}_{\bar K, n} \end{array}$$

where the right vertical arrow is reduction modulo p^n and $\upsilon_n(a_o, \ldots, a_n) = (a_o^p, \ldots, a_{n-1}^p)$. Note υ_n extends to the divided power envelopes.

We define $B^+_{cris} = K \underset{W}{\otimes} \varprojlim W_n^{DP}(\mathcal{O}_{\bar K})$; this is a topological K-algebra endowed with a continuous action of G and the structure of a filtered module. The filtration is defined by

$$\mathrm{Fil}^r B^+_{cris} = \begin{cases} B^+_{cris} & \text{if } r \le 0 \\ \text{the image of } K \underset{W}{\otimes} \varprojlim J_n^{[r]}(\mathcal{O}_{\bar K}) & \text{if } r > 0, \end{cases}$$

where $J_n^{[r]}(\mathcal{O}_{\bar K})$ denotes the r^{th} divided power of $J_n(\mathcal{O}_{\bar K})$.

1.5. For any $n \ge 1$, if $\varepsilon \in \mu_{p^n}(\bar K)$ and $\tilde\varepsilon$ is its image in $\tilde{\mathcal{O}}_{\bar K}$, the element $[\tilde\varepsilon] = (\tilde\varepsilon, 0, \ldots, 0) \in W_n(\mathcal{O}_{\bar K})$ belongs to $1 + J_n(\mathcal{O}_{\bar K})$ and hence $\log([\tilde\varepsilon])$ is an element of $W_n^{DP}(\tilde{\mathcal{O}}_{\bar K})$. This construction defines a homomorphism $\mu_{p^n}(\bar K) \longrightarrow W_n^{DP}(\mathcal{O}_{\bar K})$ and passing to the limit, we obtain an embedding $\mathbf{Q}_p(1) \longrightarrow B^+_{cris}$. We view $\mathbf{Q}_p(1)$ as included in B^+_{cris} via this map (which is G compatible) and, if t denotes any non-zero element of $\mathbf{Q}_p(1)$, we define B_{cris} to be $B^+_{cris}[t^{-1}]$. G acts on B_{cris}, $\phi(t) = p^{-1} \cdot t^{-1}$. Finally, we extend the definition of the filtration to B_{cris} by setting

$$\mathrm{Fil}^r B_{cris} = \bigcup_{s \ge 0} t^{-s} \mathrm{Fil}^{r+s} B^+_{cris} .$$

1.6. Let D be a finite dimensional filtered module. We say D is *weakly admissible* if there is a lattice M in D such that $\Sigma p^{-r} \phi(M \cap \mathrm{Fil}^r D) = M$. The category of weakly admissible modules is stable under sub and quotient objects, extensions, tensor products and taking duals; it is abelian and in fact a $\mathbf{Q}_p$-linear, neutral tannakian category (cf. [12,24]).

If D is a weakly admissible filtered module, we set

$$V(D) = \{ x \in \mathrm{Fil}^0(B_{cris} \underset{K}{\otimes} D) \mid \phi(x) = x \} .$$

This is a p-adic representation of dimension at most that of D. We say D is *admissible* provided these dimensions are equal. Conjecturally, all weakly admissible modules are admissible.

If V is a p-adic representation, we set

$$D(V) = (B_{cris} \otimes_{\mathbb{Q}_p} V)^G .$$

This is a filtered module of dimension at most that of V. We say V is a *crystalline representation* if these dimensions are equal.

1.7. PROPOSITION ([12,13,16]): *The category of admissible filtered modules (viewed as a full subcategory of the category of weakly admissible modules) is stable under sub and quotient objects, tensor products, taking duals, extensions. Any weakly admissible module whose filtration has length less than* p *(i.e. such that for some* j, $Fil^j D = D$, $Fil^{j+p} D = 0$) *is admissible. The functor V induces a ⊗-equivalence between the category of admissible filtered modules and the category of crystalline representations. The functor* D *is a quasi-inverse functor.*

2. Crystalline Representations and Etale Cohomology

2.1. Let X be a proper, smooth K-variety having good reduction; we say X is *admissible* if $H_{cris}(X)$ is admissible. We conjecture that any such X is admissible. One has the following partial result.

2.2. THEOREM A: i) *The product of two admissible varieties is admissible; the standard cellular varieties* ($P^N, Gr_{k,N}, \dots$) *are admissible.*

ii) *If* X *is a variety having good reduction and if one of the following conditions holds*

a) $p > \dim(X)$,

b) X *is a curve or an abelian variety,*

c) *there is a proper smooth model X/W with torsion-free Hodge cohomology and with ordinary* [4] *special fiber,*

then X *is admissible.*

Statement i) is a consequence of the Künneth formula, the stability of admissible modules under tensor product and the fact that certain "elementary" filtered modules (e.g. $S^i = K$ with $\phi = p^i \cdot \sigma$, $Fil^i = K$, $Fil^{i+1} = 0$) are admissible. In cases b) and c) statement ii) is established in [14;16]. We shall indicate the proof in case a) in II.2.7,2.8 (also see [22] when X is projective).

2.3. The main interest of Theorem A derives from:

THEOREM B: *There are defined on the category of admissible varieties canonical and functorial isomorphisms (compatible with multiplicative structure* *and cycle classes) of p-adic representations (resp. filtered modules)*

$$V(H_{cris}(X)) = H_{et}(X) \quad ,$$
$$D(H_{et}(X)) = H_{cris}(X) \quad ,$$

and consequently $B_{cris} \otimes_K H_{cris}(X) = B_{cris} \otimes_{\mathbf{Q}_p} H_{et}(X)$.

REMARK: If X has good reduction, then in fact the conclusion of the theorem is valid for H^m provided $m < p$.

We shall give the idea of the proof of theorem B in III.6. Here are several applications of theorems A and B.

3. "Arithmetical" Applications

3.1. Hodge-Tate Decomposition. Let $C = \hat{\overline{K}}$ and for any $r \in \mathbf{Z}$ set $C(r) = C \otimes \mathbf{Q}_p(1)^{\otimes r}$. For any admissible X, we have canonical, functorial isomorphisms compatible with ring structure and cycle classes

$$C \otimes H^m_{et}(X) = \bigoplus_{j=0}^{m} C(-j) \otimes_K H^{m-j}(X,\Omega^j_X) \quad .$$

This isomorphism is obtained by writing $gr^0(B_{cris} \otimes H^m_{et}(X)) = gr^0(B_{cris} \otimes H^m_{cris}(X))$ and using the fact [12], that $gr^r B_{cris} = C(r)$.

REMARK: This result was conjectured by Tate for all proper, smooth varieties over K and had been proved in many particular cases [1,4,5,7,15,22,31]. Recently Faltings has announced the result in the general case [9].

3.2. Action of Tame Inertia. Let V be a G-stable lattice in $H^m_{et}(X)$ and $\tilde{V}$ be the semi-simplification of V/pV; with respect to the action of I. Then, if X is admissible, the action of I (through its tame quotient) is given by characters of the form $\chi_h^{-(i_0+pi_1+\ldots+p^{h-1}i_{h-1})}$ where $h \geq 1$ and χ_h is a *fundamental character* of level h, i.e. a character of I with values in $F^{*}_{p^h}$ which factors through F^{*}_{h} (where $I^{tame} = \varprojlim F^{*}_{r}$, F_r being the subfield of p^r elements of $\bar{k}$) and extends to an isomorphism $F_h \xrightarrow{\sim} F_{p^h}$. Then the integers i_j satisfy $0 \le i_j \le m$.

This follows from a general property of the crystalline representations associated to filtered modules satisfying $Fil^0 D = D$, $Fil^{m+1} D = 0$, cf [13]. This result had been conjectured by Serre [30] and proven by Raynaud for m = 1, [27], and Kato in a more precise form provided dim (X) < p - 1 and the special fiber $\mathcal{X}_k$ is Hodge-Witt [22].

3.3. Bounds for the Discriminant. With the same notation as in 3.2, let H_n be the kernel of the representation of G on V/p^nV, $L_n = \bar{K}^{H_n}$ and $\mathfrak{D}_{L_n/K}$ the corresponding different. Then, with the valuation v of L_n normalized so that v(p) = 1, we conjecture that $v(\mathfrak{D}_{L_n/K}) < n + m/(p-1)$. For m = 1 this is proven in [17]. The general case should follow from theorem B because this inequality should hold for any crystalline representation. This is true at least when n = 1 and m < p-1, cf [18].

As a consequence, by using methods analogous to those which prove there is no non-trivial abelian scheme over **Z** [17], one deduces the following result:

If X is a proper, smooth variety over **Q** which has good reduction everywhere, and if $i,j \in \mathbf{N}$ and satisfy $i \neq j$, $i + j \le 3$, then $H^j(X,\Omega^i_X) = 0$. In particular, if the dimension of X is at most three, all its cohomology is algebraic [18].

3.4. Image of the Galois Group.

PROPOSITION: Let X *be an admissible variety and let* G_o *be the Zariski closure of the image of* I *in* $GL_{H^m_{et}(X)}$.

i) *The image of* I *is open in* $G_o(\mathbf{Q}_p)$ (*for its topology of* p-*adic Lie group*);

ii) G_o *is a connected group.*

The first property is a consequence of the fact that $H^m_{et}(X)$ is a Hodge-Tate representation [29]. The second property holds for any crystalline representation. It implies that if V is any I-stable subquotient of $H^m_{et}(X)$, then the action of I on the determinant of V is via χ^{-r} for some $r \ge 0$. For m = 1 this was proven by Raynaud [27]. Note that the fact that $H^m_{et}(X)$ is Hodge-Tate implies this last result only up to multiplication by a character of finite order.

4. Geometric Applications

4.1. Etale Cohomology of the Special Fiber. Let $\mathcal{X}$ be a proper smooth W-scheme such that $\mathcal{X}_K = X$ is admissible. The specialization map induces an isomorphism:

$$H^*_{et}(\mathcal{X}_{\bar{k}}, \mathbf{Q}_p) \xrightarrow{\sim} H_{et}(X)^I .$$

This follows because the source is naturally identified with the fixed points of ϕ in $H_{cris}(X) \otimes \mathrm{Frac}\,(W(\bar{k}))$ and the isomorphism of theorem B transforms this into the target. When m = 1, this holds even if K is not absolutely unramified and with $\mathbf{Q}_p$ replaced by $\mathbf{Z}_p$. Can it be proven for $m \geq 2$ and K an arbitrary local field by "purely" étale cohomological methods? What is the local analogue of this result?

4.2. The Crystalline Discriminant. Assume $p \neq 2$ and that k is algebraically closed. Let Y be a proper, smooth k-variety of even dimension d. Set $H_{cris}(Y) = K \underset{W}{\otimes} H_{cris}(Y/W)$.

In [25], Ogus defines an invariant, the crystalline discriminant, to be the Legendre symbol of the reduction mod p of $p^{-\mathrm{ord}<\alpha,\alpha>} \cdot <\alpha,\alpha>$ where α is a generator of the determinant of $H^d_{cris}(Y)(-d/2)$ which is fixed by ϕ. Further, he gives a conjectural formula for this in terms of the ℓ-adic Betti numbers of Y.

Suppose now that Y admits a lifting to W whose generic fiber X is admissible. by theorem B, α is then identified with an element of the determinant of $H^d_{et}(X)(d/2)$. It now follows immediately from the Hodge index theorem that Ogus' crystalline discriminant conjecture is true for Y.

4.3. Absolute Hodge and Absolute Tate Cycles. Let X be a proper smooth variety defined over an algebraically closed field E of characteristic zero. Recall [25], an element ξ of $\mathrm{Fil}^r\, H^{2r}_{DR}(X)$ is said to be *absolutely Tate* provided there is a smooth $\mathbf{Z}$-algebra $R \subset E$, a proper, smooth model of X, X_R, defined over R, having locally-free de Rham cohomology, an element $\xi_R \in H^{2r}_{DR}(X_R)$ giving back ξ and such that, for any perfect field of finite characteristic k, and any homomorphism $R \xrightarrow{\alpha} W$, the image of ξ_R, $\xi_W \in H^{2r}_{DR}(X_W)$ satisfies $\phi(\xi_W) = p^r\, \xi_W$. Given such a ξ, fix R as above; then, for almost all p, $p > \dim(X)$, and we may choose α to be an injection. Extending α to the algebraic closure of Frac(R) in E and applying theorem B, we see that ξ_W defines an element of $H^{2r}_{et}(X_{\bar{K}}, \mathbf{Q}_p(r))^G$, which by the proper base change theorem we may view as an element ξ_p of $H^{2r}_{et}(X, \mathbf{Q}_p(r))$. Conjecturally, [25], there is an absolute Hodge cycle $y = (y_{DR}, (y_\ell)_{\ell\ \mathrm{prime}})$ such that $\xi = y_{DR}$. It is reasonable to conjecture that y_p is equal to ξ_p for all p for which ξ_p is defined (and in particular that ξ_p is independent of *all* the choices made).

4.4. Transcendental Results. Let $\mathcal{X}$ be a proper, smooth W-scheme such that $\dim(\mathcal{X}_k) < p$. The proof of theorem A (part ii a) cf. II.2.6, in fact shows that $H_{DR}(\mathcal{X})$ is an object of the category MF_{tf}, [32], and that the Hodge to de Rham spectral sequence for $\mathcal{X}$

degenerates at E_1. If $\mathcal{X}$ is projective, as a consequence of the hard Lefschetz theorem for crystalline cohomology, [23], one deduces its validity for the Hodge cohomology of $\mathcal{X}_K$ (morphisms in MF_{tf} are strictly compatible with the filtrations). It follows that $h^{i,j} = h^{j,i}$ for $\mathcal{X}_K$. Now if X is any proper, smooth variety over a characteristic zero field E, we can choose $R, \mathcal{X}_R$, $\alpha: R \longrightarrow W$ (where p > dim X) as in 4.3 and immediately deduce the degeneration of the Hodge to de Rham spectral sequence (for X) at E_1, the hard Lefschetz theorem for the Hodge (or de Rham) cohomology of X (assuming X is projective) and, as Gabber suggested, using Hironaka, the Hodge symmetry even if X is "only" proper. The proof of part ii a of theorem A in fact gives the following result: if Y is a smooth variety of dimension < p defined over k and Y is liftable to W_2, then there is a canonical semi-linear quasi-isomorphism between $\oplus \Omega^j_Y[-j]$ and $\Omega^\bullet_Y$. Recently, Deligne and Illusie, [8], have found an incredibly elementary explicit proof of this fact and Raynaud has deduced from this a proof of the Kodaira-Nakano vanishing theorem for Y. By using the same method as above, this gives an *algebraic* proof of the result, valid for any X/E as above.

II. *p*-Adic Hodge Structures

1. The Syntomic Site, Crystalline Cohomology and the Cartier Isomorphism

1.1. Recall, [35], that a morphism $f: X \longrightarrow S$ of schemes is *locally a complete intersection* provided, locally on X there is a *regular* closed immersion into a smooth S-scheme through which f factors. A morphism is said to be *syntomic* provided it is flat and locally a complete intersection. This terminology is due to Mazur.

Let Y be a scheme. The big (resp. small) syntomic site Y_{SYN} (resp. Y_{syn}) of Y consists of the category of Y-schemes (resp. the full subcategory of the Y-schemes Z such that $Z \longrightarrow Y$ is syntomic) endowed with the topology (cf. [33, exposé IV]) generated by the surjective syntomic Y-morphisms of affine schemes. The big site is functorial in Y; for the small site there is the "usual" difficulty, cf. [34] or [2]; nevertheless cohomological calculations can be made using either.

1.2. For any k-scheme Z we write $\mathcal{O}_n^{cris}(Z) = H^0((Z/W_n)_{cris}, \mathcal{O}_{Z/W_n})$; this is a commutative W_n-algebra endowed with a Frobenius endomorphism, ϕ.

1.3. PROPOSITION: *The functor* $Z \longmapsto \mathcal{O}_n^{cris}(Z)$ *is a sheaf on the big syntomic site of* Spec(k). *For any* k-*scheme* Y, $H^*(Y_{SYN}, \mathcal{O}_n^{cris})$ *is canonically isomorphic* (*compatibly with Frobenius*) *to* $H^*((Y/W_n)_{cris}, \mathcal{O}_{Y/W_n})$.

The first statement follows easily from the fact that given a divided power thickening

$U \hookrightarrow T$ and a syntomic morphism $U' \longrightarrow U$ we can, locally on U', find a lifting to a syntomic morphism $T' \longrightarrow T$. The second statement follows easily from this plus the explicit description of crystalline cohomology in terms of the de Rham complex of a divided power envelope.

1.4. REMARK: If A is a k-algebra, we denote by W_n^{DP} (A) the divided power envelope (compatible with the standard divided powers on $V \cdot W_n(A)$) of the ideal formed of all $(a_0,\dots,a_{n-1}) \in W_n(A)$ such that $a_0^p = 0$; this notation is consistent with that of I.1.3 when $A = \tilde{\mathcal{O}}_{\bar{K}}$. Passing to the associated sheaf for the Zariski topology, we find a natural homomorphism: $W_n^{DP}(A) \longrightarrow \mathcal{O}_n^{cris}(A)$, defined exactly as in I.1.3. One verifies that if the Frobenius of A is surjective then $W_n^{DP}(A) \xrightarrow{\sim} \mathcal{O}_n^{cris}(A)$ and this implies that the associated sheaf for the syntomic topology, $\tilde{W}_n^{DP}$, is isomorphic to $\mathcal{O}_n^{cris}$.

1.5. We now restrict attention to the small syntomic site of k. If n and n' are integers such that $n \geq n' \geq 1$, and $n - n' = c$, there is an epimorphism, denoted generically by υ, $\mathcal{O}_n^{cris} \longrightarrow \mathcal{O}_{n'}^{cris}$, which is induced by the map $W_n(A) \longrightarrow W_{n'}(A)$ given by $(a_0,\dots,a_{n-1}) \longmapsto (a_0^{p^c},\dots,a_{n'-1}^{p^c})$. Using some standard facts about divided power envelopes of local complete intersections, it is shown that $\mathcal{O}_n^{cris}$ is flat as a W_n-algebra. Thus, given integers $n', n'' \geq 1$, we find a short exact sequence

$$0 \longrightarrow \mathcal{O}_{n''}^{cris} \xrightarrow{\pi} \mathcal{O}_{n'+n''}^{cris} \xrightarrow{\upsilon} \mathcal{O}_{n'}^{cris} \longrightarrow 0$$

where π is defined by $\pi x = p^{n'} \cdot \hat{x}$ if $\hat{x}$ is a lifting of x in $\mathcal{O}_{n'+n''}^{cris}$.

1.6. Let $\mathcal{O}_1$ be the "structure sheaf" on Spec(k), i.e. $\mathcal{O}_1(Y) = \Gamma(Y, \mathcal{O}_Y)$ for any k-scheme Y. There is an epimorphism of syntomic sheaves $\mathcal{O}_1^{cris} \longrightarrow \mathcal{O}_1$ and the kernel J_1 is a divided power ideal. For any $r \in \mathbf{N}$, let $J_1^{[r]}$ be the r^{th} divided power of J_1. The $J_1^{[r]}$ form a decreasing filtration of $\mathcal{O}_1^{cris}$,

$$\mathcal{O}_1^{cris} = J_1^{[0]} \supset J_1 = J_1^{[1]} \supset J_1^{[2]} \supset \dots,$$

and their intersection is zero. By an elementary calculation of the divided power envelopes of certain complete intersections, it is easy to check that for $r \leq p-1$ the natural map $\mathrm{Sym}^r(J_1/J_1^{[2]}) \longrightarrow J_1^{[r]}/J_1^{[r+1]}$ is an isomorphism, cf. 1.7.

For any integer $r \geq 1$, let $\hat{I}_1^{<r>}$ be the kernel of the composite mapping

$$\mathcal{O}_{r+1}^{cris} \xrightarrow{\phi} \mathcal{O}_{r+1}^{cris} \xrightarrow{\upsilon} \mathcal{O}_r^{cris}.$$

Set $\hat{I}_1^{<o>} = \mathcal{O}_1^{cris}$. There are two homomorphisms of $\hat{I}_1^{<r>}$ in $\mathcal{O}_1^{cris}$:

(i) υ_r is the composite $\hat{I}_1^{<r>} \longrightarrow \mathcal{O}_{r+1}^{cris} \longrightarrow \mathcal{O}_1^{cris}$

(ii) a σ-semi-linear homomorphism, $\hat{f}_r$, defined by $\hat{f}_r(x) = y$ if $\phi(x) = \pi(y)$.

Let $F^r\mathcal{O}_1^{cris} = \mathrm{Im}\,\upsilon_r$, $F_r\mathcal{O}_1^{cris} = \mathrm{Im}\ \hat{f}_r$ and set $F_{-1}\mathcal{O}_1^{cris} = 0$.

One verifies that for all $r \in \mathbf{N}$, $F^r\mathcal{O}_1^{cris} \supset F^{r+1}\mathcal{O}_1^{cris}$ and $F_{r-1}\mathcal{O}_1^{cris} \subset F_r\mathcal{O}_1^{cris}$ and that $\hat{f}_r$ induces a σ-semi-linear isomorphism:

$$f_r\colon F^r\mathcal{O}_1^{cris}/F^{r+1}\mathcal{O}_1^{cris} \xrightarrow{\sim} F_r\mathcal{O}_1^{cris}/F_{r-1}\mathcal{O}_1^{cris} .$$

1.7. PROPOSITION:

i) *For all* r, $F^r\mathcal{O}_1^{cris} = J_1^{[r]}$.

ii) $\cup F_r\mathcal{O}_1^{cris} = \mathcal{O}_1^{cris}$.

Working locally for the syntomic topology, we are lead to consider a k-algebra A of the following type. Let $(P_1,\ldots,P_d)$ be a regular sequence in a polynomial ring $k[X_1,\ldots,X_m]$. Let A be the quotient of the perfection of this polynomial ring by the ideal generated by $P_1,\ldots,P_d$. If π_j denotes the image in A of the p^{th} root of P_j in $k[X_1,\ldots,X_m]$, then one checks that

$$\mathcal{O}_1^{cris}(A) = W_1^{DP}(A) = \bigoplus_{m_1,\ldots,m_d \in \mathbf{N}} A\gamma_{pm_1}(\pi_1)\ldots\gamma_{pm_d}(\pi_d) .$$

and that $J_1^{[r]}(A)$ is the sub-A-module of $\mathcal{O}_1^{cris}(A)$ generated by the $\gamma_{m_1}(\pi_1)\ldots\gamma_{m_d}(\pi_d)$ with $\Sigma m_j \geq r$, and that, if $\tilde{A} = A/(\pi_1,\ldots,\pi_d)$ then

$$J_1^{[r]}/J_1^{[r+1]} = \bigoplus_{m_1+\ldots+m_d = r} \tilde{A}\gamma_{m_1}(\pi_1)\ldots\gamma_{m_d}(\pi_d) .$$

Let π'_j be a p^{rth} root of π_j in A and let $\hat{\pi}_j$ be the image of $(\pi'_j,\ldots,0) \in W_{r+1}(A)$ in $W_{r+1}^{DP}(A) = \mathcal{O}_{r+1}^{cris}(A)$. As $\phi(\hat{\pi}_j) = p!\gamma_p(\hat{\pi}_j)$, we find that if $x \in A$ and $\hat{x} \in \mathcal{O}_{r+1}^{cris}(A)$ is a lifting and $\Sigma m_j = r$, then $\hat{u} = \hat{x}\gamma_{m_1}(\hat{\pi}_1)\ldots\gamma_{m_d}(\hat{\pi}_d)$ is a lifting of $u = x\gamma_{m_1}(\pi_1)\ldots\gamma_{m_d}(\pi_d)$ in $\mathcal{O}_{r+1}^{cris}(A)$ which satisfies:

$$\phi(\hat{u}) = (p!)^r\phi(\hat{x})\eta_{m_1}\ldots\eta_{m_d}\gamma_{pm_1}(\hat{\pi}_1)\ldots\gamma_{pm_d}(\hat{\pi}_d)$$

(where $\eta_m = (pm)!/(p!)^m m!$ is a *p*-adic unit).

This immediately gives statement ii). Using this calculation one shows inductively that $F_r\mathcal{O}_1^{cris}(A) = \bigoplus_{\Sigma m_j \leq r} A\gamma_{pm_1}(\pi_1)\ldots\gamma_{pm_d}(\pi_d)$ and this easily gives the proposition.

2. Crystalline and de Rham Cohomology

2.1. Let s be an integer ≥ 1. If $i: \mathrm{Spec}(k)_{SYN} \longrightarrow \mathrm{Spec}(W_s)_{SYN}$ then one verifies that i_* is exact. We shall (by abuse of notation) continue to write $\mathcal{O}_n^{cris}$ for $i_*(\mathcal{O}_n^{cris})$ restricted to $\mathrm{Spec}(W_s)_{syn}$. Denote by $\mathcal{O}_s$ the "structural sheaf" on $\mathrm{Spec}(W_s)_{syn}$ and for any $n \leq s$ set $\mathcal{O}_n = \mathcal{O}_s/p^n.\mathcal{O}_s$. There is a homomorphism of W_n-algebras

$$\mathcal{O}_n^{cris} \longrightarrow \mathcal{O}_n$$

which, in terms of Witt vectors, is induced by the homomorphism $W_n(\mathcal{O}_1) \longrightarrow \mathcal{O}_n$ given by $(a_0,\dots,a_{n-1}) \longmapsto \hat{a}^{p^n}+\dots+p^{n-1}\hat{a}_{n-1}^{p}$, where $\hat{a}_j$ is a lifting of a_j to $\mathcal{O}_n$; this homomorphism was considered in the special case of $\mathcal{O}_{\bar{K},n}$ in I.1.3. This is an epimorphism of sheaves and its kernel J_n is a divided power ideal whose r^{th} divided power we denote by $J_n^{[r]}$.

For all $n \leq s$ we thus have a decreasing filtration

$$\mathcal{O}_n^{cris} = J_n^{[0]} \supset J_n = J_n^{[1]} \supset \dots$$

and $\cap J_n^{[r]} = (0)$. For n = 1, the just defined $J_1^{[r]}$ is the direct image by i of the $J_1^{[r]}$ of the preceding paragraph.

One checks that the $J_n^{[r]}$ are flat as sheaves of W_n-modules and thus if $n',n'' \in \mathbf{N}$ and $n' + n'' \leq s$ we have a short exact sequence

$$0 \longrightarrow J_{n''}^{[r]} \longrightarrow J_{n'+n''}^{[r]} \longrightarrow J_{n'}^{[r]} \longrightarrow 0 \quad .$$

2.2. Let Y_s be a smooth W_s-scheme of dimension d and (for $n \leq s$) set $Y_n = Y_s \otimes W_n$ and let $\mathcal{O}_{Y_n}^{cris}$ (resp. $J_{Y_n}^{[r]}$) be the restriction to $(Y_s)_{syn}$ of $\mathcal{O}_n^{cris}$ (resp. $J_n^{[r]}$). Denote by α the evident morphism of sites

$$\alpha: Y_{s,syn} \longrightarrow Y_{s,Zar} \ .$$

PROPOSITION: *There are canonical isomorphisms* (for $1 \leq n \leq s$)

i) $R\alpha_* \mathcal{O}_{Y_n}^{cris} \xrightarrow{\sim} \Omega^{\bullet}_{Y_n}$.

ii) for $r \in \mathbf{N}$, $R\alpha_* J_{Y_n}^{[r]} \xrightarrow{\sim} \sigma_{\geq r}\Omega^{\bullet}_{Y_n}$ (where $\sigma_{\geq r}\Omega^{\bullet}_{Y_n} = 0 \to \dots \to 0 \to \Omega^r_{Y_n} \to \dots$) and $R\alpha_*(J_{Y_n}^{[r]}/J_{Y_n}^{[r+1]}) \xrightarrow{\sim} \Omega^r_{Y_n}[-r]$.

This is an easy consequence of the fact that crystalline cohomology can be calculated using the syntomic site and (of course) Berthelot's fundamental comparison theorem between crystalline and de Rham cohomology, [2].

2.3. Since $\phi(J_1) = 0$ it follows that $\phi(J_n^{[r]}) \subset p^r \cdot \mathcal{O}_n^{cris}$ provided $r \leq p-1$. Let r and n be two integers which satisfy $r \leq p-1$ and $n+r \leq s$. If $x \in J_n^{[r]}$ and $\hat{x} \in J_{n+r}^{[r]}$ is a lifting then $\phi\hat{x} = p^r \cdot \hat{y}$ with $\hat{y} \in \mathcal{O}_{n+r}^{cris}$ and the image of $\hat{y}$ in $\mathcal{O}_n^{cris}$ is well-defined (independent of the choice of $\hat{x}$ and $\hat{y}$). We thus obtain a σ-semi-linear map

$$\phi_r : J_n^{[r]} \longrightarrow \mathcal{O}_n^{cris} .$$

REMARK: Observe that for $n = 1$, $\phi_r J_1^{[r+1]} = 0$ provided $r<p-1$ and since $\mathrm{Sym}^r(J_1/J_1^{[2]}) \xrightarrow{\sim} J_1^{[r]}/J_1^{[r+1]}$ we see that ϕ_r is determined by ϕ_1 and thus may be defined provided $s \geq 2$. When $r = p-1$, this no longer applies, but nevertheless, we may define a map $J_1^{[p-1]}/J_1^{[p]} \longrightarrow \mathcal{O}_1^{cris}$ directly using only ϕ_1.

2.4. Fix two integers t and n such that $t \leq p-1$ and $n+t \leq s$. Let Λ_n^t be the cokernel of the injective map

$$\bigoplus_{r=1}^{t} J_n^{[r]} \longrightarrow \bigoplus_{r=0}^{t} J_n^{[r]}$$

given by $(x_1,\ldots,x_t) \longmapsto (x_1, x_2 - p\,x_1, \ldots, x_t - p\,x_{t-1}, -p\,x_t)$. Since $\phi_{r-1}(x) = p\,\phi_r(x)$ if $x \in J_n^{[r]}$, the maps ϕ_r induce a semi-linear map

$$\bar{\phi} : \Lambda_n^t \longrightarrow \mathcal{O}_n^{cris} .$$

LEMMA: $\bar{\phi}(\Lambda_1^t) = F_t\mathcal{O}_1^{cris}$ *and the kernel of* $\bar{\phi}$ *is isomorphic to* $J_1^{[t+1]}$.

This is proved by induction on t, the case $t = 0$ being obvious. We have

$$\Lambda_1^t = \Big(\bigoplus_{r=0}^{t-1} J_1^{[r]}/J_1^{[r+1]}\Big) \oplus J_1^{[t]}$$

and it is clear that the following diagram commutes:

$$\begin{array}{ccc}
J_1^{[t]} & \xrightarrow{\ \phi_t\ } & F_t\mathcal{O}_1^{cris} \\
\downarrow & & \downarrow \\
J_1^{[t]}/J_1^{[t+1]} & \xrightarrow{\ f_t\ } & F_t\mathcal{O}_1^{cris}/F_{t-1}\mathcal{O}_1^{cris}
\end{array} .$$

If $t+1<p$, as $\phi_t(J_1^{[t+1]})=0$, we immediately find a short exact sequence

$$0 \longrightarrow J_1^{[t+1]} \longrightarrow \Lambda_1^t \longrightarrow F_t\,\mathcal{O}_1^{cris} \longrightarrow 0$$

where $J_1^{[t+1]} \longrightarrow \Lambda_1^t$ is the map $x \longmapsto ((0,\ldots,0),x))$. If $t+1=p$, we still have $\bar{\phi}(\Lambda_1^{p-1}) = F_{p-1}\,\mathcal{O}_1^{cris}$ and if $x \in J_1^{[p]}$, one checks easily that there is a $y \in \mathcal{O}_1^{cris}$ such that $\phi_{p-1}(x) = \phi_0(y)$ and that the image of y in $\mathcal{O}_1^{cris}/J_1$ is independent of the choice of y. Denoting this image by $\beta(x)$ we define a morphism $J_1^{[p]} \longrightarrow \Lambda_1^{p-1}$ by $x \longmapsto ((-\beta(x),\ldots,0),x)$ and obtain as above the short exact sequence for $t = p-1$ also.

REMARK: If $t = p-1$, we can also use the remark of 2.3 to define a map $J_1^{[p-1]}/J_1^{[p]} \longrightarrow F_{p-1}$ which lifts f_{p-1} and is defined in terms of ϕ_1. This gives an isomorphism

$$\bigoplus_{r=0}^{p-1} J_1^{[r]}/J_1^{[r+1]} \xrightarrow{\sim} F_{p-1}\,\mathcal{O}_1^{cris} .$$

2.5. From now on by abuse of notation we will write $H^m(F)$ for $H^m(Y_{s,syn},F)$ if F is a syntomic sheaf on Y_s. The exact sequence, deduced from 1.7,

$$0 \longrightarrow F_{j-1}\mathcal{O}_1^{cris} \longrightarrow F_j\mathcal{O}_1^{cris} \longrightarrow J_1^{[j]}/J_1^{[j+1]} \longrightarrow 0$$

yields a long exact cohomology sequence and thus isomorphisms $H^m(F_{j-1}\mathcal{O}_1^{cris}) \xrightarrow{\sim} H^m(F_j\mathcal{O}_1^{cris})$ provided $H^{m-1}(J_1^{[j]}/J_1^{[j+1]}) = H^m(J_1^{[j]}/J_1^{[j+1]}) = 0$. By 2.2 this is equivalent to the vanishing of $H^{m-1-j}(\Omega^j_{Y_1})$ and $H^{m-j}(\Omega^j_{Y_1})$ and hence holds provided $j > \inf(m,d)$. This condition insures that $H^m(F_{j-1}\mathcal{O}_1^{cris}) = H^m(\mathcal{O}_1^{cris})$. Similarly, if we consider the short exact sequence of 2.4, we obtain sufficient conditions for the map induced on cohomology by $\bar{\phi}$ to be injective (resp. bijective). We summarize:

LEMMA: *The mapping* $H^m(\Lambda_1^t) \longrightarrow H^m(\mathcal{O}_1^{cris})$ *induced by* $\bar{\phi}$ *is injective if* $m \le t$ *and bijective if* $d \le t$. *If* $t \ge \inf(m,d)$, $H^m(F_t\mathcal{O}_1^{cris}) \xrightarrow{\sim} H^m(\mathcal{O}_1^{cris})$.

As a consequence of this we see that if Y_2 is a smooth scheme over W_2 of dimension $d < p$, then the natural map $\bigoplus_{r=0}^{d} J_1^{[r]}/J_1^{[r+1]} \longrightarrow \mathcal{O}_1^{cris}$ induced by $\bar{\phi}$ if $d < p-1$ (or as in the remark of 2.4 if $d = p-1$) is a quasi-isomorphism. This implies:

COROLLARY: *If* X *is a smooth* k-*scheme of dimension* $< p$, *which is liftable to*

W_2, then each choice of lifting defines in the derived category a quasi-isormorphism :

$$\oplus\, \Omega^j_X[-j] \xrightarrow{\sim} \Omega^{\cdot}_X \ .$$

If X is proper, then its Hodge to de Rham spectral sequence degenerates at E_1.

REMARK: A completely elementary and explicit proof of this last result has been given by Deligne and Illusie [8], who in addition obtain interesting results about the liftability of smooth varieties of dimension $< p$.

2.6. THEOREM: *Let Y_s be a proper, smooth W_s-scheme whose special fiber has dimension* d. *For all* $m \geq 0$ *and all pairs* $(t,n) \in \mathbf{N}^2$ *which satisfy*

a) $\inf(m,d) \leq t \leq p-1$

b) $1 \leq n \leq s-t$

we have

i) *The mapping* $\bar{\phi}$ *induces a semi-linear isomorphism*

$$H^m(Y_{s,syn}, \Lambda^t_n) \xrightarrow{\sim} H^m(Y_{s,syn}, \mathcal{O}^{cris}_n) \ .$$

ii) *The exact sequence*

$$0 \longrightarrow \bigoplus_{r=1}^{t} J^{[r]}_n \longrightarrow \bigoplus_{r=0}^{t} J^{[r]}_n \longrightarrow \Lambda^t_n \longrightarrow 0$$

induces an exact sequence

$$0 \longrightarrow \bigoplus_{r=1}^{t} H^m(J^{[r]}_n) \longrightarrow \bigoplus_{r=0}^{t} H^m(J^{[r]}_n) \longrightarrow H^m(\Lambda^t_n) \longrightarrow 0 \ .$$

We prove this by induction on m (it being clearly true for m = -1). First, we show that the map induced by $\bar{\phi}$, $H^m(\Lambda^t_n) \longrightarrow H^m(\mathcal{O}^{cris}_n)$, is injective. To do this we use an induction on n, the case n = 1 having been established in the lemma of 2.5. Suppose n ≥ 2; the flatness of the $J^{[r]}_n$'s as W_n-modules implies that of Λ^t_n and thus we have a commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Lambda^t_{n-1} & \longrightarrow & \Lambda^t_n & \longrightarrow & \Lambda^t_1 & \longrightarrow & 0 \\ & & \downarrow \bar{\phi} & & \downarrow \bar{\phi} & & \downarrow \bar{\phi} & & \\ 0 & \longrightarrow & \mathcal{O}^{cris}_{n-1} & \longrightarrow & \mathcal{O}^{cris}_n & \longrightarrow & \mathcal{O}^{cris}_1 & \longrightarrow & 0 \ ; \end{array}$$

which induces another such diagram:

$$\begin{array}{ccccccc} H^{m-1}(\Lambda^t_1) & \longrightarrow & H^m(\Lambda^t_{n-1}) & \longrightarrow & H^m(\Lambda^t_n) & \longrightarrow & H^m(\Lambda^t_1) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H^{m-1}(\mathcal{O}^{cris}_1) & \longrightarrow & H^m(\mathcal{O}^{cris}_{n-1}) & \longrightarrow & H^m(\mathcal{O}^{cris}_n) & \longrightarrow & H^m(\mathcal{O}^{cris}_1) \ . \end{array}$$

As the left hand vertical arrow is an isomorphism and the second and fourth vertical arrows are injective, it follows that the third vertical arrow is injective too.

Thus $\lg_{W_n} H^m(\Lambda_n^t) \le \lg_{W_n} H^m(\mathcal{O}_n^{cris})$. The induction hypothesis implies the exactness of the sequence

$$0 \longrightarrow \bigoplus_{r=1}^{t} H^m(J_n^{[r]}) \longrightarrow \bigoplus_{r=0}^{t} H^m(J_n^{[r]}) \longrightarrow H^m(\Lambda_n^t)$$

Thus $\lg_{W_n} H^m(\mathcal{O}_n^{cris}) = \lg_{W_n} H^m(J_n^{[o]}) \le \lg_{W_n} H^m(\Lambda_n^t)$ giving an equality between these lengths and the two assertions of the theorem follow for m.

2.7. COROLLARY: *Let* $s \in \mathbf{N} \cup \{\infty\}$ *and let* Y_S *be a proper smooth (formal) scheme on* W_S *(with the convention that* $W_\infty = W$*) whose special fiber has dimension* d.

i) *If* n *(finite or not)* $\le$ s-inf(m,d) *and if* inf(m,d) $\le$ inf(p-1,s-1), *then, for all* r, *the mapping*

$$H^m(Y_{n,\sigma \ge r+1}\Omega^{\bullet}_{Y_n}) \longrightarrow H^m(Y_{n,\sigma \ge r}\Omega^{\bullet}_{Y_n})$$

is injective,

ii) *under the same hypothesis,* $H^m_{DR}(Y_n)$ *endowed with the filtration defined by* $H^m(Y_{n,\sigma \ge r}\Omega^{\bullet}_{Y_n})$ *and the mappings* ϕ_r: $H^m(Y_{n,\sigma \ge r}\Omega^{\bullet}_{Y_n}) \longrightarrow H^m_{DR}(Y_n)$ *is an object of the category* MF_{tf} *of* [32].

iii) *If* $d \le p-1$ *and if* n *(finite or not)* $\le$ s-d, *the Hodge to de Rham spectral sequence*

$$E_1^{r,m} = H^m(Y_n, \Omega^r_{Y_n}) \Rightarrow H^*_{DR}(Y_n),$$

degenerates at E_1.

Let t = inf(m,d). When s is finite statement i) follows from the injectivity of

$$\bigoplus_{r=1}^{t} H^m(J_n^{[r]}) \longrightarrow \bigoplus_{r=0}^{t} H^m(J_n^{[r]})$$

and iii) follows from i). Assertion ii) is equivalent to the exactness of the sequence

$$0 \longrightarrow \bigoplus_{r=1}^{t} H^m(J_n^{[r]}) \longrightarrow \bigoplus_{r=0}^{t} H^m(J_n^{[r]}) \xrightarrow{\ \bar{\phi}\ } H^m(\mathcal{O}_n^{cris}) \longrightarrow 0 .$$

The case where $n = s = \infty$ follows from this by passing to the limit.

2.8. REMARK: i) When Y_S is projective this result was obtained by Kato [22].

ii) Assertion ii) of 2.7 enables us to define a canonical splitting of the Hodge filttration of $H^m_{DR}(Y_n)$, the Wintenberger splitting, [32]. Further, one may associate to $H^m_{DR}(Y_n)$ a representation of G, cf. [13]. When $n = s = \infty$ we obtain the weak admissibility and hence, by [13] the admissibility of $K \otimes_W H^m_{DR}(Y_\infty)$.

III. *p*-Adic Periods and the Syntomic-Etale Topology

1. The Sheaves $\widetilde{S}^r_n$ and Crystalline Cohomology

1.1. We shall work in this paragraph on the site $\mathrm{Spf}(W)_{nil,SYN}$ whose objects consist of all W-schemes on which p is locally nilpotent, this category being endowed with the syntomic topology. For any X in this category, we denote by X_n its reduction modulo p^n and we recall that we have sheaves $\mathcal{O}_n$, $\mathcal{O}_n^{cris}$, J_n, $J_n^{[r]}$ defined as follows:

$$\mathcal{O}_n(X) = \Gamma(X_n, \mathcal{O}_{X_n})$$

$$\mathcal{O}_n^{cris}(X) = \mathcal{O}_n^{cris}(X_1) = H^0((X_1/W_n)_{cris}, \mathcal{O}_{X_1/W_n})$$

$$J_n = \mathrm{Ker}(\mathcal{O}_n^{cris} \longrightarrow \mathcal{O}_n)$$

$$J_n^{[r]} = r^{th} \text{ divided power of } J_n .$$

Recall that $\mathcal{O}_n^{cris}$ is endowed with a σ-semi-linear endomorphism φ, the Frobenius, and we define $\widetilde{S}^r_n$ as follows

$$\widetilde{S}^r_n = \mathrm{Ker}\,(J_n^{[r]} \xrightarrow{\phi - p^r} \mathcal{O}_n^{cris}) .$$

Thus we have a short exact sequence (denoting by $\mathcal{O}_n^{cris,r}$ the image of $\phi - p^r$)

$$0 \longrightarrow \widetilde{S}^r_n \longrightarrow J_n^{[r]} \xrightarrow{\phi - p^r} \mathcal{O}_n^{cris,r} \longrightarrow 0 .$$

LEMMA: $\mathcal{O}_n^{cris} / \mathcal{O}_n^{cris,r}$ *is killed by* p^r.

This is proved by explicit (laborious) computations using the fact that $\mathcal{O}_n^{cris} = \widetilde{W}_n^{DP}$ (cf. II.1.4) and that for an appropriate finite extension L of K, there is an explicitly constructed element $u \in W_n^{DP}(\mathcal{O}_{L,1})$ such that $u \in J^{[p-1]}$ and $\phi u = p^{p-1}u$. (Viewing u in $W_n^{DP}(\mathcal{O}_{\overline{K}})$, it is the reduction modulo p^n of $t^{p-1}/p \in H^0_{cris}(\widetilde{\mathcal{O}}_{\overline{K}}/W)$, where t is a generator for the Tate module $\mathbf{Z}_p(1)$ inside this ring.)

1.2. Let X be a proper, smooth W-scheme and $\overline{X} = X \otimes \mathcal{O}_{\overline{K}}$. Using the fact that syntomic morphisms can, locally, always be lifted modulo a nilpotent ideal, it follows that $H^*(\overline{X}_m, \widetilde{S}^r_n)$, $H^*(\overline{X}_m, J_n^{[r]})$, $H^*(\overline{X}_m, \mathcal{O}_n^{cris})$, $H^*(\overline{X}_m, \widetilde{\mathcal{O}}_n^{cris,r})$ are all independent of the choice of $m \geq n$ and we will denote them by $H^*(\overline{X}, \widetilde{S}^r_n)$ (resp...). Further, given any projective system of sheaves of $\mathbf{Z}_p$-modules $(\mathcal{F}_n)_{n\geq 1}$ we will write

$$H^*(\overline{X}, \mathcal{F}_{\mathbf{Z}_p}) = \varprojlim H^*(\overline{X}, \mathcal{F}_n)$$

$$H^*(\bar{X}, \mathcal{F}_{\mathbb{Q}_p}) = \mathbb{Q}_p \underset{\mathbb{Z}_p}{\otimes} H^*(\bar{X}, \mathcal{F}_{\mathbb{Z}_p}) \ .$$

1.3. PROPOSITION: i) $H^*(\bar{X}, \mathcal{O}_n^{cris}) = W_n^{DP}(\tilde{\mathcal{O}}_{\bar{K}}) \otimes H^*_{DR}(X_n)$

ii) $H^*(\bar{X}, \mathcal{O}_{\mathbb{Q}_p}^{cris}) = B^+_{cris} \underset{K}{\otimes} H^*_{DR}(X_K)$.

Statement ii) follows immediately from i); statement i) is the Künneth formula for $H^*_{cris}(X_n \otimes \mathcal{O}_{\bar{K},n}/W_n)$ and is proved by an elementary computation using the de Rham complex of a divided power envelope as well as the fact that $\mathrm{Spec}(\tilde{\mathcal{O}}_{\bar{K}})$ has trivial higher cohomology, [16].

1.4. LEMMA: i) $H^*(\bar{X}, \mathcal{O}_{\mathbb{Q}_p}^{cris,r}) = H^*(\bar{X}, \mathcal{O}_{\mathbb{Q}_p}^{cris})$

ii) *there is an integer* j (*independent of* n) *such that* p^j *kills the cokernel of* $H^*(\bar{X}, \mathcal{O}_{\mathbb{Z}_p}^{cris,r}) \longrightarrow H^*(\bar{X}, \mathcal{O}_n^{cris,r})$.

The first statement follows immediately from the lemma of 1.1 while the second statement follows from the fact that the cokernel of $H^*_{DR}(X) \longrightarrow H^*_{DR}(X_n)$ is killed by a power of *p* independent of n, together with statement i) of the proposition of 1.3.

1.5. Denote by π the morphism $\bar{X}_n \longrightarrow X_n$. One verifies directly that $R^j\pi_*J_n^{[r]} = 0$ for $j > 0$, and that one has a short exact sequence on $X_{n,syn}$

$$0 \longrightarrow \bigoplus_{\substack{i+j=r+1 \\ i,j\geq 1}} J_n^{[i]} \underset{W_n}{\otimes} J_n^{[j]}(\mathcal{O}_{\bar{K},n}) \longrightarrow \bigoplus_{i+j=r} J_n^{[i]} \underset{W_n}{\otimes} J_n^{[j]}(\mathcal{O}_{\bar{K},n}) \longrightarrow \pi_*(J_n^{[r]}) \longrightarrow 0 \ .$$

PROPOSITION: i) *If the dimension of* X_1 *is strictly less than* p, *then for any* m

$$\mathrm{Fil}^r(W_n^{DP}(\tilde{\mathcal{O}}_{\bar{K}}) \otimes H^m_{DR}(X_n)) \xrightarrow{\ \sim\ } H^m(\bar{X}_n, J_n^{[r]}) \ .$$

ii) $\mathrm{Fil}^r(B^+_{cris} \otimes H^m_{DR}(X)) \xrightarrow{\ \sim\ } H^m(\bar{X}, J_{\mathbb{Q}_p}^{[r]})$.

The first statement is an immediate consequence of the degeneration of the Hodge to de Rham spectral sequence (for X_n) at E_1 because then the long exact cohomology sequence associated to the above short exact sequence will decompose into short exact sequences. The second statement is a consequence of the fact that

$$\bigoplus_{\substack{i+j=r+1 \\ i,j\geq 1}} H^{m+1}(X, J_{\mathbb{Z}_p}^{[i]}) \otimes \mathrm{Fil}^j \varprojlim W_n^{DP}(\tilde{\mathcal{O}}_{\bar{K}}) \longrightarrow \bigoplus_{i+j=r} H^m(\bar{X}, J_{\mathbb{Z}_p}^{[i]}) \otimes \mathrm{Fil}^j \varprojlim W_n^{DP}(\tilde{\mathcal{O}}_{\bar{K}})$$

has its kernel killed by some power of *p*.

1.6. THEOREM: *Assume* X_K *is admissible and let* $r \geq m$. *The sequence*

$$0 \longrightarrow H^m(\bar{X}, \tilde{S}^r_{\mathbb{Q}_p}) \xrightarrow{\ \iota\ } H^m(\bar{X}, J^{[r]}_{\mathbb{Q}_p}) \longrightarrow H^m(\bar{X}, \mathcal{O}^{cris,r}_{\mathbb{Q}_p}) \longrightarrow 0$$

is exact.

COROLLARY: *With the above hypothesis we have (for* $r \geq m$*) an exact sequence*

$$0 \longrightarrow H^m(\bar{X}, \tilde{S}^r_{\mathbb{Q}_p}) \longrightarrow \mathrm{Fil}^r(B^+_{cris} \otimes_K H^m_{DR}(X_K)) \xrightarrow{\phi - p^r} H^m_{DR}(X_K) \longrightarrow 0 \ .$$

The corollary follows immediately from the theorem and 1.3 and 1.5. For the proof of the theorem we use the following:

LEMMA: *If* X_K *is admissible and* $r \geq m$ *then there is an* $s \geq 0$ *such that the cokernel of* $H^m(\bar{X}, J^{[r]}_{\mathbb{Z}_p}) \xrightarrow{\phi - p^r} H^m(\bar{X}, \mathcal{O}^{cris}_{\mathbb{Z}_p})$ *is killed by* p^s.

The lemma implies the theorem for $m = 0$ and thus, if $m \geq 1$, it suffices to verify the exactness of the sequence obtained by omitting the right hand zero. Using the lemma together with 1.4.ii) it is easy to check that there is an integer s' such that for any n the kernel of $H^m(\bar{X}, \tilde{S}^r_n) \longrightarrow H^m(\bar{X}, J^{[r]}_n)$ is killed by $p^{s'}$. Assuming this, an elementary and standard argument gives the theorem.

2. An Indication of the Proof of the Lemma of 1.6

2.1. LEMMA: *For* $i \geq 0$ $\mathrm{coker}(\mathrm{Fil}^i(\varprojlim W_n^{DP}(\tilde{\mathcal{O}}_{\bar{K}})) \xrightarrow{\phi - p^i} \varprojlim W_n^{DP}(\tilde{\mathcal{O}}_{\bar{K}}))$ *is killed by* p^i.

This follows from the lemma of 1.1 but may also be proved directly.

2.2. Recall we have defined in I.1.6. "*admissible filtered module* ". If D is such a module, then we have $B_{cris} \otimes_{\mathbb{Q}_p} V(D) = B_{cris} \otimes_K D$ (in a manner compatible with filtrations, Frobenius and the action of G) .

2.3. PROPOSITION: *If* D *is admissible, then the map*

$$\mathrm{Fil}^0(B_{cris} \otimes D) \xrightarrow{\phi - 1} B_{cris} \otimes D$$

is surjective.

This follows immediately from 2.1, and implies that for any $i \in \mathbb{Z}$, $\mathrm{Fil}^i(B_{cris} \otimes D) \xrightarrow{\phi - p^i} B_{cris} \otimes D$ is surjective.

2.4. PROPOSITION: *If* D *is admissible and satisifies* $\mathrm{Fil}^0 D = D$, $\mathrm{Fil}^{i+1} D = 0$, *then*

$\mathrm{Fil}^i(B^+_{cris} \otimes D) \xrightarrow{\phi - p^i} B^+_{cris} \otimes D$ *is surjective.*

This is proved by a computation, using the basic property of B^+_{cris} that $t \cdot B^+_{cris} = \{x \in B^+_{cris} \mid \phi^r x \in \mathrm{Fil}^1 B^+_{cris}, \forall r\}$. In fact the proof yields the fact that $V(D) \subset t^{-i} B^+_{cris} \otimes D$.

2.5. Let D satisfy the hypothesis of 2.4 and choose (cf. [12,14]) a strongly divisible lattice M in D.

PROPOSITION: *There is an integer* $s \geq 0$ *such that the cokernel of* $\mathrm{Fil}^i(\varprojlim W_n^{DP}(\tilde{\mathcal{O}}_{\bar{K}}) \otimes M) \xrightarrow{\phi - p^i} \varprojlim W_n^{DP}(\tilde{\mathcal{O}}_{\bar{K}}) \otimes M$ *is killed by* p^s.

This is proved by a long involved computational argument, which we do not attempt to elucidate here.

2.6. Let $M \subset H^m_{DR}(X)/\text{torsion}$ be a strongly divisible lattice, chosen sufficiently small so that the left hand vertical arrow in the following commutative diagram is defined:

$$\begin{array}{ccc} \mathrm{Fil}^r(\varprojlim W_n^{DP}(\tilde{\mathcal{O}}_{\bar{K}}) \otimes M) & \xrightarrow{\phi - p^r} & \varprojlim W_n^{DP}(\tilde{\mathcal{O}}_{\bar{K}}) \otimes M \\ \downarrow & & \downarrow \\ \text{Image of } H^m(\bar{X}, J^{[r]}_{\mathbb{Z}_p}) \text{ in } H^m(\bar{X}, \mathcal{O}^{cris}_{\mathbb{Z}_p})/\text{torsion} & \xrightarrow{\phi - p^r} & H^m(\bar{X}, \mathcal{O}^{cris}_{\mathbb{Z}_p})/\text{torsion} \end{array}$$

As the upper horizontal and right hand vertical arrows both have cokernels killed by fixed powers of p, the lemma of 1.6 follows.

2.7. REMARK: The proof actually shows that the results of 1.6 are valid provided $r \geq \inf(m, \text{length of the Hodge filtration of } H^m_{DR}(X_K))$.

3. The Sheaves S_n^r

3.1. The natural epimorphism $J^{[r]}_{n+r} \longrightarrow J^{[r]}_n$ induces a map $\tilde{S}^r_{n+r} \longrightarrow \tilde{S}^r_n$. We denote its image by S^r_n. It follows easily from the lemma of 1.1 that for $m \geq n+r$ this image coincides with $\mathrm{im}(\tilde{S}^r_m \longrightarrow \tilde{S}^r_n)$. The following is obvious:

LEMMA: $H^*(\bar{X}, \tilde{S}^r_{\mathbb{Z}_p}) \xrightarrow{\sim} H^*(\bar{X}, S^r_{\mathbb{Z}_p})$.

Notice that we have natural maps $S^r_n \times S^{r'}_n \longrightarrow S^{r+r'}_n$ which endow $\bigoplus_{r \geq 0} S^r_n$ with the structure of an associative, commutative $\mathbf{Z}/p^n\mathbf{Z}$-algebra with unit. We now define a map $\mu_{p^n} \longrightarrow S_n^1$ as follows: For any W-algebra A, we write A_{n+1} as a quotient of the polynomial ring over W_{n+1} and let $\mathfrak{D}$ be the corresponding divided power envelope so that we have an epimorphism $\mathfrak{D} \longrightarrow A_{n+1}$. Let $\zeta' \in \mathfrak{D}$ be a lifting of an element of $\mu_{p^{n+1}}(A_{n+1})$. Then $\zeta'^{p^{n+1}} \in 1 + J_{n+1}(A)$ and its logarithm $\log(\zeta'^{p^{n+1}})$ is well-defined. We define in this way a homomorphism $\mu_{p^{n+1}}(A) \longrightarrow \mu_{p^{n+1}}(A_{n+1}) \longrightarrow J_{n+1}(A)$ and it is clear that the image is contained in $\tilde{S}_{n+1}^1(A)$, as well as that the image of $\mu_p(A)$ is contained in $\operatorname{Ker}(\tilde{S}_{n+1}^1(A) \longrightarrow S_n^1(A))$. Passing to the associated sheaves, we thus obtain a map $\mu_{p^n} \longrightarrow S_n^1$.

3.2. PROPOSITION: *If A is a p-adically separated and complete flat W-algebra, such that A_1 is a syntomic k-algebra, then* $\mu_{p^n}(A) \xrightarrow{\sim} S_n^1(A)$.

This may be proved by explicit calculations using the Artin-Hasse logarithm.

4. The Syntomic Etale Site

4.1. We will say that a morphism $\mathfrak{U} \longrightarrow \mathfrak{V}$ of p-adic formal schemes over SpfW is syntomic provided $\mathfrak{U}_n \longrightarrow \mathfrak{V}_n$ is syntomic for all $n \geq 1$. Recall that if L is a complete non-archimedean valued field with ring of integers $\mathcal{O}_L$ then we may associate to any formal scheme of finite type over $\mathcal{O}_L$, a rigid analytic space over L, its generic fiber, cf. [28]. If $\mathfrak{Z}$ is any (finite type) formal $\mathcal{O}_L$-scheme we define the *syntomic-étale* site of $\mathfrak{Z}$, denoted $\mathfrak{Z}_{syn,et}$ as follows: Objects are morphisms $\mathfrak{U} \longrightarrow \mathfrak{Z}$ which are syntomic, quasi-finite and have étale generic fiber (in the sense of rigid geometry). We shall be primarily concerned with the following situation. Given X our proper smooth W-scheme and $\bar{X} = X \otimes \mathcal{O}_{\bar{K}}$, let $\mathfrak{Z} = \hat{\bar{X}} = \varinjlim \bar{X}_n$ (so here L = C).

PROPOSITION: *If for any* $n \geq 1$, i *denotes the inclusion* $\bar{X}_n \longrightarrow \mathfrak{Z}$, *then* i_* *is exact for the syntomic-étale topology.*

It suffices to show that, if $\mathfrak{A}$ is a p-adically complete W-algebra and $\mathfrak{A}_n \longrightarrow B$ is syntomic and quasi-finite, then, locally, we may lift B to $\mathfrak{B}$ such that $\mathfrak{A} \longrightarrow \mathfrak{B}$ is syntomic, quasi-finite with étale generic fiber. If $B = \mathfrak{A}_n[X_1,\ldots,X_d]/(f_1,\ldots,f_d)$. then we may replace B by $B' = \mathfrak{A}_n[Y_1,\ldots,Y_d]/(f_1(X),\ldots,f_d(X))$ where $X_i \longmapsto Y_i^{p^{n+1}}$ defines the map

$B \longrightarrow B'$ and we lift B' to $\mathcal{B}' = \mathcal{A}\{Y_1,\ldots,Y_d\}/(\hat{f}_i(Y^{p^{n+1}}) + p^n.Y_i)$, where $\hat{f}_i$ lifts f_i.

REMARK: By abuse of notation, we will write S_n^r instead of $i_*S_n^r$ where $i:\bar{X}_{n+r} \longrightarrow \mathfrak{X}$.

4.2 We introduce also the syntomic -étale site of $\bar{X}$, where objects are morphisms $U \longrightarrow \bar{X}$ which are syntomic, quasi-finite and with $U \otimes \bar{K}$ étale. We have then morphisms (where $\mathfrak{X} = \hat{\bar{X}}$)

$$\mathfrak{X} \xrightarrow{\ i\ } \bar{X} \xleftarrow{\ j\ } X_{\bar{K}}$$

and these induce morphisms of topoi: $j: X_{\bar{K},et} \longrightarrow \bar{X}_{syn\text{-}et}$, $i: \mathfrak{X}_{syn\text{-}et} \longrightarrow \bar{X}_{syn\text{-}et}$. The definition of j is obvious since the category of schemes étale over $X_{\bar{K}}$ is a full sub-category of the category underlying the syntomic-étale site of $\bar{X}$. The definition of i_* is determined by $i_*\mathcal{F}(\mathrm{Spec}(B)) = \mathcal{F}(\mathrm{Spf}\ \hat{B})$ where Spec(B) is an affine object of the syntomic-étale site of $\bar{X}$ and $\hat{B} = \varprojlim B_n$. The definition of i^* is more subtle and requires some preliminary results.

4.3. PROPOSITION: *Let* Spec(A) *be Zariski open in* $\bar{X}$, $\mathcal{A} = \hat{A} = \varprojlim A_n$ *and* $\mathcal{A} \longrightarrow \mathcal{B}$ *be a syntomic, quasi-finite morphism (where* $\mathcal{B}$ *is a p-adically separated, complete algebra) with étale generic fiber. Then there is a syntomic, quasi-finite morphism* $A \longrightarrow B$ *with étale generic fiber such that* $\hat{B} = \mathcal{B}$.

If we write $\mathcal{B} = \mathcal{A}\{X_1,\ldots,X_d\}/(P_1,\ldots,P_d)$ where the P_i's are restricted power series and if we take polynomials $Q_1,\ldots,Q_d$ in $A[X_1,\ldots,X_d]$ which are congruent to the P's modulo p^r for some $r >> 0$, then $B = A[X_1,\ldots,X_d]/(Q_1,\ldots,Q_d)$ satisfies $\hat{B} = \mathcal{B}$, [17]. The facts that B is finitely presented and $\mathcal{B}$ is p-torsion-free imply that B is p-torsion-free. If j is the jacobian, then for some $g \in \mathcal{B}$, $j \cdot g = p^n$. If we write $g = g' + p^{n+1}g''$ with $g' \in B$, $g'' \in \mathcal{B}$, then $jg' = p^n(1 - pg'')$ and $1 - pg'' \in B$ so replacing B by $B[1/1 - pg'']$ we force the generic fiber to be étale. It now follows that $A \longrightarrow B$ is flat and hence this morphism satisfies all the conditions of the proposition.

We refer to a B as in the above proposition as a *decompletion* of $\mathcal{B}$.

4.4. If B is as constructed in 4.3, then we denote its henselization with respect to p by B^h. It is now easy to show, using the results of [26], that this ring depends only on $\mathcal{B}$ and neither on the choice of $\hat{A} \longrightarrow \mathcal{B}$ nor on the choice of B. We define the inverse image functor i^*, applied to a $\mathcal{A}$ on $\bar{X}_{syn\text{-}ét}$, by sheafifying the presheaf $\mathcal{B} \longmapsto \mathcal{A}(B^h)$ where this last term is defined as the (filtered) inductive limit, $\varinjlim \mathcal{A}(B')$, the limit taken over all commutative diagrams:

$$\begin{array}{ccc} & & \mathcal{B} \\ & \nearrow & \uparrow \\ B & \longrightarrow & B' \end{array} \quad .$$

where $B \longrightarrow B'$ is étale.

Now notice that given a B' such as in the diagram, then the pair B' and B_K constitute a covering of B. This implies that the following diagram is cartesian for any sheaf $\mathcal{A}$ on $\bar{X}_{\text{syn-ét}}$

$$\begin{array}{ccc} \mathcal{A} & \longrightarrow & j_*j^*\mathcal{A} \\ \downarrow & & \downarrow \\ i_*i^*\mathcal{A} & \xrightarrow[i_*(\alpha)]{} & i_*i^*j_*j^*\mathcal{A} \end{array}$$

and hence, just as in the case of the étale topology, we deduce:

PROPOSITION: *The functor* $\mathcal{A} \longmapsto (i^*\mathcal{A}, j^*\mathcal{A}, \alpha)$ *establishes an equivalence of categories between sheaves on* $\bar{X}_{\text{syn,ét}}$ *and the category of triples* $(\mathcal{F},\mathcal{H},\alpha)$ *consisting of a sheaf* $\mathcal{F}$ *on* $\mathcal{Y}_{\text{syn-ét}}$, *a sheaf* $\mathcal{H}$ *on* $X_{\bar{K},et}$ *and a morphism* $\alpha: \mathcal{F} \longrightarrow i^*j_*\mathcal{H}$. *The functor* i_* *is exact as is the functor* $j_!$ *defined by* $j_!(\mathcal{H}) = (0,\mathcal{H},0)$. *For any sheaf* $\mathcal{A}$ *we have an exact sequence*

$$0 \longrightarrow j_!j^*\mathcal{A} \longrightarrow \mathcal{A} \longrightarrow i_*i^*\mathcal{A} \longrightarrow 0 \quad .$$

REMARK: It is useful to note that given $X' \longrightarrow \bar{X}$ a syntomic, quasi-finite morphism with étale generic fiber, there is a canonical decomposition $X' = X'_1 \coprod X'_2$ where X'_1 is *p*-adically separated and on X'_2 *p* is invertible.

5. Construction of the Sheaves $\mathcal{S}_n^r$

5.1. We construct sheaves, $\mathcal{S}_n^r$ on $\bar{X}_{\text{syn-ét}}$ such that $i^*\mathcal{S}_n^r = S_n^r$, $j^*\mathcal{S}_n^r = Z/p^nZ(r)$. By 4.4 this is equivalent to defining a map $\alpha: S_n^r \longrightarrow i^*j_* Z/p^nZ(r)$. Given $\hat{A} \longrightarrow \mathcal{B}$, B and B^h as in 4.3 and 4.4, it suffices to define a map $S_n^r(\mathcal{B}) \longrightarrow Z/p^nZ(r)(B^h[1/p])$. If we descend A to an A_o defined over $\mathcal{O}_L$, L a finite extension of K in $\bar{K}$ and we descend B also, then we may write $B = \varinjlim B_i$ where

$B_i = B_o \otimes_{A_o} \mathcal{O}_{L_i}$ where L_i runs through the finite extensions of L contained in $\bar{K}$.

Then $S_n^r(\mathcal{B}) = S_n^r(\mathcal{B}_{n+r}) = S_n^r(B_{n+r}) = \varinjlim S_n^r(B_{i,n+r})$. In addition, we have $B^h = \varinjlim B_i^h$ and thus $\mathbf{Z}/p^n\mathbf{Z}(r)(B^h[1/p]) = \varinjlim \mathbf{Z}/p^n\mathbf{Z}(r)(B_i^h[1/p])$. Thus, we may "work at finite level" and it suffices to define a map $S_n^r(B_{i,n+r}) \longrightarrow \mathbf{Z}/p^n\mathbf{Z}(r)(B_i^h[1/p])$.
We change notation, and now write A for a smooth, p-adically separated $\mathcal{O}_L$-algebra, B for a p-adically separated syntomic, quasi-finite A-algebra having étale generic fiber. Further, we may assume that if π is a uniformizing parameter for $\mathcal{O}_L$ that $A/\pi A$ is an integral domain. Thus, A is endowed with a valuation, $\upsilon(a) = \max\{j | a \in \pi^j A\}$, inducing its p-adic topology. Let $\mathcal{B} = \hat{B}, \mathcal{A} = \hat{A}$. Because $\pi_0(\mathrm{Spec}(B^h[1/p])) = \pi_0(\mathrm{Spec}(\mathcal{B}[1/p]))$, [10], we may work with $\mathcal{B}$. $\mathcal{B}[1/p]$ is an étale $\mathcal{A}[1/p]$-algebra, and thus is a finite product of integral domains, each of which is regular and hence normal. We write $\mathcal{B}[1/p] = B_1 \times \ldots \times B_t$. Each B_j is a Tate algebra, and we denote by $\overset{\circ}{B}_j$ the subring of elements whose spectral norm is at most equal to 1, cf. [6]. Each $\overset{\circ}{B}_j$ is integrally closed in B_j and hence is a p-adically separated normal domain which contains the image of $\mathcal{B}$ in B_j. We have a natural map

$$S_n^r(\mathcal{B}_{n+r}) \longrightarrow \prod_{j=1}^{t} S_n^r(\overset{\circ}{B}_j/p^{n+r}).$$

Assume now without loss of generality, that L contains $\mu_{p^n}(\bar{K})$ so that

$$\mathbf{Z}/p^n\mathbf{Z}(r)(\mathcal{B})[1/p] = \prod_{j=1}^{t} (\mu_{p^n}(\bar{K})^{\otimes r})_j .$$

Thus, to define α, we must define a map $S_n^r(\overset{\circ}{B}_j) \longrightarrow \mu_{p^n}(\bar{K})^{\otimes r}$.

5.2. Since B_j is a $W[\zeta_n]$-algebra, where ζ_n is a primitive $p^{n\,th}$ root of 1, we have a natural map

$$\mu_{p^n}(\bar{K})^{\otimes r} \longrightarrow \mathrm{Sym}^r(\mu_{p^n}(\overset{\circ}{B}_j)) \longrightarrow S_n^r(\overset{\circ}{B}_j/p^{n+r})$$

where the last map is induced from $\mu_{p^n} \longrightarrow S_n^1$, cf. 3.1. We refer to this homomorphism as the *natural homomorphism*.

THEOREM: *Let* C *be a* $W[\zeta_n]$*-algebra which is a normal domain and such that* p *is not invertible in* C. *Set* $r = (p-1)a + b$ *with* $a, b \geq 0$, $b < p-1$ *and let*

$c = a + \upsilon_p(a!)$. *Then there is a functorial isomorphism of* $\mu_{p^n}(\overline{K})^{\otimes r}$ *onto* $S_n^r(C/p^{n+r})$ *such that the natural homomorphism is* p^c *times this isomorphism.*

We indicate the definition of this isomorphism. Let $(\zeta_i)_{i\geq 1}$ be a generator for the Tate module $Z_p(1)(\mathcal{O}_{\overline{K}})$ and let $t \in H^o_{cris}(\mathrm{Spec}(\widetilde{\mathcal{O}}_{\overline{K}})/W)$ be the corresponding element. Consider the element $u = t^{p-1}/p \in \mathrm{Fil}^{p-1} H^o_{cris}(\mathrm{Spec}(\widetilde{\mathcal{O}}_{\overline{K}})/W)$ and let t_n (resp. u_n) be the image in $W_n^{DP}(\widetilde{\mathcal{O}}_{\overline{K}})$. In fact, they belong to $W_n^{DP}(W[\zeta_n]/p^n)$. Then $t_n^b\gamma_a(u_n) \in S_n^r(W[\zeta_n]/p^{n+r})$ and if $a! = p^{\upsilon(a!)}a'$, then $t_n^r = a' p^c t_n^b \gamma_a(u_n)$. The homomorphism $\mu_{p^n}(\overline{K})^{\otimes r} \longrightarrow S_n^r(C/p^{n+r})$ is now defined by sending $\zeta_n^{\otimes r}$ to $a' t_n^b \gamma_a(u_n)$ viewed as an element in $S_n^r(C/p^{n+r})$.

5.3. From the theorem, the definition of α is immediate. There is an alternative procedure for defining α. Namely, in the notation of 5.1, we let E be the fraction field of A, a discretely valued field, and $\mathbf{C}$ be the completion of the algebraic closure of $\hat{E}$. For each $j = 1,\ldots,t$ we choose an embedding of $\mathrm{Frac}(B_j)$ into $\mathbf{C}$ which induces an embedding of $\overset{o}{B}_j$ into $\mathcal{O}_{\mathbf{C}}$. Thus, we obtain a map from $S_n^r(\overset{o}{B}_j/p^{n+r})$ to $S_n^r(\mathcal{O}_{\mathbf{C},n+r})$. But the field $\mathbf{C}$ contains $C = \hat{\overline{K}}$ and is completely analogous in the sense that, if $\mathbf{B}_{cris}$ is the ring associated to the perfect closure of $k(X_1)$, then $\mathbf{C}$ plays the role of C. Thus, the results of [14] can be applied in this context, and they imply that $S_n^r(\mathcal{O}_{\mathbf{C},n+r}) = \mu_{p^n}(\overline{K})^{\otimes r}$.

6. Construction of *p*-Adic Etale Cohomology

6.1. We shall indicate how the above results allow us to prove theorem B of I.2.3. Recall X is a proper, smooth W-scheme such that X_K is admissible. For any admissible filtered module D and any integer i write $V_i(D)$ for $\{x \in B_{cris} \otimes D \mid \phi(x) = p^i x, x \in \mathrm{Fil}^i\}$. Multiplication by t induces an isomorphism $V_i(D) \xrightarrow{\sim} V_{i+1}(D)$ which can be viewed intrinsically as a canonical isomoprhism $V_i(D)(-i) \xrightarrow{\sim} V_{i+1}(D)(-i-1)$. Assume $D = \mathrm{Fil}^o D$ and $\mathrm{Fil}^{i+1}(D) = 0$. Then, the proof of 2.4 gives the fact that $V_i(D) = \{x \in \mathrm{Fil}^i B^+_{cris} \otimes D) \mid \phi x = p^i x\}$. Applying this with $D = H^m_{cris}(X_K)$ and taking $r \geq \inf(m$, length of the Hodge filtration) we obtain from 1.6 and 3.2 an isomorphism (where we write $V_r^m(X)$ for $V_r(H^m_{cris}(X_K))$)

$$H^m(\bar{X}, S^r_{Q_p}) = V^m_r(X) .$$

Note that, since X is admissible, $\bigoplus_{m=0}^{2d} V^m_m(X)(-m)$ is an anti-commutative graded algebra which satisfies Poincaré duality, cf [12,14]. We wish to prove that for r and m as above $V^m_r(X) \xrightarrow{\sim} H^m_{et}(X_{\bar{K}}, Q_p(r))$.

6.2. PROPOSITION: *The natural map* $H^*(\bar{X}_{syn\text{-}et}, \mathcal{S}^r_n) \longrightarrow H^*(\mathcal{Y}_{syn\text{-}et}, S^r_n)$ *is an isomorphism.*

By 4.4 this is equivalent to the assertion that $H^*(j_! Z/p^nZ) = 0$. But using the exact sequence

$$0 \longrightarrow j_! Z/p^nZ \longrightarrow Z/p^nZ \longrightarrow i_* Z/p^nZ \longrightarrow 0$$

and recalling that i_* is exact, we see that this is equivalent to the fact that $H^*(\bar{X}_{syn\text{-}et}, Z/p^nZ) \xrightarrow{\sim} H^*(\mathcal{Y}_{syn\text{-}et}, Z/p^nZ)$. Analysis of the proof of Grothendieck's comparison theorem, [20], shows that we may replace syn-et by étale in this last assertion. Now the proper base change theorem for étale cohomology yields the desired conclusion.

6.3. Using 6.1 and 4.1 we see that the map α constructed in 5 permits us to define a map

$$\beta: V^m_r(X) \longrightarrow H^m_{ét}(X_{\bar{K}}, Q_p(r)) .$$

PROPOSITION: *The morphism* β *is an isomorphism.*

Since source and target have the same dimension, it suffices to show β is injective. The morphisms β are compatible with the multiplicative structure. Thus, using Poincaré duality on the source, it suffices to show that on $V^{2d}_d(X)$, β is injective (here $\dim(X_K = d)$. After replacing W by a finite unramified extension, we may assume that X has a W-valued point, x. Its crystalline cohomology class satisfies $\phi(cl(x)) = p^d\,(cl(x))$ and $cl(x) \in Fil^d H^{2d}_{cris}(X_K)$. Thus it suffices to show β takes this crystalline cycle class to the corresponding étale cycle class. Let us blow up the point x in X to obtain $\tilde{X}$ with exceptional divisor equal to P^{d-1}_W. Observe that $\tilde{X}_K$ is admissible and that the étale cohomology of $X_{\bar{K}}$ injects into that of $\tilde{X}_{\bar{K}}$. Hence it suffices to prove that $\beta_{\tilde{X}}$ takes the crystalline class of a point to the étale cycle class. Since the exceptional divisor has self-intersection equal to - H, where H is a hyperplane in P^{d-1}_W, it follows that the class of a point in either theory is given by $-c_1(\mathcal{O}_{\tilde{X}}(1))^d$. Thus, it suffices to show β transforms the crystalline Chern classes of a line bundle into its étale Chern class. To verify this, we recall that the crystalline Chern class (relative to W_n) is defined using the "exponential sequence":

$$\begin{array}{ccccccccc} 0 & \longrightarrow & 1+J_n & \longrightarrow & \mathcal{O}_n^{cris*} & \longrightarrow & \mathcal{O}_n^{*} & \longrightarrow & 0 \\ & & \big\downarrow \log & & & & & & \\ & & J_n & , & & & & & \end{array}$$

while the étale Chern class ("modulo p^n") is defined using the Kummer sequence:

$$0 \longrightarrow \mu_{p^n} \longrightarrow G_m \xrightarrow{p^n} G_m \longrightarrow 0 \ .$$

But, working over W_n, the Kummer sequence maps to the exponential sequence, since there is the natural map $\mathcal{O}_n^* \longrightarrow \mathcal{O}_n^{cris*}$ given by $\zeta \longmapsto \hat{\zeta}^{p^n}$ where $\hat{\zeta}$ (locally) lifts ζ. Hence, the desired compatibility follows from the fact that $S_n^1 = \mu_{p^n}$.

6.4. REMARK: Combining our techniques with those of Kato, [22], we can prove, even if K is no longer assumed absolutely unramified, that, if X is a proper and smooth $\mathcal{O}_K$-scheme, then, for $r < p - 1$ and $i \le r$, we have $H^i(\overline{X}, S_n^r) \xrightarrow{\sim} H^i_{et}(X_{\overline{K}}, Z/p^nZ(r))$. There remains though, the problem of relating the source to crystalline cohomology. When $e = 1$ this has been done and thus, using [13], we obtain the fact that, for $m < p - 1$, the invariant factors for $H^m_{et}(X_{\overline{K}}, Z/p^nZ)$ coincide with those of $H^m_{DR}(X_n)$, and in particular that for $m \le p - 1$ the invariant factors of the torsion in $H^m_{et}(X_{\overline{K}}, Z_p)$ depend only on the special fiber.

BIBLIOGRAPHY

1. Artin, M., Mazur, B., Formal groups arising from algebraic varieties, Ann. Sci. Ec. Norm. Sup., 4e série t.10 , 1977, pp. 87-132.

2. Berthelot, P. Cohomologie cristalline des schémas de caracteristique p > 0, Lect. Notes in Math. 407, Springer-Verlag, Berlin, 1974.

3. Berthelot,P., Breen, L., Messing, W., Theorie de Dieudonné cristalline II, Lect. Notes in Math. 930, Springer-Verlag, Berlin, 1982.

4. Bloch, S., Kato, K. *p*- Adic Etale Cohomology, to appear in Publ. Math. IHES.

5. Bogomolov, F., Sur l'algébricité des représentations ℓ-adiques, C.R. Acad. Sci. Paris, 290, 1980, pp.701-704.

6. Bosch, S., Güntzer, U., Remmert, R., Non-Archimedian Analysis, Springer Verlag, Berlin, 1984.

7. Coleman, R., Hodge-Tate periods and *p*- adic abelian integrals, Inv. Math., 78, 1984, pp. 351-379.

8. Deligne, P., Illusie, L., Opération de Cartier et Décomposition du Complexe de de Rham (Etat 0), preprint.

9. Faltings, G. *p*- Adic Hodge Theory, preprint, 1985.

10. Ferrand, D., Raynaud, M., Fibres formelles d'un anneau local noethérien, Ann. Sci. Ec. Norm. Sup., 4e série, t.3, 1970, pp. 295-311.

11. Fontaine, J.-M. , Groupes p- divisibles sur les corps locaux, Astérisque 47-48 , Soc. Math. de France, 1977.

12. Fontaine, J.-M., Modules galoisiens, modules filtrés et anneaux de Barsotti-Tate, in Journées de géométrie algébrique de Rennes, Astérisque 65, Soc. Math. de France, 1979, pp. 3-80.

13. Fontaine, J.-M., Laffaille, G., Construction de représentations p- adiques, Ann. sci. Ec. Norm. Sup., 4e série, t.15, 1982, pp. 547-608.

14. Fontaine, J.-M. Sur certains types de représentations p- adiques du groupe de Galois d'un corps local, construction d'un anneau de Barsotti-Tate, Ann. of Math., 115, 1982, pp. 529-577.

15. Fontaine,J.-M. Formes différentielles et modules de Tate des variétés abéliennes sur les corps locaux, Inv. Math. 65, 1982, pp. 379-409.

16. Fontaine, J.-M., Cohomologie de de Rham, cohomologie cristalline et représentations p- adiques, in Lect. Notes in Math., 1016, Springer-Verlag, Berlin, 1983, pp. 86-108.

17. Fontaine, J.-M., Il n'y a pas de variété abélienne sur Z, Inv. Math. 81, 1985, pp. 515-538.

18. Fontaine, J.-M., Schémas propres et lisses sur Z, in preparation.

19. Gillet, H., Messing, W., Cycle classes and Riemann-Roch for crystalline cohomology, to appear in Duke Journal of Mathematics.

20. Grothendieck, A., Le groupe de Brauer III, in Dix exposés sur la cohomologie des schémas, North-Holland, Amsterdam, 1968, pp. 88-188.

21. Grothendieck, A. Groupes de Barsotti-Tate et cristaux, Actes Congrès Intern. Math., Nice 1970, t.1, Gauthiers-Villars, Paris, 1971, pp. 431-436.

22. Kato, K., On p- adic vanishing cycles (application of ideas of Fontaine-Messing), preprint, 1985.

23. Katz, N., Messing, W., Some consequences of the Riemann hypothesis for varieties over finite fields, Inv. Math. 23, 1974, pp. 73-77.

24. Laffaille, G., Groupes p- divisible et modules filtrés: le cas peu ramifié, Bull. Soc. Math. de France, 108, 1980, pp. 187-206.

25. Ogus, A., Hodge cycles and crystalline cohomology, in Hodge cycles, Motives and Shimura varieties, Lect. Notes in Math 900, Springer-Verlag, Berlin, 1982, pp. 357-414.

26. Raynaud, M., Anneaux locaux henséliens, Lect. Notes in Math, 169, Springer-Verlag, Berlin, 1970.

27. Raynaud, M., Schémas en groupes de type $(p,\ldots,p)$, Bull. Soc. Math. de France, 102, 1974, pp. 241-280.

28. Raynaud, M., Géometrie analytique rigide d'après Tate, Kiehl,..., Bull. Soc. Math. de France, 102, 1974, pp. 241-280.

29. Sen, S., Lie algebras of Galois groups arising from Hodge-Tate modules, Ann. of Math., 97, 1973, pp. 160-170.

30. Serre, J.-P., Propriétés galoisiennes des points d'ordre fini des courbes elliptiques, Inv. Math., 15, 1972, pp. 259-331.

31. Tate, J., p- divisible groups, in Proc. of a Conf. on Loc. Fields, Springer-Verlag, Berlin, 1967, pp. 158-183.

32. Wintenberger, J.-P., Un scindage de la filtration de Hodge pour certaines variétés algébriques sur les corps locaux, Ann. Math., 119, 1984, pp. 511-548.

33. SGA_3, Demazure, M., Grothendieck, A., Schémas en groupes, t.1, Lect. Notes in Math, 151, Springer-Verlag, Berlin, 1970.

34. SGA_4, Artin, M., Grothendieck, A., Verdier, J.-L., Théorie de topos et cohomologie étale des schémas, t.1, Lect. Notes in Math., 269, Springer-Verlag, Berlin, 1972.

35. SGA_6, Berthelot, P., Grothendieck, A., Illusie, L., Théorie des intersections et théorème de Riemann-Roch, Lect. Notes in Math, 225, Springer-Verlag, Berlin, 1971.

J.-M.F.
UNIVERSITE de GRENOBLE I
INSTITUT FOURIER
(laboratoire associé au C.N.R.S.)
B.P. 74
38402 SAINT-MARTIN d'HERES
FRANCE

W.M.
SCHOOL of MATHEMATICS
UNIVERSITY OF MINNESOTA
MINNEAPOLIS, MN 55455
U.S.A.