INTERSECTION HOMOLOGY AND PERVERSE SHEAVES

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Sommaire

1. Intersection (co)homology

1.1. **The idea of homology.**

Let **Top** denote the category of topological spaces with continuous morphisms and $\mathbf{Top_2}$ the category of pairs in **Top**. Thus objects of $\mathbf{Top_2}$ are pairs (X, A) of $X, A \in \mathbf{Top}$ with an inclusion $A \subset X$, morphisms are the obvious ones. We denote by

$$T : \mathbf{Top_2} \longrightarrow \mathbf{Top_2}$$

the functor associating to a pair $(X, A) \in \mathbf{Top_2}$ the pair $(A, \emptyset)$.

At the end of the 19th century Poincaré associated simple algebraic invariants to (pairs of) spaces. Ordinary homology theory (with $\mathbb{Z}$-coefficients) is a collection of functors

$$H^c_\bullet : \mathbf{Top_2} \longrightarrow \mathbf{Ab} \ \ ,$$

and natural transformations

$$\partial_\bullet : H^c_\bullet \longrightarrow H^c_{\bullet-1} \circ T$$

(where **Ab** denotes the category of Abelian groups) satisfying the following axioms :

- *Homotopy invariance :* If $f : (X, A) \longrightarrow (Y, B)$ is a homotopy equivalence then
$$f_* : H^c_\bullet(X, A) \longrightarrow H^c_\bullet(Y, B)$$
is an isomorphism.
- *Exactness :* for every pair $(X, A) \in \mathbf{Top_2}$ there is a long exact sequence
$$\cdots \longrightarrow H^c_q(A) \longrightarrow H^c_q(X) \longrightarrow H^c_q(X, A) \xrightarrow{\partial_q} H^c_{q-1}(A) \longrightarrow \cdots \ \ ,$$
where $H^c_q(X) := H^c_q(X, \emptyset)$.
- *Excision :* if $(X; A, B)$ is an excisive triad (i.e. A and B are two subspaces of X such that X is the union of the interiors of A and B) then the inclusion
$$(A, A \cap B) \hookrightarrow (X, B)$$
induces an isomorphism
$$H^c_\bullet(A, A \cap B) \simeq H^c_\bullet(X, B) \ \ .$$
- *Additivity :* If $(X, A) = \coprod_i (X_i, A_i)$ then the inclusions $(X_i, A_i) \subset (X, A)$ induce an isomorphism
$$\oplus_i H^c_\bullet(X_i, A_i) \simeq H^c_\bullet(X, A) \ \ .$$
- *Dimension :* If $X = \{*\}$ then $H^c_0(X) = \mathbb{Z}$ and $H^c_i(X) = 0$ for any $i \neq 0$.

Remarks 1.1.1. (1) For simplicity of notations we used the notation $H^c_\bullet(X)$ rather than the correct one $H^c_\bullet(X, \mathbb{Z})$.

(2) We chose the unusual notation $H^c_\bullet$ to emphasize the use of finite chains (or homology with compact support). The notation $H_\bullet$ will be reserved to the Borel-Moore homology, which is more natural from the point of view of sheaf theory.

One can show that these axioms uniquely define ordinary homology $H^c_\bullet$ ("dual" axioms define ordinary cohomology theory $H^\bullet(\cdot)$). There are many equivalent geometric ways to define it : via homotopy theory, simplicial homology, cellular homology, singular homology, $\cdots$.

1.2. **Homology and manifolds : Poincaré duality.**

Definition 1.2.1. *A (topological) manifold is a paracompact Hausdorff space M such that each point $x \in M$ has an open neighbourhood homeomorphic to $\mathbb{R}^n$ for some fixed integer n. We refer to n as the dimension of M.*

Ordinary homology theory $H^c_\bullet$ is particularly efficient when studying a *closed (i.e. compact without boundary) oriented n-manifold* X. Indeed in this case Poincaré and Lefschetz showed that $H^c_\bullet$ satisfies two crucial properties :

1. There is a functorial intersection product
$$H^c_i(X) \times H^c_j(X) \xrightarrow{\cap} H^c_{i+j-n}(X) \ \ .$$

2. *Poincaré duality :* Let $H^c_\bullet(X, \mathbb{Q}) := H^c_\bullet(X) \otimes_{\mathbb{Z}}, \mathbb{Q})$. When $i + j = n$ then the pairing
$$H^c_i(X, \mathbb{Q}) \times H^c_j(X, \mathbb{Q}) \xrightarrow{\cap} H^c_0(X, \mathbb{Q}) \xrightarrow{\varepsilon} \mathbb{Q}$$
is non-degenerate (here ε is the "augmentation" counting the points of a 0-cycle with multiplicities).

However these properties miserably fails for singular spaces !

Example 1.2.2. Let $X = S^2 \vee S^2$ be the union of two 2-spheres glued at a point. Algebraically X can be realized as the complex projective variety
$$\{(x : y : z) \in \mathbf{P}^2\mathbb{C} \;/\; yz = 0\} \quad ,$$
union at the point $(1 : 0 : 0)$ of the two $\mathbf{P}^1\mathbb{C} \simeq S^2$
$$U := \{(x : y : z) \in \mathbf{P}^2\mathbb{C} \;/\; y = 0\} \qquad \text{and} \qquad V := \{(x : y : z) \in \mathbf{P}^2\mathbb{C} \;/\; z = 0\} \quad .$$
Using the Mayer-Vietoris exact sequence
$$\cdots \longrightarrow H^c_q(U \cap V) \longrightarrow H^c_q(U) \oplus H^c_q(V) \longrightarrow H^c_q(X) \longrightarrow H^c_{q-1}(U \cap V) \longrightarrow \cdots$$
for the excisive triad $(X, U + \varepsilon, V + \varepsilon)$ one obtains
$$\begin{cases} H^c_0(X) = \mathbb{Z} \\ H^c_1(X) = 0 \\ H^c_2(X) = \mathbb{Z} \oplus \mathbb{Z} \quad . \end{cases}$$
Thus Poincaré duality does not hold.

Example 1.2.3. The following was Poincaré's original example of how singularities can cause the failure of Poincaré's duality.

Let $X^3 = \Sigma(S^1 \times S^1)$ be the suspension of the 2-torus (where $\Sigma X = (X \times \{0\}) \backslash X \times [0, 1]/(X \times \{1\}))$. The two isolated singularities of X^3 are the two cone-points. Let $x = \Sigma(\{*\} \times S^1)$ and $y = \Sigma(S^1 \times \{*\})$, these are two 2-cycles. The intersection $x \cap y = \Sigma(\{*\} \times \{*\})$ is still a $2 + 2 - 3 = 1$-chain, but it has a boundary, namely the two cone-points. This boundary does not change as we move x and y in their homology classes.

One easily computes $H^c_0(X) = \mathbb{Z}$, $H^c_1(X) = 0$, $H^c_2(X) = \mathbb{Z} \oplus \mathbb{Z}$ and $H^c_3(X) = \mathbb{Z}$, thus Poincaré duality does not hold.

Intersection (co)homology theory, developed by Goresky and MacPherson in the 1970's (cf. [17] for an interesting historical account), is a generalization to a large category of singular spaces of the Poincaré-Lefschetz intersection theory for compact oriented manifolds. Naturally enough, although originally defined in classical topological terms, this theory develops all its power only once interpreted in sheaf-theoretical terms. It naturally leads to the theory of perverse sheaves and $\mathcal{D}$-modules which will be our main topic of study.

1.3. **Simplicial complexes and polyhedra.** A good reference for this section is [19].

The easiest, if not the most elegant, way of understanding $H^c_\bullet$ and intersection product, is to restrict ourselves to a subcategory of **Top** : the category of polyhedra, for which $H^c_\bullet$ is conveniently defined via simplicial homology (with compact support). The very down-to-earth definition of simplicial homology is the most natural way of guessing the "right" definition for intersection homology.

1.3.1. *Simplicial complexes.* A polyhedron is the geometric realization of a combinatorial gadget : a simplicial complex. We first recall some standards definitions :

- an *n-simplex* σ in $\mathbb{R}^N$ is the convex hull of independent points $v_0, \cdots v_n$ (i.e. $v_1 - v_0, \cdots v_n - v_0$ are linearly independent vectors). These points are called the vertices of the n-simplex.
- the faces of σ are the $(n-1)$-simplices whose vertices are those of σ.
- an *orientation* of σ is an ordering of its vertices modulo even permutations.
- a *simplicial complex* in $\mathbb{R}^N$ is a set $\mathcal{N}$ of simplices in $\mathbb{R}^N$ such that :
  (i) if $\sigma \in \mathcal{N}$ then the faces of σ are in $\mathcal{N}$.
  (ii) if $\sigma, \tau \in \mathcal{N}$ and $\sigma \cap \tau \neq \emptyset$ then $\sigma \cap \tau$ is a simplex whose vertices are also vertices of σ and τ.

(iii) if $x \in \sigma \in \mathcal{N}$ there exists a neighbourhood U of x in $\mathbb{R}^N$ such that $U \cap \tau \neq \emptyset$ for only finitely many simplices τ in $\mathcal{N}$.

Note that simplicial complexes naturally form a category. A morphism $f : \mathcal{N} \longrightarrow \mathcal{M}$ is a collection of linear maps on simplices compatible with faces.

1.3.2. *Polyhedra.*

- The *support* $|\mathcal{N}|$ of a simplicial complex $\mathcal{N}$ is the union of its simplices :

$$|\mathcal{N}| := \cup_{\sigma \in \mathcal{N}} \sigma \quad .$$

- a *triangulation* of $X \in \mathbf{Top}$ is a homeomorphism

$$T : |\mathcal{N}| \longrightarrow X$$

  with $\mathcal{N}$ a simplicial complex. A space X admitting a triangulation is called triangulable.
- a polyhedron is a topological space equipped with a class of triangulation stable under passing to finer and finer subdivisions.

Polyhedra form a subcategory of **Top**, morphisms being piecewise linear maps.

1.3.3. *PL-manifolds.*

Definition 1.3.1. *Let K be a polyhedron. We will say that K is a piecewise linear (PL) manifold of dimension n if every point $x \in K$ admits an open neighbourhood U with a piecewise linear homeomorphism $U \simeq \mathbb{R}^n$.*

Let K be a polyhedron and x a point of K. Choose a triangulation of K containing x as a vertex. The *star* of x is the union of those simplices containing x. The link $lk(x)$ of x consists of those simplices of the star of x which do not contain x.

As a subset of K the link $lk(x)$ depends on the choice of the triangulation of K. However one can show that as an abstract polyhedron $lk(x)$ is independent of the triangulation up to piecewise linear homeomorphism. Moreover $lk(x)$ depends only on a neighbourhood of x in K.

If $K = \mathbb{R}^n$ and $x \in \mathbb{R}^n$ is the origin then $lk(x)$ can be identified with the sphere S^{n-1} seen as the polyhedron $\partial\Delta^n$. It follows that if K is any PL-manifold the link $lk(x)$ is equivalent to S^{n-1} for ever point $x \in K$. Conversely if K is a polyhedron such that every link in K is an $(n-1)$-sphere then K is a PL-manifold. Indeed for any polyhedron K and for every $x \in K$ the star of x can be identified with the cone on $lk(x)$. If $lk(x) \simeq S^{n-1}$ then the star of x is a closed PL-ball so that x has a neighbourhood which admits a PL-homeomorphism to $\mathbb{R}^n$. Thus :

Proposition 1.3.2. *Let K be a polyhedron. The following are equivalent :*

(i) *K is a PL-manifold.*
(ii) *For each $x \in K$ the link $lk(x)$ is a PL-sphere.*

Warning : If K is a n-polyhedron whose underlying topological space is a n-manifold then K need not be a PL manifold : it is not possible in general to choose *piecewise linear* local charts in $\mathbb{R}^n$!

Indeed let K be such a polyhedron. As K is a topological n-manifold we deduce that $H_\bullet(K, K \setminus \{x\}; \mathbb{Z})$ is isomorphic to $\mathbb{Z}$ in degree n and zero elsewhere. It is equivalent to saying that $lk(x)$ is an homology $(n-1)$-sphere. However it does not imply that $lk(x)$ is itself a PL-sphere.

An explicit example is obtained as follows. Let P be the Poincaré homology 3-sphere. Thus P is obtained from a dodecahedron by identifying the opposite pentagonal faces after a rotation of $\pi/5$. Alternatively $P = SU(2)/I \simeq S^3/U$ where I is the binary dodecahedral group, a group of order 120 (the spin extension of the group $A_5 \subset SO(3)$ of isometries of the dodecahedron). The suspension ΣP is a 4-dimensional polyhedron whose link is isomorphic to P at precisely 2 points x and y. Notice that ΣP is not a manifold : one can show that the local fundamental group of $P \setminus \{x\}$ near X is isomorphic to $\pi_1(P) = I$. However $\Sigma^2 P$ is a topological manifold, as a particular case of the following surprising result :

Theorem 1.3.3 (Cannon-Edwards)**.** *Let P be an n-dimensional homology sphere. Then $\Sigma^2 P$ is homeomorphic to an $(n+2)$-sphere.*

But then $\Sigma^2 P$ is a 5-dimensional polyhedron which is a manifold but not a PL-manifold : it contains two points whose links are ΣP which is not even a topological manifold (let alone a PL 4-sphere).

1.3.4. *Remarks on triangulability.* We mention some results without proofs concerning triangulability.

- any *differentiable* manifold can be triangulated in an essentially unique way (Whitehead (1940)).
- any compact topological manifold of dimension ≤ 3 can be triangulated (Moïse (1951)).
- there are 4-dimensional compact topological manifolds which cannot be triangulated (Casson (1985)).
- Nothing known in dimension larger than 5.

Is any compact topological manifold at least homotopically triangulable (i.e. homotopically equivalent to a compact polyhedron) ? First notice that such a manifold M is a Euclidean Neighboorhood Retract (i.e. there exists $i : M \hookrightarrow \mathbb{R}^n$ with $i(M)$ retract of some neighbourhood) : it easily follows from local contractibility and the fact that any compact manifold can be embedded in $\mathbb{R}^n$. Now one checks that a compact space is an ENR if and only if it is a retract of a compact polyhedron. In particular M is dominated by a finite polyhedron L : there exists $f : M \longrightarrow L$, $g : L \longrightarrow M$ and a homotopy $fg \simeq 1 : M \longrightarrow M$ (Borsuk (1933)). Thus M has the homotopy type of the non-compact polyhedron

$$(\cup_{k=-\infty}^{\infty} L \times [k, k+1])/((x, k) \simeq (gf(x), k+1), x \in L, k \in \mathbb{Z}) \ .$$

In general it is not true that a compact space dominated by a finite polyhedron is homotopy equivalent to a finite polyhedron. However it holds true for compact manifolds (Kirby-Siebenmann (1970)). Moreover if we enlarge the category of polyhedra to the category of CW-complexes then any compact topological n-manifold, $n \neq 4$, admits a structure of a finite CW-complex (Kirby-Siebenmann (1970)).

1.4. **Simplicial homology with compact support.** From now on X will be a triangulable space. Fix $T : |\mathcal{N}| \longrightarrow X$ a triangulation. Note that $\mathcal{N}$ is finite if and only if X is compact. Let

$$\mathcal{N}^{(i)} := \{\sigma \in \mathcal{N}, \sigma \text{ is an } i\text{-simplex}\} \ .$$

For each $\sigma \in \mathcal{N}$ we fix an orientation of σ.

Definition 1.4.1. *We denote by $C_i^{c,T}(X)$ the $\mathbb{Z}$-module of compactly supported i-chains of (X, T), namely the Abelian free group freely generated by $\mathcal{N}^{(i)}$. Thus an i-chain is a linear combination*

$$\xi = \sum_{\sigma \in \mathcal{N}^{(i)}} \xi_\sigma \sigma$$

with $\xi_\sigma \in \mathbb{Z}$ is non-zero for only finitely many σ's.

Definition 1.4.2. *We define*

$$\partial : C_i^{c,T}(X) \longrightarrow C_{i-1}^{c,T}(X)$$

by

$$\partial\sigma = \sum_{\tau \text{ face of } \sigma} \pm\tau$$

for any $\sigma \in \mathcal{N}^{(i)}$ and the requirement that ∂ is R-linear. In this formula the sign $\pm$ is 1 if the orientation of τ is obtained from the one of σ by omitting an even vertex, -1 otherwise.

Lemma 1.4.3. $\partial^2 = 0$.

Definition 1.4.4. $H_i^{c,T}(X) := (\ker \partial : C_i^{c,T}(X) \longrightarrow C_{i-1}^{c,T}(X))/(\mathrm{Im}\, \partial : C_{i+1}^{c,T}(X) \longrightarrow C_i^{c,T}(X))$.

Of course we want to get rid of the choice of the triangulation

$$T : \mathcal{N} \longrightarrow X \ .$$

Definition 1.4.5. *A triangulation $T : |\mathcal{N}| \longrightarrow X$ is a refinement of a triangulation $T' : |\mathcal{N}'| \longrightarrow X$ if for any $\sigma \in \mathcal{N}$ there exists $\sigma' \in \mathcal{N}'$ such that $T(\sigma) \subset T'(\sigma')$.*

In this case one easily checks that the natural map $C_i^{c,T'}(X) \longrightarrow C_i^{c,T}(X)$ which associates to $\sigma' \in \mathcal{N}'^{(i)}$ the sum

$$\sum_{\substack{\sigma \in \mathcal{N}^i \\ T(\sigma) \subset T'(\sigma')}} \pm\sigma$$

(where the sign $\pm$ is 1 if the orientations of σ and σ' are compatible) is compatible with the differentials ∂.

Definition 1.4.6. *The simplicial chain complex with compact support of X is defined as $C_i^c(X) = \mathrm{colim}_T C_i^{c,T}(X)$ with the induced differential ∂. The simplicial homology with compact support of X is*

$$H_i^c(X) = \frac{\ker \partial : C_i^c(X) \longrightarrow C_{i-1}^c(X)}{\mathrm{Im}\ \partial : C_{i+1}^c(X) \longrightarrow C_i^c(X)} \quad .$$

Remarks 1.4.7.
- Note that this definition is independent of any triangulation of X but *a priori* impossible to compute.
- It is not obvious that it is functorial, namely that a continuous $f : X \longrightarrow Y$ induces a linear map

$$f_* : H_\bullet^c(X) \longrightarrow H_\bullet^c(Y) \quad .$$

However one has the following result :

Theorem 1.4.8. *If $T : |\mathcal{N}| \longrightarrow X$ is any triangulation of $X \in$* **Top** *then there is a natural isomorphism $H_\bullet^c(X) \simeq H_\bullet^{c,T}(X)$.*

Proof. (sketch of) : one shows that both coïncide with the usual singular homology $H_\bullet^{\mathrm{sing}}(X)$. Let $T : |\mathcal{N}| \longrightarrow X$ be any triangulation of X. Let $\sigma : \Delta_i \longrightarrow X$ be a singular i-simplex of X. One shows that there exists $\sigma_{pl} : \Delta_i \longrightarrow X$, which is piecewise linear with respect to T and a refinement of the obvious triangulation on Δ_i and approximates σ in the sense that

$$\sigma = \sigma_{pl} + \partial_{\mathrm{sing}}\Sigma$$

for some singular $(i+1)$-chain Σ. Since $\partial_{\mathrm{sing}}^2 = 0$ one has $\partial_{\mathrm{sing}}\sigma = \partial_{\mathrm{sing}}\sigma_{pl}$. It follows that the natural map $H_\bullet^{c,T}(X) \longrightarrow H^{\mathrm{sing}}(X)$ is an isomorphism. Taking colimits over T we get the result. ☐

1.5. **Local coefficients.** In the definition of homology with compact support we can replace $\mathbb{Z}$ by any Abelian group R, thus obtaining $H_\bullet^c(X, R)$, homology with compact support and constant coefficients R. Homology with constant coefficients loses a lot of topological information. A key idea of this course will be that one should always consider (co)homology with *local coefficients*.

Definition 1.5.1. *Assume X is connected. A* local system *$\mathcal{L}$ on X is a left $\mathbb{Z}[\pi_1(X)]$-module L.*

Suppose $T : |\mathcal{N}| \longrightarrow X$ is a triangulation. It induces a triangulation $\tilde{T} : |\tilde{\mathcal{N}}| \longrightarrow \tilde{X}$ which is naturally $\pi_1(X)$-equivariant. Hence the complex $C_\bullet^{c,\tilde{T}}(\tilde{X})$ is naturally a complex of right $\mathbb{Z}[\pi_1(X)]$-modules.

Definition 1.5.2. *The complex of singular $\mathcal{L}$-chains with compact support on X is $C_\bullet^{c,T}(X, \mathcal{L}) := C_\bullet^{c,\tilde{T}}(\tilde{X}) \otimes_{\mathbb{Z}[\pi_1(X)]} L$ with the differential ∂ induced from the differential on $C_\bullet^{c,T}(\tilde{X})$.*

Once more taking the colimit over T one obtains a complex $(C_\bullet^c(X, \mathcal{L}), \partial)$ whose cohomology is $H_\bullet^c(X, \mathcal{L})$.

Examples 1.5.3.
- Of course considering $\mathbb{Z}$ as the trivial left $\pi_1(X)$-module one recovers $H_\bullet^c(X) = H_\bullet^c(X, \mathbb{Z})$.
- If $L = \mathbb{Z}[\pi_1(X)]$ then $H_\bullet^c(X, \mathcal{L}) \simeq H_\bullet^c(\tilde{X})$.

1.6. **Simplicial homology (= Borel-Moore homology).** Let $\mathcal{L}$ be a local system on X. Let $T : |\mathcal{N}| \longrightarrow X$ be a triangulation. Rather than using the complex $(C_\bullet^{c,T}(X, \mathcal{L}), \partial)$ of finite linear combination of simplices, Borel and Moore noticed that one can work with chains which are *formal infinite* linear combinations of simplices to obtain a complex $(C_\bullet^T(X, \mathcal{L}), \partial)$. Taking colimits one obtains the inclusion of complexes

$$C_\bullet^c(X, \mathcal{L}) \hookrightarrow C_\bullet(X, \mathcal{L})$$

and a homology map

$$H_\bullet^c(X, \mathcal{L}) \longrightarrow H_\bullet(X, \mathcal{L}) \ \ ,$$

which is an isomorphism if X is compact.

1.7. **Cohomology and cohomology with compact support.** Given a right $\mathbb{Z}[\pi_1(X)]$-module M we denote by $\overline{M}$ the corresponding left $\mathbb{Z}[\pi_1(X)]$-module. We define

$$C^i(X, \mathcal{L}) := \mathrm{Hom}_{\mathbb{Z}[\pi_1(X)]}(\overline{C_i^c(\tilde{X}), L})$$

with the dual differential ∂^* which makes $(C^\bullet(X, \mathcal{L}), \partial^*)$ a complex. Its cohomology is denoted $H^\bullet(X, \mathcal{L})$.

One define similarly the Borel-Moore cohomology or cohomology with compact support $H_c^\bullet(X, \mathcal{L})$ as the cohomology of the complex

$$C_c^i(X, \mathcal{L}) := \mathrm{Hom}_{\mathbb{Z}[\pi_1(X)]}(\overline{C_i(\tilde{X})}, L) \ \ .$$

1.8. **Stratified spaces.** References are [18], [12], [1].

Singular spaces are naturally associated with many important mathematical objects (for example in representation theory). We are essentially interested only in singular spaces of "finite type", i.e. spaces that require finitely many data to characterize them (like finite simplicial complexes). Such spaces should always be embeddable in manifolds. Hence a natural idea for defining singular spaces is rather to define a reasonable notion for a singular subspace X of a finite dimensional manifold Y.

A desirable property would be the existence of a disjoint decomposition (called *stratification*) of X into smooth submanifolds of Y (called *strata*). The underlying idea is the following : for a singular space X of "finite type" the group $\mathrm{Homeo}(X)$ should act on X with finitely many orbits. These orbits should be the natural strata of X. A transitive action should define a manifold.

Whitney proposed the following definition :

Definition 1.8.1. *A Whitney stratification of a manifold Y is a disjoint decomposition $Y = \cup_\alpha S_\alpha$ of Y into submanifolds that satisfies the following four axioms :*

(i) *The decomposition is locally finite : every point $x \in Y$ has a neighbourhood U such that $U \cap S_\alpha$ is empty for all but finitely many α.*
(ii) *If S_β has a non-empty intersection with the closure $\overline{S_\alpha}$ then S_β is contained in $\overline{S_\alpha}$.*
(iii) Whitney's condition A. *Suppose $S_\beta \subset \overline{S_\alpha}$. If a sequence of points $a_k \in S_\alpha$ tends to a point $x \in S_\beta$ then*
$$T_x S_\beta \subset \lim_k T_{a_k} S_\alpha \quad .$$
(iv) Whitney's condition B. *Suppose $S_\beta \subset \overline{S_\alpha}$. If two sequences $(a_k) \in S_\alpha$ and $(b_k) \in S_\beta$ both converge to the same $x \in S_\beta$ then*
$$\lim_k [a_k, b_k] \subset \lim_k T_{a_k} S_\alpha \quad ,$$
provided both limits exist (here the chords are understood in the Grassmannian manifold of a local coordinates system in Y, the validity of the condition is independent of the system chosen).

If Y is a complex manifold then a *complex Whitney stratification* is a decomposition into complex submanifolds which satisfies the same four conditions. In this case tangent spaces and chords can be taken complex or real.

It is not clear at all why the Whitney conditions are the right ones (Whitney introduces these conditions in 1965, they were recognized to be natural 20 years later...).

Accordingly one can define a "good singular space" X as a *stratified space* : a closed subspace of a manifold Y, union of strata of a Whitney stratification of Y. The decomposition of X in these strata will still be called a Whitney stratification of X.

Let us try to give a more formal definition of a stratified space.

First notice that if we are given a Whitney stratification of the sphere $S^{n-1} \subset \mathbb{R}^n$, it defines a Whitney stratification on $\mathbb{R}^n$, the *conical stratification* attached to the one on S^{n-1} : the strata are the open cones on the strata of S^{n-1}, plus the origin.

An important result is then the following local structure theorem :

Theorem 1.8.2. *Let Y be a n-dimensional manifold with a Whitney stratification (S_α). Let x be a point of a k-dimensional stratum S_α. Then there exists a Whitney stratification of the sphere S^{n-k-1} and a homeomorphism, taking strata to strata, of a neighbourhood N_x of x :*
$$N_x \overset{\phi}{\simeq} \mathbb{R}^k \times \mathbb{R}^{n-k} \quad .$$
Here $\mathbb{R}^{n-k}$ is given the conical stratification and $\mathbb{R}^k$ has only one stratum.

The stratification of the sphere S^{n-k-1} at x is called the link stratification at x.

This enable us to formalize our definition of stratified space :

Definition 1.8.3 (Stratified spaces)**.** *A 0-dimensional stratified space is a countable set of points with the discrete topology.*

For $n > 0$ an n-dimensional stratified space is a paracompact Hausdorff space X with a filtration

$$X = X_n \supseteq X_{n-1} \supseteq \cdots X_1 \supseteq X_0$$

of X by closed subsets X_i such that :

(a) *$X \setminus X_{n-1}$ is dense in X.*

(b) *any $x \in X_i \setminus X_{i-1}$ admits a neighbourhood N_x in X of the form*

$$N_x \overset{\phi}{\simeq} \mathbb{R}^i \times C(L)$$

where

$$L = L_{n-i-1} \supseteq \cdots \supseteq L_1 \supseteq L_0$$

is an $(n-i-1)$-stratified space and $C(L)$ denotes the cone $L \times [0,1)/L \times \{0\}$.

One requires that ϕ preserves the stratification :

$$N_x \cap X_{i+j+1} \overset{\phi}{\simeq} \mathbb{R}^i \times C(L_j) \ \ ,$$

for $n - i - 1 \geq j \geq 0$.

Remarks 1.8.4. One easily checks the following facts :

- up to homeomorphism the space L, called the *link at x*, only depends on the stratification at the point x.
- necessarily $X^i := X_i \setminus X_{i-1}$, if non-empty, is a topological manifold of dimension i. The strata of X are the connected components of these manifolds X^i and we recover our previous picture. In particular X^0 is the union of the open strata, called the *regular strata*.

Definition 1.8.5. *Let X be an n-dimensional stratified space. We denote by Σ the set of singular strata, i.e. all the strata except the regular ones.*

1.8.1. *Complex quasi-projective varieties are stratified spaces.*

Theorem 1.8.6 (Whitney)**.** *Any complex quasi-projective variety of pure dimension admits a (complex) Whitney stratification, hence is a stratified space.*

Example 1.8.7. Let X be a quasi-projective variety of pure dimension n. Let $S(X)$ be the singular locus of X. Define a filtration of X by

$$X_{n-1} = S(X), \qquad X_{n-k} = S(X_{n-k+1}) \ \ .$$

In general this does not define a Whitney stratification but some refinement will (cf. [20]).

1.9. **(Simplicial) Intersection homology.** Let X be an n-dimensional stratified space. Intersection homology will associate to the data $(X, \mathcal{L}, p)$ (where $\mathcal{L}$ is a local system on X^0 and $p : \Sigma \longrightarrow \mathbb{Z}$ a function called *perversity*) a graded $\mathbb{Z}$-module $I^pH_\bullet(X, \mathcal{L})$.

1.9.1. *PL stratified spaces.* We will give the definition of intersection homology in the simplicial context. We thus need a notion of triangulation for stratified spaces.

Definition 1.9.1. *A PL stratified space is a stratified space admitting a triangulation $T : |\mathcal{N}| \longrightarrow X$ satisfying :*

- *each X_i is a union of simplices.*
- *at each point $x \in X$ the link L is a PL stratified space and the homeomorphism*

$$N_x \overset{\phi}{\simeq} \mathbb{R}^i \times C(L)$$

is a PL-homeomorphism.

Theorem 1.9.2 (Lojasiewicz, Goresky)**.** *Let*

$$X = X_n \supseteq X_{n-1} \supseteq \cdots X_1 \supseteq X_0$$

be a Whitney filtration of a complex quasi-projective variety X of pure dimension n. Then the stratified space X is PL.

1.9.2. *Intersection homology : the constant coefficients case.* To give the intuition without technical details we start with the constant coefficients case. Let X be a PL stratified space and let $T : |\mathcal{N}| \longrightarrow X$ be a compatible triangulation.

Definition 1.9.3. *Let $\xi = \sum_{\sigma \in \mathcal{N}^{(i)}} \xi_\sigma \sigma \in C_i^T(X)$ be a simplicial i-chain. Its support is*

$$|\xi| = \cup_{\xi_\sigma \neq 0} T(\sigma) \ \ .$$

There are two ideas in the definition of intersection homology.

First one considers only chains $\xi \in C_i^T(X)$ such that every simplex σ in ξ satisfies : *the interior of σ is contained in X^0*. Because of this condition it will make sense to consider such chains with coefficients in a local system $\mathcal{L}$ over X^0. We denote by $C_i^T(X^0)$ the $\mathbb{Z}$-module of such chains. Notice that this does not form a subcomplex of $C_\bullet^T(X)$ as ∂ does not naturally map $C_i^T(X^0)$ to $C_{i-1}^T(X^0)$. We modify ∂ by neglecting all simplices of the boundary which do not satisfy our condition, thus obtaining a complex $C_\bullet^T(X^0)$.

Second, we will consider only chains "sufficiently dimensionally transverse" to the strata $S_\alpha \in \Sigma$. If ξ is dimensionally transverse to S_α then

$$\dim(|\xi| \cap S_\alpha) \leq i - \operatorname{codim} S_\alpha \ \ .$$

We will allow "less transverse" chains. This default of transversality will be encoded in a function called *perversity.*

Definition 1.9.4 (perversity)**.** *A perversity is a function $p : \Sigma \longrightarrow \mathbb{Z}$.*

Definition 1.9.5. *A i-chain $\xi = \sum_{\sigma \in \mathcal{N}^{(i)}} \xi_\sigma \sigma \in C_i^T(X^0)$ is said p-allowable if for all each simplex $\sigma \in \mathcal{N}^{(i)}$ with $\xi_\sigma \neq 0$ and each $S_\alpha \in \Sigma$*

$$\dim(|\xi| \cap S_\alpha) \leq i - \operatorname{codim} S_\alpha + p(S_\alpha) \ \ .$$

We define

$$I^pC_i^T(X) := \{\xi \in C_i^T(X^0) \ / \ \xi \text{ is } p\text{-allowable and } \partial\xi \text{ is } p\text{-allowable } .\}$$

Similarly one defines $I^pC_i^{c,T}(X) \subset C_i^{c,T}(X)$.

Remarks 1.9.6.
- (i) In this definition saying that a set has negative dimension should be taken as saying that the set is empty.
- (ii) Notice that if ξ is an i-chain then it is not every $i-1$-face of every i-simplex of ξ that must be checked for its allowability but only those that survive in $\partial\xi$. Boundary pieces that cancel out do not need to be checked for allowability.

If T' is a refinement of T the induced map

$$C_i^T(X^0) \longrightarrow C_i^{T'}(X^0)$$

preserves the support thus restricts to maps

$$I^pC_i^T(X) \longrightarrow C_i^{T'}(X) \ \ .$$

Similarly for $I^pC_i^{c,T}(X)$.

Definition 1.9.7 (perverse chains)**.** *$I^pC_i(X) := \operatorname{colim}_T I^pC_i^T(X)$ where the colimit is taken over all triangulations compatible with the given stratification.*

Define similarly $I^pC_i^c(X)$.

Notice that the differential $\partial : C_i(X^0) \longrightarrow C_{i-1}(X^0)$ induces differentials $\partial : I^pC_i(X) \longrightarrow I^pC_{i-1}(X)$ and $\partial : I^pC_i^c(X) \longrightarrow I^pC_{i-1}^c(X)$

Definition 1.9.8. *The i-th intersection homology group of X with perversity p is defined as*

$$I^pH_i(X) := \frac{\ker \partial : I^pC_i(X) \longrightarrow I^pC_{i-1}(X)}{\operatorname{Im} \partial : I^pC_{i+1}(X) \longrightarrow I^pC_i(X)} \ \ .$$

On defines similarly $I^pH_\bullet^c(X)$.

1.9.3. *Local coefficients.* Let $\mathcal{L}$ be a local system on X^0. For simplicity we assume that X^0 is connected (i.e. contains only one strata).

The embedding $X^0 \hookrightarrow X$ gives a natural group morphism $\pi_1(X^0) \longrightarrow \pi_1(X)$. In general this morphism is neither injective nor surjective. In particular $\mathcal{L}$ does not necessarily extends to a local system on X.

Definition 1.9.9. *The complex of allowable p-chains with coefficients in $\mathcal{L}$ is*

$$I^pC_i^T(X,\mathcal{L}) := I^pC_\bullet^{\tilde{T}}(\tilde{X}) \otimes_{\mathbb{Z}[\pi_1(X^0)]} L$$

with the differential ∂ induced from the differential on $I^pC_\bullet^T(\tilde{X})$. Here the natural $\mathbb{Z}[\pi_1(X)]$-module $I^pC_\bullet^{\tilde{T}}(\tilde{X})$ is seen as a $\mathbb{Z}[\pi_1(X^0)]$-module through the natural map $\pi_1(X^0) \longrightarrow \pi_1(X)$.

Similarly one defines $I^pC_i^{T,c}(X,\mathcal{L})$.

Taking colimits over admissible triangulation one defines similarly to the constant coefficients case the intersection homology groups $I^pH_\bullet(X,\mathcal{L})$ and $I^pH_\bullet^c(X,\mathcal{L})$.

1.9.4. *Intersection cohomology.* In a dual way one defines intersection cohomology groups $I^pH^i(X,\mathcal{L})$ and intersection cohomology groups with compact support $I^pH_c^i(X,\mathcal{L})$.

2. Properties of intersection homology

In this section we enumerate and try to motivate several properties of intersection homology. Most of them were proved in [9] using topological methods. We will prove some of them later in the sheaf theoretic context.

2.1. **Restricting perversities.** In [18, p.30-31] Macpherson claims that the computation of the groups $I^pH_\bullet(X,\mathcal{L})$ can always be (non-trivially, by changing spaces !) reduced to the case where the perversity p satisfies $p(S_\alpha) \geq 0$ and $p(S_\alpha) \leq \operatorname{codim} S_\alpha - 2$. Thus usually we will restrict ourselves to such perversity functions.

2.2. **PL-pseudomanifolds.** Consider now the case where $\operatorname{codim} S_\alpha = 1$. Then any value of $p(S_\alpha)$ is either < 0 or $> \operatorname{codim} S_\alpha - 2 = -1$. By the previous remark the intersection homology group brings nothing new to a codimension one stratum of X. For this reason one sometimes assume that X has no codimension one stratum.

Definition 2.2.1 (PL-pseudomanifold). *A PL-stratified space is a PL- pseudomanifold if it admits an admissible triangulation with no codimension one stratum.*

Notice that any quasi-projective variety with a (complex) Whitney stratification is a PL-pseudomanifold.

2.3. **Finite generation.** If X is a compact stratified space then $I^pH_\bullet(X)$ is a finitely generated $\mathbb{Z}$-module.

2.4. **Considering all compatible triangulations is not necessary.** Theorem 1.4.8 does not hold for intersection homology : it is not true that $I^pH_\bullet(X)$ and $I^pH_\bullet^T(X)$ (with the obvious definition) coïncide for any compatible triangulation T. However Goresky and MacPherson proved this is true if T is *flaglike*, meaning that for each i the intersection of any simplex σ with X_i is a single face of σ, cf. [11].

Notice that given any compatible triangulation its second barycentric subdivision is flaglike.

2.5. **Comparing perversities.** The zero perversity 0 is the function $0(S_\alpha) = 0$ for all $S_\alpha \in \Sigma$, the top perversity t is $t(S_\alpha) = \operatorname{codim} S_\alpha - 2$ for all $S_\alpha \in \Sigma$.

If p and q are two perversities we write $p \leq q$ if $p(S_\alpha) \leq q(S_\alpha)$ for all $\alpha \in \Sigma$.

If $p \leq q$ we have an inclusion

$$I^pC_\bullet(X) \hookrightarrow I^qC_\bullet(X)$$

which induces a canonical morphism

$$I^pH_\bullet(X) \longrightarrow I^qH_\bullet(X) \quad .$$

2.6. **Failure of universal coefficients.** If $R \subset R'$ is a morphism of Abelian groups one has a natural inclusion

$$I^p C_\bullet(X,R) \otimes_R R' \subset I^p C_\bullet(X,R')$$

which is not surjective in general because of the allowability condition on $\partial\xi$. Thus there is no universal coefficients theorem in general.

2.7. **Normalization and top perversity.** An PL-stratified space X is said to be *normal* if any point $x \in X$ has a fundamental system of neighbourhoods U such that $U \setminus X_{n-2}$ is connected. Equivalently X is normal if the link at any point is connected.

Notice that if X is a normal complex analytic space then X with an adequate filtration is a normal pseudomanifold.

For any stratified space X there is a normal stratified space $\hat{X}$ and a projection $\pi : \hat{X} \longrightarrow X$ uniquely characterized by the property that the point of $\pi^{-1}(x)$ are in bijection with the connected components of $U \setminus X_{n-2}$. The stratified space $\hat{X}$ (with $\hat{X}_{n-k} = \pi^{-1}(X_{n-k})$) is called the normalization of X. It is easily defined as follows :

$$\hat{X} := (\cup_{x\in X} \hat{N}_x)/R \ \ ,$$

where if $x \in X$ has a neighbourhood $N_x \simeq \mathbb{R}^j \times C(L)$ and the link L has connected components $L_0, \cdots, L_n$ one puts

$$\hat{N}_x := \cup_i \mathbb{R}^j \times C(L_j)$$

and the equivalence relation R identifies two points in $\hat{N}_x$ and $\hat{N}_y$ with the same image in $X \setminus X_{n-2}$.

Theorem 2.7.1. [9, theor.4.3] *If X is a normal pseudomanifold and $\mathcal{L}$ is the restriction to X^0 of a local system $\mathcal{L}_X$ on X then the map $I^t H_\bullet(X,\mathcal{L}) \longrightarrow H_\bullet(X,\mathcal{L}_X)$ for the maximal perversity is an isomorphism.*

2.8. **Cap product with the fundamental class.** Let X be an $\mathbb{Z}$-oriented n-dimensional PL stratified space : there exists a cycle $[X]$ with support X and associating to each oriented n-simplex ± 1.

Let T be a compatible triangulation of X. Let T' be the first barycentric subdivision of T.

Definition 2.8.1. *For a $n-k$-simplex σ of T let $D'(\sigma)$ be the union of those simplices of T' whose vertices are the barycenters of the simplices τ of T containing σ. The k-simplex $D'(\sigma)$ is called the dual block of σ.*

Remark 2.8.2. Let $L'(\sigma)$ be the union of those simplices of T' whose vertices are the barycenters of the simplices τ of T containing σ and different from σ. Then $D'(\sigma)$ is the cone over $L'(\sigma)$.

Definition 2.8.3. *The Poincaré duality map is the chain map*

$$C_T^{n-\bullet}(X) \xrightarrow{\cap[X]} C_\bullet^T(X)$$

which maps the cochain associating

$$\begin{cases} 1 & \text{to the oriented } (n-k)-\text{simplex } \sigma \\ 0 & \text{to the others} \end{cases}$$

to the k-chain of $C_k^{T'}(X)$ with support $D'(\sigma)$ and multiplicity 1 for a suitable orientation.

This map induces homomorphisms

$$H^{n-k}(X) \xrightarrow{\cap[X]} H_k(X)$$

and

$$H_c^{n-k}(X) \xrightarrow{\cap[X]} H_k^c(X) \ .$$

If X is an n-manifolds then these maps are the Poincaré isomorphisms. The point in this case is that the dual block $D'(\sigma)$ of σ is a cell so the chain complex based on dual blocks computes the homology of X. For stratified spaces on the other hand, the dual blocks $D'(\sigma) = C(L'(\sigma))$ are cones on spaces that may not be spheres, thus are not cells in general.

Proposition 2.8.4. *Let X be a $\mathbb{Z}$-oriented n-dimensional PL stratified space . Then the cap product with the fundamental class*

$$H^{n-k}(X) \xrightarrow{\cap[X]} H_k(X)$$

factors canonically for any perversity $0 \leq p \leq t$ through

$$H^{n-k}(X) \longrightarrow I^0H_k(X) \longrightarrow I^pH_k(X) \longrightarrow I^tH_k(X) \longrightarrow H_k(X) \ .$$

Proof. First notice that for the complex $C_\bullet^T(X^0)$ is a subcomplex of $C_\bullet^T(X)$ thus the map $I^tH_k(X) \longrightarrow H_k(X)$ is well-defined.

Let T be an admissible triangulation on X, T' its first barycentric subdivision. Let σ be an $n-k$-simplex in T.

There exist a unique open stratum of X^s containing the interior of σ, $s \geq n-k$. We have $D'(\sigma) \subset X \setminus X_{s-1}$ and if $n-l \geq s$ then

$$D'(\sigma) \cap T'_{|X_{n-l}} = D'_{X_{n-l}}(\sigma) \ .$$

Let S_α be any stratum of X^{n-l}. If $n-l \geq s$ then

$$\dim(|D'(\sigma)| \cap S_\alpha) = \dim |D'_{S_\alpha}(\sigma)| = (n-l) - (n-k) = k-l \ ,$$

and if $n-l < s$ then

$$D'(\sigma) \cap S_\alpha \subset (X \setminus X_{s-1}) \cap X_{n-l} = \emptyset \ .$$

Arguing similarly for $\partial D'(\sigma)$ one obtains that $D'(\sigma)$ is 0-allowable and the result. □

Theorem 2.8.5. [9, theor.4.3] *If X is a normal $\mathbb{Z}$-oriented n-dimensional PL-pseudomanifold then*

$$H^{n-k}(X) \longrightarrow I^0H_k(X)$$

is an isomorphism.

2.9. **Intersection pairing and Poincaré duality.**

Definition 2.9.1. *Two perversities p and q are dual if $p+q=t$ (i.e. for each $S_\alpha \in \Sigma$ we have $p(S_\alpha)+q(S_\alpha) = \operatorname{codim} S_\alpha - 2$).*

Definition 2.9.2. *Two (finite dimensional) $\mathbb{Q}$-local systems $\mathcal{L}$ and $\mathcal{L}'$ on X^0 are said to be dual if there exist a perfect pairing*

$$\mathcal{L} \otimes_{\mathbb{Z}} \mathcal{L}' \longrightarrow \mathcal{O}r_{X^0} \quad ,$$

where $\mathcal{O}r_{X^0}$ is the orientation local system on X^0 (thus we have a class $[X^0] \in H_n(X^0, \mathcal{O})$ which for any $x \in X^0$ restricts to a generator of $H_n(X^0, X^0 - \{x\}, \mathbb{Q}) \simeq \mathbb{Q}$).

Theorem 2.9.3. [9, theor.1] *Let X be an n-dimensional stratified space. Let p, q, r be perversities such that $p+q \leq r$. Let $\mathcal{L}$ and $\mathcal{L}'$ be two dual $\mathbb{Q}$-local systems on X^0.*

Then there is a canonical intersection pairing

$$I^pH_i(X,\mathcal{L}) \times I^qH_j(X,\mathcal{L}') \xrightarrow{\cap} I^rH_{i+j-n}(X,\mathbb{Q}) \quad .$$

Similarly with compact support and also

$$I^pH^c_i(X,\mathcal{L}) \times I^qH_j(X,\mathcal{L}') \xrightarrow{\cap} I^rH^c_{i+j-n}(X,\mathbb{Q}) \quad .$$

Theorem 2.9.4. [9, theor.3.3] *If $p+q=t$ then the intersection pairing*

$$I^pH_i(X,\mathcal{L}) \times I^qH^c_{n-i}(X,\mathcal{L}') \xrightarrow{\cap} I^tH^c_0(X,\mathbb{Q}) \xrightarrow{\varepsilon} \mathbb{Q}$$

is non-degenerate.

2.10. **Topological invariance.** From now on we assume X is an n-dimensional pseudomanifold.

The groups $I^pH_\bullet(X,\mathcal{L})$ are defined using a lot of non-topological baggage : first triangulation, second and more importantly stratification.

We will see an example (cf. example **??**) where intersection homology depends on the choice of the stratification. On the other hand we already stated that for the zero and the top perversity (and trivial local system) intersection homology is just ordinary cohomology or homology, thus topological invariant.. In general some restrictions on the perversity will be needed to obtain topological invariance.

First we make some remarks. In [9] Goresky and MacPherson consider another complex $I^pC^{GM}_\bullet(X)$ (which we call the GM complex) defined as a subcomplex of $C_\bullet(X)$:

$$I^pC_i^{T,GM} := \{\xi \in C_i^T(X) \;/\; \xi \text{ is p-allowable and } \partial\xi \text{ is p-allowable } .\}$$

In general this GM-complex $I^pC^{GM}_\bullet(X)$ does not coïncide with $I^pC_\bullet(X)$ for two reasons :

1. a simplex of a GM-chain does not necessarily have its interior contained in X^0, thus there is no natural inclusion from $I^pC_i^{T,GM}(X)$ to $I^pC_i^T(X)$.
2. conversely, even if $I^pC_i^T(X) \subset C_i^T(X)$, $I^pC_\bullet(X)$ is not a subcomplex of $C^T_\bullet(X)$ as the differential is modified.

The first problem can be solved by imposing the condition $p(S_\alpha) \leq 1$ for any codimension 2 stratum S_α. In this case any simplex of a GM-chain has its interior contained in X^0 and

$$I^pC_i^{GM}(X) \subset I^pC_i(X) \quad .$$

To fix the second problem one has to ensure that given a chain $\xi \in I^pC_i(X) \subset C_i(X)$, *any face of any simplex* in its support is not entirely contained in the singular strata : in this case the usual boundary of ξ will have no simplex entirely contained in the singular strata, the two boundary operators will coïncide on ξ and the two complexes $I^pC_\bullet(X)$ and $I^pC^{GM}_\bullet(X)$ will coïncide.

Thus let $\xi = \sum_{i\in\mathbb{N}^{(i)}} \xi_\sigma \sigma \in I^pC_i^T(X)$ and S_α be any codimension 2 stratum. The condition

$$\dim(|\sigma| \cap S_\alpha) \; < i-1$$

will force any codimension 1 face of a simplex σ with $\xi_\sigma \neq 0$ to have its interior in X^0. As the p-allowability condition forces

$$\dim(|\sigma| \cap S_\alpha) \leq i - 2 + p(S_\alpha) \quad ,$$

it is enough to require $p(S_\alpha) \leq 0$.

Goreski and MacPherson consider in [9] even more restricted perversity functions, depending only on the codimensions of the strata :

Definition 2.10.1. *A GM-perversity (for Goresky-MacPherson) is a function $p : \mathbb{N}_{\geq 2} \longrightarrow \mathbb{N}$ with the properties that $p(2) = 0$ and $p(i+1) = p(i)$ or $p(i)+1$ for $i > 2$. Given a classical perversity p the associated perversity is defined by*

$$\forall S_\alpha \in \Sigma, \quad p(S_\alpha) = p(\operatorname{codim} S_\alpha) \ \ .$$

Theorem 2.10.2. [9] *Let X be an n-dimensional PL pseudomanifold and p a GM-perversity. Then $I^pH_\bullet(X)$ is a topological invariant (in particular independant of the stratification).*

2.11. **Singular intersection homology.** The independance from the stratification has also proven by King [16] without sheaf theory. He developed a version of *singular* intersection homology and proved an analog of theorem 1.4.8.

2.12. **Intersection homology is not functorial under continuous maps.** Let $f : X \longrightarrow Y$ be a continuous map of pseudomanifolds. Composition with f does not in general map perverse simplices to perverse simplices. Thus in contrast to ordinary homology f does not induce any map $f_* : I^pH_\bullet(X) \longrightarrow I^pH_\bullet(Y)$.

A continuous map $f : X \longrightarrow Y$ which is *stratum preserving* (i.e. the image of each stratum of X is contained in a single stratum of Y) and *placid* (i.e. for each stratum T of Y one has $\operatorname{codim} f^{-1}(T) \geq \operatorname{codim} T$) induces a linear map $f_* : I^pH_\bullet(X) \longrightarrow I^pH_\bullet(Y)$.

2.13. **Intersection homology is not an homotopy invariant.** We will show in the next section that the intersection homology of a cone (thus a contractible space) is in general non-trivial (for a non-trivial perversity). Hence intersection homology is not a homotopy invariant.

However G. Friedman [4] showed that composition with *stratum preserving, codimension preserving, homotopy equivalence* induces isomorphism in intersection homology.

Gajer [6] defined a version of intersection homotopy and proved an intersection version of the classical Dold-Thom theorem :

$$I^pH_\bullet(X, \mathbb{Z}) \simeq I^p\pi_\bullet(AG(X)) \ \ ,$$

where $AG(X)$ denotes the free abelian topological group generated by X.

3. Examples

We just copy the examples given by MacPherson [18].

3.1. **Example 6 : intersection cohomology of pseudomanifolds with isolated singularities.** Let X be an n-pseudomanifold with singular set X_0 of dimension 0. Given a perversity p the only relevant information is the number $p(n)$, which satisfies $0 \leq p(n) \leq n-2$. We fix once for all a ring R of coefficients that we won't mention.

A chain ξ of dimension i is p-allowable if and only if

$$\begin{cases} \dim |\xi| \cap X - 0 \leq i - n + p(n) \ , \\ \dim |\partial\xi| \cap X_0 \leq i - 1 - n + p(n) \ . \end{cases}$$

Thus, if we denote by ϕ the set of closed sets in X contained in $X \setminus X_0$, one has :

$$I^pC_i(X) = \begin{cases} C_i^\phi(X \setminus X_0) & \text{if } i < n - p(n) \\ C_i(X) \cap \partial^{-1}(C_{i-1}^\phi(X \setminus X_0)) & \text{if } i = n - p(n) \\ C_i(X) & \text{if } i > n - p(n) \ . \end{cases}$$

Hence

$$I^pH_i(X) = \begin{cases} H_i^\phi(X \setminus X_0) & \text{if } i < n - p(n) - 1 \\ \text{Im } : H_i^\phi(X \setminus X_0) \longrightarrow H_i(X) & \text{if } i = n - p(n) - 1 \\ H_i(X) & \text{if } i > n - p(n) - 1 \ . \end{cases}$$

Similarly

$$I^qH_i^c(X) = \begin{cases} H_i^c(X \setminus X_0) & \text{if } i < n - q(n) - 1 \\ \text{Im } : H_i^c(X \setminus X_0) \longrightarrow H_i^c(X) & \text{if } i = n - q(n) - 1 \\ H_i^c(X) & \text{if } i > n - q(n) - 1 \ . \end{cases}$$

3.2. **Example 7 : cone over a manifold.** Let L be a compact manifold and $X = c^0L$ the open cone $L \times [0,1[/L \times \{0\}$. By the previous computation one obtains :

$$I^pH_i(X) = \begin{cases} 0 & \text{if } i \leq n - p(n) - 1 \\ H_{i-1}(L) & \text{if } i > n - p(n) - 1 \ . \end{cases}$$

Similarly

$$I^pH_i^c(X) = \begin{cases} H_i(L) & \text{if } i \leq n - p(n) - 1 \\ 0 & \text{if } i > n - p(n) - 1 \ . \end{cases}$$

4. Sheaves on locally compact spaces

4.1. Presheaves.

Definition 4.1.1. *Let $\mathcal{S}$ and $\mathcal{C}$ be two categories.*

A $\mathcal{C}$-valued presheaf on $\mathcal{S}$ is a functor

$$\mathcal{F} : \mathcal{S}^{\mathrm{op}} \longrightarrow \mathcal{C} \quad .$$

A morphism of presheaves is a morphism of such functors.

We will denote by $\mathbf{PSh}(\mathcal{S}, \mathcal{C})$ the category of $\mathcal{C}$-valued presheaves on $\mathcal{S}$.

Definition 4.1.2. *Let $X \in \mathbf{Top}$ and $\mathcal{C}$ a category.*

A $\mathcal{C}$-valued presheaf on X is a $\mathcal{C}$-valued presheaf on $\mathbf{Op}_X$, where $\mathbf{Op}_X$ denotes the category of open subsets of X (morphisms being inclusions of open subsets).

We denote by $\mathbf{PSh}(X, \mathcal{C})$ the category of $\mathcal{C}$-valued presheaves on X.

Thus $\mathcal{F} \in \mathbf{PSh}(X, \mathcal{C})$ associates to each open set $U \subset X$ an object $\mathcal{F}(U) \in \mathcal{C}$ and to each inclusion $V \subset U$ of open subsets a restriction morphism $\rho_{V,U} : \mathcal{F}(U) \longrightarrow \mathcal{F}(V)$ in $\mathcal{C}$ with the conditions :

$$\begin{cases} \rho_{U,U} = \mathrm{id}_{\mathcal{F}(U)} \quad . \\ \rho_{W,U} = \rho_{W,V} \circ \rho_{V,U} \quad \text{for} \quad W \subset V \subset U \in \mathbf{Op}_X \quad . \end{cases}$$

Definition 4.1.3 (localization). *Let $\mathcal{F} \in \mathbf{PSh}(X, \mathcal{C})$. Assume $\mathcal{C}$ has colimits.*

For $x \in X$ we define $\mathcal{F}_x = \mathrm{colim}_{x \in U} \mathcal{F}(U)$, where U runs through the family of open neighbourhoods of x and the transition maps are the $\rho_{V,U}$.

The object $\mathcal{F}_x \in \mathcal{C}$ is called the stalk of $\mathcal{F}$ at x and the image $s_x \in \mathcal{F}_x$ of any $s \in F(U)$ (for $x \in U$) is called the germ of s at x.

4.2. Sheaves.

A $\mathcal{C}$-valued sheaf on X can be thought (at least) in two different equivalent ways :

- either as a $\mathcal{C}$-valued presheaf with good glueing properties.
- or as a space with *horizontal topological structure* and *vertical $\mathcal{C}$-structure.*

We start with the first viewpoint and will come back to the second viewpoint in section 4.5.

Definition 4.2.1. *A presheaf $\mathcal{F} \in \mathbf{PSh}(X, \mathcal{C})$ is a sheaf if for any family $(U_i)_{i \in I} \in \mathbf{Op}_X$ and denoting by U the union $\cup_{i \in I} U_i \in \mathbf{Op}_X$ the following two conditions hold :*

(a) *Given two sections $s', s'' \in \mathcal{F}(U)$ whose restrictions $s'_{|U_i} (:= \rho_{U_i,U}(s))$ and $s''_{|U_i}$ coïncide for every $i \in I$, then $s' = s''$.*

(b) *Given $s_i \in \mathcal{F}(U_i)$, $i \in I$, such that $s_{i|U_i \cap U_j} = s_{i|U_i \cap U_j}$ for any $i, j \in I$ then there exists $s \in \mathcal{F}(U)$ such that $s_{|U_i} = s_i$.*

A morphism of $\mathcal{C}$-valued sheaves $\mathcal{F} \longrightarrow \mathcal{G}$ is a morphism of presheaves.

We denote by $\mathbf{Sh}(X, \mathcal{C}) \subset \mathbf{PSh}(X, \mathcal{C})$ the full subcategory of $\mathcal{C}$-valued sheaves on X.

Examples 4.2.2.

(1) Let $C^i(U)$ denote the $\mathbb{C}$-vector space of $\mathbb{C}$-valued i-th differentiable functions on U. Then the presheaf $U \mapsto C^i(U)$ (with the usual restrictions morphisms) is a sheaf $C^i_X \in \mathbf{Sh}(X, \mathbb{C} - \mathrm{Vect})$. Similarly one defines the sheaf $\mathcal{O}_X$ of holomorphic functions on an analytic variety X.

(ii) Let A be an Abelian group and X a topological space. If X contains two disjoint open sets the constant presheaf $U \mapsto A$ is not a sheaf (the glueing property of sheaves would imply that its value on the union of two disjoints open sets is $A \oplus A$, not A).

(iii) Let $X = \mathbb{R}$. The presheaf $\mathcal{F}$ defined by

$$\mathcal{F}(U) := \{\text{bounded continuous functions on}\, U\}$$

is not a sheaf : the constant function x defines a section of $\mathcal{F}$ on each interval $]-i, i[$ but these sections can't be glue together into a global section of $\mathcal{F}$ on $\mathbb{R}$.

4.3. **Abelian sheaves.** From now on we will work with presheaves of Abelian groups (one could work with $\mathbf{PSh}(X, \mathcal{A})$ where $\mathcal{A}$ is any Abelian category).

Lemma 4.3.1. *The category* $\mathbf{PSh}(X) := \mathbf{PSh}(X, \mathbf{Ab})$ *is Abelian.*

Proof. As everything in $\mathbf{PSh}(X)$ is defined objectwise this follows immediately from the fact that $\mathbf{Ab}$ is Abelian. □

Lemma 4.3.2. *Let* $f : \mathcal{F} \longrightarrow \mathcal{G} \in \mathbf{Sh}(X)$*. Then the presheaf* $\ker f$ *defined by*

$$(\ker f)(U) = \ker(\mathcal{F}(U) \xrightarrow{f_U} \mathcal{G}(U))$$

is a sheaf.

Proof. Let $U = \cup_{i \in I} U_i$. Let $s_i \in (\ker f)(U_i) := \ker(\mathcal{F}(U_i) \xrightarrow{f} \mathcal{G}(U_i))$ such that

$$s_{i|U_i \cap U_j} = s_{j|U_i \cap U_j} \quad .$$

As $\mathcal{F}$ is a sheaf there exists a unique $s \in \mathcal{F}(U)$ such that $s_{|U_i} = s_i$. The section $f(s) \in \mathcal{G}(U)$ vanishes on the U_i's. As $\mathcal{G}$ is a sheaf $f(s) = 0$. Thus $s \in \ker f$ and $\ker f$ is a sheaf. □

Proposition 4.3.3. *Let* $f : \mathcal{F} \longrightarrow G \in \mathbf{Sh}(X)$*. Then :*

(i) *f is a monomorphism in $\mathbf{Sh}(X)$ if and only if f is a monomorphism in $\mathbf{PSh}(X)$ if and only if for all $x \in X$ the morphism $f_x : \mathcal{F}_x \longrightarrow \mathcal{G}_x \in \mathbf{Ab}$ is injective.*

(b) *f is an isomorphism in $\mathbf{Sh}(X)$ if and only if f is an isomorphism in $\mathbf{PSh}(X)$ if and only if for all $x \in X$ the morphism $f_x : \mathcal{F}_x \longrightarrow \mathcal{G}_x \in \mathbf{Ab}$ is an isomorphism.*

Proof. For (i) : that f is a monomorphism in $\mathbf{Sh}(X)$ if and only if f is a monomorphism in $\mathbf{PSh}(X)$ follows from lemma 4.3.2. In particular the condition that $f_x : \mathcal{F}_x \longrightarrow \mathcal{G}_x \in \mathbf{Ab}$ is injective is clearly necessary. Assume conversely that f_x is injective for all $x \in X$ and let us prove that $f : \mathcal{F}(U) \longrightarrow \mathcal{G}(U)$ is injective. Let $s \in \mathcal{F}(U)$ with $f(s) = 0$. Then $(f(s))_x = 0 = f_x(s_x)$. As f_x is injective $s_x = 0$ for all $x \in U$. Thus there exists an open covering $U = \cup_{i \in I} U_i$ with $s_{|U_i} = 0$. As $\mathcal{F}$ is a sheaf $s = 0$.

For (ii) : the fact that f is an isomorphism in $\mathbf{Sh}(X)$ if and only if f is an isomorphism in $\mathbf{PSh}(X)$ follows from the fact that $\mathbf{Sh}(X)$ is a full subcategory of $\mathbf{PSh}(X)$. The condition that $f_x : \mathcal{F}_x \longrightarrow \mathcal{G}_x \in \mathbf{Ab}$ is an isomorphism is clearly necessary. Assume conversely that f_x is an isomorphism for all $x \in X$ and let us prove that $f : \mathcal{F}(U) \longrightarrow \mathcal{G}(U)$ is surjective. Let $t \in \mathcal{G}(U)$. Thus there exists an open covering $U = \cup_{i \in I} U_i$ and $s_i \in \mathcal{F}(U_i)$ such that $t_{|U_i} = f(s_i)$. In particular $f(s_i)_{|U_i \cap U_j} = f(s_j)_{|U_i \cap U_j}$. Hence by part (i) of the lemma, $s_{i|U_i \cap U_j} = s_{j|U_i \cap U_j}$. As $\mathcal{F}$ is a sheaf there exists $s \in \mathcal{F}(U)$ with $s_{|U_i} = s_i$. Since $f(s)_{|U_i} = t_{|U_i}$ we have $f(s) = t$ as $\mathcal{G}$ is a sheaf. □

While kernels coïncide in $\mathbf{PSh}(X)$ and $\mathbf{Sh}(X)$ this is no more true for images and cokernels : if $f : \mathcal{F} \longrightarrow \mathcal{G} \in \mathbf{Sh}(X)$ in general the presheaves

$$U \mapsto \operatorname{coker}(\mathcal{F}(U) \longrightarrow \mathcal{G}(U))$$

and

$$U \mapsto \operatorname{Im}\ (\mathcal{F}(U) \longrightarrow \mathcal{G}(U))$$

are not sheaves.

Examples 4.3.4. Let us give two similar examples.

(a) Let $X = \mathbb{C}$, $\mathcal{F} = \mathcal{G} = \mathcal{O}_X$ the sheaf of holomorphic functions on $\mathbb{C}$ and consider the sheaf morphism $f = \partial/\partial z : \mathcal{O}_X \longrightarrow \mathcal{O}_X$. Then on any open disk U of $\mathbb{C}$ one has $\operatorname{coker}(f(U)) = 0$ has the equation $\partial f/\partial z = g$ always admits a solution on such a disk. On the other hand the cokernel presheaf is non-trivial as $\operatorname{coker}(f(\mathbb{C}^*)) \neq 0$: indeed the equation $\partial f/\partial z = 1/z$ does not have any solution on $\mathbb{C}^*$. Hence $U \mapsto \operatorname{coker}(\mathcal{F}(U) \longrightarrow \mathcal{G}(U))$ is not a sheaf.

(b) Let $X = \mathbb{C}^*$, $\mathcal{F} = C^0_X$, $\mathcal{G} = \mathcal{F}^*$ the sheaf of complex invertible continuous functions on $\mathbb{C}^*$, and $f = \exp : \mathcal{F} \longrightarrow \mathcal{G}$. Then f is locally surjective but the invertible continuous function z on $\mathbb{C}^*$ is not the exponential of a continuous function defined on $\mathbb{C}^*$. Thus $U \mapsto \operatorname{Im}\ (\mathcal{F}(U) \longrightarrow \mathcal{G}(U))$ is not a sheaf.

4.4. **Sheafification; cokernel; image.**

Proposition 4.4.1. *The inclusion* $\mathbf{PSh}(X) \supset \mathbf{Sh}(X)$ *admits a left-adjoint functor*

$$^+ : \mathbf{PSh}(X) \rightleftarrows Sh(X) : i$$

called sheafification.

The sheaf $\mathcal{F}^+$ *is called the sheaf associated to the presheaf* $\mathcal{F}$.

Proof. For any $U \in \mathbf{Op}_X$ let $\mathcal{F}^+(U)$ be the set of functions

$$s : U \longrightarrow \coprod_{x \in U} \mathcal{F}_x$$

such that for any $x \in U$ one has $s(x) \in \mathcal{F}_x$ and there exists an open neighbourhood $x \in V \subset U$, $t \in \mathcal{F}(V)$ such that $s(y) = t_y$ for any $y \in V$. Then $\mathcal{F}^+$ is clearly a sheaf, there is a natural morphism $\mathcal{F} \longrightarrow \mathcal{F}^+$ which induces an isomorphism $\mathcal{F}_x \longrightarrow (\mathcal{F}^+)_x$ for any $x \in X$. In particular if $\mathcal{F}$ is a sheaf then the natural map $\mathcal{F} \longrightarrow \mathcal{F}^+$ is an isomorphism of sheaves by proposition 4.3.3. One easily checks this is the required adjoint. □

Definition 4.4.2. *Let* $f : \mathcal{F} \longrightarrow \mathcal{G} \in \mathbf{Sh}(X)$. *We define* $\mathrm{coker} f$ *as the sheaf associated to the presheaf* $U \mapsto \mathrm{coker}(\mathcal{F}(U) \longrightarrow \mathcal{G}(U))$. *Similarly one defines* $\mathrm{Im}\, f = \ker(\mathcal{G} \longrightarrow \mathrm{coker} f)$ *and* $\mathrm{Coim} f = \mathrm{coker}(\ker f \longrightarrow \mathcal{F})$.

Notice that for all $x \in X$ one has :

$$(\ker f)_x = \ker f_x \qquad \text{and} \qquad (\mathrm{coker} f)_x = \mathrm{coker} f_x \quad .$$

It implies immediately the following :

Proposition 4.4.3. *The category* $\mathbf{Sh}(X)$ *is Abelian and for any* $x \in X$ *the functor* $\mathcal{F} \mapsto \mathcal{F}_x$ *from* $\mathbf{Sh}(X)$ *to* $\mathbf{Ab}$ *is exact.*

4.5. **Espaces étalés.** Let $\mathcal{F} \in \mathbf{PSh}(X$. We denote by $\mathcal{F}^{\mathrm{et}}$ the set $\coprod_{x \in X} \mathcal{F}_x$ (here the F_x are seens as sets and the coproduct is the disjoint union in **Set**) and define a topology on $\mathcal{F}^{\mathrm{et}}$ by deciding that for $U \in \mathbf{Op}_X$ and $s \in \mathcal{F}(U)$ the subset

$$\{s_x \in \mathcal{F}_x\}$$

is open in $\mathcal{F}^{\mathrm{et}}$.

Then the map $\pi : \mathcal{F}^{\mathrm{et}} \longrightarrow X$ mapping $\mathcal{F}_x$ to x is continuous and a local homeomorphism. One easily checks that the topology on $\mathcal{F}^{\mathrm{et}}$ is the coarsest making the sections $s \in \mathcal{F}(U)$ of $\pi_{|U}$ continuous.

Remark 4.5.1. Even if X is Hausdorff in general $\mathcal{F}^{\mathrm{et}}$ is not. Consider for example the presheaf $\mathcal{F}(U) = 0$ if $0 \notin U$ and $\mathcal{F}(U) = \mathbb{Z}/2\mathbb{Z}$ if $0 \in U$ on $\mathbb{R}$. The associated space $\pi : \mathcal{F}^{\mathrm{et}} \longrightarrow \mathbb{R}$ is a global homeomorphism on $\mathbb{R}^*$ with fiber $\mathbb{Z}/2\mathbb{Z}$ over 0.

With these notations $\mathcal{F}^+(U)$ is nothing else than the set of continous sections of $\mathcal{F}^{\mathrm{et}}$ over U. This provides the alternative definition :

Definition 4.5.2. *A sheaf on* X *is a pair* $(\mathcal{F}^{\mathrm{et}}, \pi : \mathcal{F}^{\mathrm{et}} \longrightarrow X)$ *satisfying :*

(a) $\mathcal{F}^{\mathrm{et}} \in \mathbf{Top}$.
(b) $\pi : \mathcal{F}^{\mathrm{et}} \longrightarrow X$ *is a local homeomorphism (also called :* espace étalé*).*
(c) *every* $\mathcal{F}_x := \pi^{-1}(x)$ *is an Abelian group.*
(d) *The group theoretic operations are continuous.*

This viewpoint naturally generalizes the definition of a locally constant $\mathbb{Z}$-sheaf.

4.6. **Direct and inverse image.**

Definition 4.6.1. *Let $f : Y \longrightarrow X \in \mathbf{Top}$. Let $\mathcal{G} \in \mathbf{Sh}(Y)$. The presheaf*

$$U \mapsto \mathcal{G}(f^{-1}(U))$$

on X is a sheaf, denoted $f_*\mathcal{G}$ and called the direct image of $\mathcal{G}$.

Examples 4.6.2. (i) Let $p_X : X \longrightarrow *$ the map to the final object $* \in \mathbf{Top}$. Let $\mathcal{F} \in \mathbf{Sh}(X)$. Then $(p_X)_*\mathcal{F} = \Gamma(X, \mathcal{F}) = \mathcal{F}(X)$.

(ii) Let $X = Y = \mathbb{C}^*$ and $f : Y \longrightarrow X$ be the map $z \mapsto z^2$. If D is an open disk in X then $f^{-1}D$ is a disjoint union of two copies of D. Hence the sheaf $(f_*A_Y)_{|D}$ is isomorphic to A_D^2, the constant sheaf of rank two on D. However $\Gamma(X, f_*A_Y) = \Gamma(Y, A_Y) = A$ thus f_*A_Y is locally constant but not constant.

Definition 4.6.3. *Let $f : Y \longrightarrow X \in \mathbf{Top}$. Let $\mathcal{F} \in \mathbf{Sh}(X)$. The inverse image $f^{-1}\mathcal{F} \in \mathbf{Sh}(Y)$ is the sheaf on Y associated to the presheaf :*

$$V \mapsto \mathrm{colim}_{U \supset f(V)}\mathcal{F}(U) \quad .$$

Examples 4.6.4. (i) Let $i_x = \{x\} \hookrightarrow X$. Then $i_x^{-1}\mathcal{F} \simeq \mathcal{F}_x$.

(ii) Let A be an Abelian group, thus $A \in \mathbf{Ab} \simeq \mathbf{Sh}(*)$. Then $A_X = (p_X)^{-1}A$. In particular if $f : Y \longrightarrow X$ then $A_Y \simeq f^{-1}A_X$.

One defines in an obvious way the direct and the inverse image of morphisms. Hence we get two functors :

$$f_* : \mathbf{Sh}(Y) \longrightarrow \mathbf{Sh}(X) \quad ,$$
$$f^{-1} : \mathbf{Sh}(X) \longrightarrow \mathbf{Sh}(Y) \quad .$$

Lemma 4.6.5. $(f^{-1}\mathcal{F})_y = \mathcal{F}_{f(y)}$. *In particular the functor $f^{-1} : \mathbf{Sh}(X) \longrightarrow \mathbf{Sh}(Y)$ is exact.*

Proof. Follows immediately from the definitions. □

Proposition 4.6.6. $f^{-1} : \mathbf{Sh}(X) \overrightarrow{\longleftarrow} \mathbf{Sh}(Y) : f_*$ *is an adjonction i.e. there is a natural isomorphism*

$$\mathrm{Hom}_{\mathbf{Sh}(Y)}(f^{-1}\mathcal{F}, \mathcal{G}) \simeq \mathrm{Hom}_{\mathbf{Sh}(X)}(\mathcal{F}, f_*\mathcal{G}) \quad .$$

Proof. Let us define morphisms of functors :

$$(1) \qquad f^{-1} \circ f_* \longrightarrow \mathrm{id} \qquad \text{and} \qquad \mathrm{id} \longrightarrow f_* \circ f^{-1} \quad .$$

For the first one : notice that

$$\begin{aligned}(f^{-1} \circ f_*)(\mathcal{G})(U) &= \mathrm{colim}_{V \supset f(U)} f_*(\mathcal{G})(V) \\ &= \mathrm{colim}_{V \supset f(U)}\mathcal{G}(f^{-1}(V)) \quad .\end{aligned}$$

As $V \supset f(U)$ one has $U \subset f^{-1}(V)$ and there are natural restriction maps $\mathcal{G}(f^{-1}(V)) \longrightarrow \mathcal{G}(U)$. These maps are compatible with the colimit. This defines $f^{-1} \circ f_* \longrightarrow \mathrm{id}$.

For the second one :

$$\begin{aligned}(f_* \circ f^{-1})(\mathcal{F})(U) &= (f^{-1}(\mathcal{F}))(f^{-1}(U)) \\ &= \mathrm{colim}_{V \supset f(f^{-1}(U))}\mathcal{F}(V) \quad .\end{aligned}$$

As $U \supset f(f^{-1}(U)$ there is a natural map $\mathcal{F}(U) \longrightarrow \mathrm{colim}_{V \supset f(f^{-1}(U))}\mathcal{F}(V)$ which defines $\mathrm{id} \longrightarrow f_* \circ f^{-1}$.

We thus have homomorphisms :

$$\mathrm{Hom}(\mathcal{F}, f_*\mathcal{G}) \xrightarrow{\alpha} \mathrm{Hom}(f^{-1}\mathcal{F}, f^{-1} \circ f_*\mathcal{G}) \xrightarrow{\beta} \mathrm{Hom}(f^{-1}\mathcal{F}, \mathcal{G}) \quad ,$$

$$\mathrm{Hom}(f^{-1}\mathcal{F}, \mathcal{G}) \xrightarrow{\gamma} \mathrm{Hom}(f_*f^{-1}, f_*\mathcal{G}) \xrightarrow{\delta} \mathrm{Hom}(\mathcal{F}, f_*\mathcal{G}) \quad ,$$

where α and γ are defined in the obvious way and β and δ are deduced from (1).

One checks easily that $\beta \circ \alpha$ and $\delta \circ \gamma$ are inverse one to each other. □

Corollary 4.6.7. *The functor $f_* : \mathbf{Sh}(Y) \longrightarrow \mathbf{Sh}(X)$ is left-exact and maps injectives to injectives.*

Proof. The first property follows formally from the fact that f_* is right adjoint to the right-exact functor f^{-1}.

The second one follows formally from the fact that f_* is right adjoint to the left-exact functor f^{-1}. Indeed let $I \in \mathbf{Sh}(Y)$ an injective object. We would like to complete the diagram in $\mathbf{Sh}(X)$:

$$\begin{tikzcd} A \arrow[r, hook] \arrow[dr] & B \arrow[d, dashed] \\ & f_*I \end{tikzcd} .$$

By the adjunction property and as f^{-1} is left-exact this is equivalent to the completion of the following diagram in $\mathbf{Sh}(Y)$, which holds as I in injective :

$$\begin{tikzcd} f^{-1}A \arrow[r, hook] \arrow[dr] & f^{-1}B \arrow[d, dashed] \\ & I \end{tikzcd} .$$

□

We can say more if we consider $i : Z \hookrightarrow X$ a closed subspace :

Lemma 4.6.8. *Let $i : Z \hookrightarrow X$ be a closed subspace. Then $i_* : \mathbf{Sh}(Z) \longrightarrow \mathbf{Sh}(X)$ is exact.*

Proof. By definition for $\mathcal{F} \in \mathbf{Sh}(Z)$ and $x \in X$ we have

$$(i_*\mathcal{F})_x = \mathrm{colim}_{U \ni x}\mathcal{F}(Z \cap U) \ .$$

Hence

$$(i_*\mathcal{F})_x = \begin{cases} \mathcal{F}_x & \text{if } x \in Z \\ 0 & \text{if } x \notin Z \ . \end{cases}$$

The result follows immediately. □

Corollary 4.6.9. *Let $i : Z \hookrightarrow X$ be a closed subspace. Then $H^\bullet(X, i_*\mathcal{F}) = H^\bullet(Z, \mathcal{F})$ for any $\mathcal{F} \in \mathbf{Sh}(Z)$.*

Proof. Choose $\mathcal{F} \longrightarrow I^\bullet$ an injective resolution in $\mathbf{Sh}(Z)$. Then $i_*\mathcal{F} \longrightarrow i_*I^\bullet$ is still a resolution as i_* is exact, injective as i_* maps injective to injective. □

Lemma 4.6.10. *Direct and inverse images are compatible with composition :*

$$(f \circ g)_* = f_* \circ g_* \qquad \textit{and} \qquad (f \circ g)^{-1} = g^{-1} \circ f^{-1} \ .$$

Proof. The first equality follows easily from the definition, the second by adjunction. □

4.7. **Restriction of sheaves and direct image.** Let $j : W \longrightarrow X$ be the inclusion of any subset W of X with the induced topology. We define

$$\mathcal{F}_{|W} := j^{-1}\mathcal{F} \qquad \text{and} \qquad \Gamma(W, \mathcal{F}) := \Gamma(W, F_{|W}) \ .$$

Notice that these definitions agree with the previous ones if W is open in X.

The morphism $\mathcal{F} \longrightarrow j_*j^{-1}\mathcal{F}$ defines a morphism $p_{X*}\mathcal{F} \longrightarrow p_{W*}j^{-1}\mathcal{F}$ i.e. a morphism $\Gamma(X, \mathcal{F}) \longrightarrow \Gamma(W, \mathcal{F})$. Replacing X par U open containing W we get a natural morphism

$$\mathrm{colim}_{U \supset W}\Gamma(U, \mathcal{F}) \longrightarrow \Gamma(W, \mathcal{F}) \ . \tag{2}$$

Notice that this morphism is always injective : if $s \in \Gamma(U, \mathcal{F})$ is zero in $\Gamma(W, F)$ this implies $s_x = 0$ for all $x \in W$, hence $s = 0$ on a neighbourhood of W. This morphism is not an isomorphism in general but will be under reasonable assumptions :

Definition 4.7.1. *Recall that $W \subset X$ is relatively Hausdorff if two distinct points in W admit disjoint neighbourhoods in X. If $W = X$ one says that X is Hausdorff. Recall the convention that a compact set is Hausdorff.*

A paracompact space is a Hausdorff space such that each open covering admits a refinement which is locally finite. Closed subspaces of paracompact spaces, as well as locally compact spaces countable at infinity or metrizable spaces are paracompact.

A normal space is a Haudorff space such that for any closed disjoint subsets A and B there exist disjoint open sets $U \supset A$ and $V \supset B$.

Any paracompact space is normal but there exist normal spaces which are not paracompact.

Proposition 4.7.2. *Assume one of the following :*

(i) $W \in \mathbf{Op}_X$.
(ii) W *is a relatively Hausdorff compact subset of* X.
(iii) W *is closed and* X *is paracompact.*

Then the natural morphism

$$\mathrm{colim}_{U \supset W} \Gamma(U, \mathcal{F}) \longrightarrow \Gamma(W, \mathcal{F})$$

is an isomorphism.

Proof. Case (i) is obvious.

For case (ii) : let $s \in \Gamma(W, \mathcal{F})$. As W is compact and by definition of $\Gamma(W, \mathcal{F})$ there exists a finite family of open subsets $(U_i)_{1 \leq i \leq n}$ of X covering W and sections $s_i \in \Gamma(U_i, \mathcal{F})$ such that $s_{i|U_i \cap W} = s_{|U_i \cap W}$. Moreover we can find another family of open set $(V_i)_{1 \leq i \leq n}$ covering W such that $W \cap \overline{V_i} \subset U_i$. We shall glue the sections S_i on a neighbourhood of W. We argue by induction on the number n of the U_i's. Thus let $n = 2$ and define $W_i = W \cap \overline{V_i}$. Then $s_{1|W_1 \cap W_2} = s_{2|W_1 \cap W_2}$. Let E be an open subset of X such that $s_{1|E} = s_{2|E}$. We may find open subsets $E_i \subset U_i$, $i = 1, 2$, such that $W \supset E_1 \cup E \cup E_2$ and $E_1 \cap E_2 = \emptyset$. Set $E'_i = E_i \cup E$. Then $s_{1|E'_1 \cap E'_2} = s_{2|E'_1 \cap E'_2}$. This defines $t \in \Gamma(E'_1 \cup E'_2, \mathcal{F})$ with image s in $\Gamma(W, \mathcal{F})$.

For case (iii) : adapt case (ii), exercice (cf. [15]). $\square$

Let $f : Y \longrightarrow X$ a continuous map and $\mathcal{G} \in \mathbf{Sh}(Y)$. Let $x \in X$. The natural morphism

$$\mathrm{colim}_{U \ni x} \Gamma(f^{-1}U, \mathcal{G}) \longrightarrow \Gamma(f^{-1}(x), \mathcal{G}_{|f^{-1}(x)})$$

defines a morphism

$$(f_* \mathcal{G})_x \longrightarrow \Gamma(f^{-1}(x), \mathcal{G}_{|f^{-1}(x)}) \quad . \tag{3}$$

This morphism is not an isomorphism in general.

Example 4.7.3. Let $f : U \hookrightarrow X$ be an open inclusion. Choose $x \in \overline{U} \setminus U$. Then $f^{-1}(x) = \emptyset$ and $\Gamma(f^{-1}(x), \mathcal{G}_{|f^{-1}(x)} = 0$ but $(f_* \mathcal{G})_x = \mathrm{colim}_{V \ni x} \Gamma(U \cap V, \mathcal{G})$ is non-zero in general.

Definition 4.7.4. *A morphism $f : Y \longrightarrow X$ is called proper if f is closed (i.e. the image of any closed subset of Y is closed in X) and its fiber are relatively Hausdorff and compact.*

If X and Y are locally compact then f is proper if and only if the preimage of a compact subset of X is compact in Y. If $X = *$ then f is proper if and only if Y is compact.

Proposition 4.7.5. *Assume that $f : Y \longrightarrow X$ is proper and let $\mathcal{G} \in \mathbf{Sh}(Y)$. Then the morphism*

$$(f_* \mathcal{G})_x \longrightarrow \Gamma(f^{-1}(x), \mathcal{G}_{|f^{-1}(x)})$$

is an isomorphism.

Proof. When U ranges over the family of open neighbourhoods of x then $f^{-1}(U)$ ranges over a neighbourhood system of $f^{-1}(X)$. Hence

$$\mathrm{colim}_{U \ni x} \Gamma(f^{-1}(U), \mathcal{G}) \simeq \Gamma(f^{-1}(x), \mathcal{G}_{|f^{-1}(x)})$$

by proposition 4.7.2[(ii)]. $\square$

As a corollary one obtains the following generalization of lemma 4.6.8 :

Corollary 4.7.6. *If f is finite (i.e. proper and the preimage of any point is a finite set) then the functor f_* is exact.*

4.8. **Internal Hom and tensor product.** Let $\mathcal{F}, \mathcal{F}', \mathcal{F}'' \in \mathbf{Sh}(X)$.

Lemma 4.8.1. *The presheaf $\mathcal{H}om(\mathcal{F}, \mathcal{F}')$ defined by*

$$\mathcal{H}om(\mathcal{F}, \mathcal{F}')(U) = \mathrm{Hom}_{\mathbf{Sh}(U)}(\mathcal{F}_{|U}, \mathcal{F}'_{|U})$$

is a sheaf.

Proof. Easy exercice. □

In general the morphism $\mathcal{H}om(\mathcal{F}, \mathcal{F}')_x \longrightarrow \mathrm{Hom}(\mathcal{F}_x, \mathcal{F}'_x)$ is neither injective nor surjective.

By definition one has $\Gamma(\mathcal{H}om(\mathcal{F}, \mathcal{F}')) = \mathrm{Hom}(\mathcal{F}, \mathcal{F}')$. As the bifunctor $\mathrm{Hom}(\cdot, \cdot)$ is left-exact, this implies that the bifonctor

$$\mathcal{H}om(\cdot, \cdot) : \mathbf{Sh}(X)^{\mathrm{op}} \times \mathbf{Sh}(X) \longrightarrow \mathbf{Sh}(X)$$

is left-exact.

Definition 4.8.2. *We define $\mathcal{F} \otimes \mathcal{F}'$ as the sheaf associated to the presheaf*

$$U \mapsto \mathcal{F}(U) \otimes \mathcal{F}'(U) \quad .$$

Since tensor products commute with colimits we get that

$$(\mathcal{F} \otimes \mathcal{F}')_x = \mathcal{F}_x \otimes \mathcal{F}'_x$$

hence

$$\cdot \otimes \cdot : \mathbf{Sh}(X) \otimes \mathbf{Sh}(X) \longrightarrow \mathbf{Sh}(X)$$

is right-exact.

Most classical formulas relating $\otimes$ and Hom for $\mathbb{Z}$-modules have their counterpart for $\mathbf{Sh}(X)$. In particular one has an adjunction formula :

$$\begin{aligned}\mathcal{H}om(\mathcal{F} \otimes \mathcal{F}', \mathcal{F}'') &\simeq \mathcal{H}om(\mathcal{F}, \mathcal{H}om(\mathcal{F}', \mathcal{F}'')) \\ &\simeq \mathcal{H}om(\mathcal{F}', \mathcal{H}om(\mathcal{F}, \mathcal{F}'')) \quad .\end{aligned}$$

In particular $\mathrm{Hom}(\mathcal{F} \otimes \mathcal{F}', \mathcal{F}'') \simeq \mathrm{Hom}(\mathcal{F}, \mathcal{H}om(\mathcal{F}', \mathcal{F}''))$.

The adjunction between direct and inverse images generalizes as

$$\mathcal{H}om(\mathcal{F}, f_*\mathcal{G}) \simeq f_*\mathcal{H}om(f^{-1}\mathcal{F}, \mathcal{G}) \quad .$$

4.9. **Direct image with proper support.**

Definition 4.9.1. *Let $\mathcal{F} \in \mathbf{Sh}(X)$ and $U \in \mathbf{Op}_X$. The support of a section $s \in \mathcal{F}(U)$ is the closed set*

$$\mathrm{supp}\,(s) = \{x \in U \;/\; s_x \neq 0\} \quad .$$

Definition 4.9.2. *Let $f : Y \longrightarrow X \in \mathbf{Top}$.*

We define the sub-presheaf $f_!\mathcal{G} \subset f_*\mathcal{G}$ by

$$f_!\mathcal{G}(V) = \{s \in \mathcal{G}(f^{-1}(V)) \;/\; f_{|\mathrm{supp}\, s} : \mathrm{supp}\, s \longrightarrow V \text{ is proper}\} \quad .$$

Proposition 4.9.3. *$f_!\mathcal{G}$ is a sheaf.*

Proof. It is enough to show that given $(U_i)_{i \in I}$ a family of open subsets of X with union U, and S a closed subset of $f^{-1}(U)$ such that $f : S \cap f^{-1}(U_i) \longrightarrow U_i$ is proper for all $i \in I$ then $f : S \longrightarrow U$ is proper. Any fiber of $f : S \longrightarrow U$ is the fiber of some $f : S \cap f^{-1}(U_i) \longrightarrow U_i$, thus is relatively Hausdorff and compact. Hence it is enough to check that $f : S \longrightarrow U$ is closed. This follows once more from the fact that the maps $f : S \cap f^{-1}(U_i) \longrightarrow U_i$, $i \in I$, are closed. □

Remarks 4.9.4.
1. notice that if f is proper (for example if Y and X are compact, or $f : Z \hookrightarrow X$ is a closed subspace) then $f_! = f_*$.
2. As $f_!$ is a subfunctor of f_* it is left exact.
3. One showed that $(f \circ g)_* = f_* \circ g_*$. Similarly one can check that $(f \circ g)_! = f_! \circ g_!$.

Definition 4.9.5. *The sections of $\mathcal{G}$ with proper support are defined by*

$$\Gamma_c(Y, \mathcal{G}) := p_{Y!}\mathcal{G} \quad .$$

Equivalently : $\Gamma_c(Y, \mathcal{G}) = \{s \in \Gamma(Y, \mathcal{G}) \ / \ \mathrm{supp}\,(s) \text{ is compact and relatively Hausdorff}\}$.

Remark 4.9.6. In particular $\Gamma_c(X, f_!\mathcal{G}) = \Gamma_c(Y, \mathcal{G})$.

Last time we proved that the natural morphism

$$(f_*\mathcal{G})_x \longrightarrow \Gamma(f^{-1}(x), \mathcal{G}_{|f^{-1}(x)}) \tag{4}$$

is an isomorphism if f is proper. Similarly one obtains :

Proposition 4.9.7. *Let $f : Y \longrightarrow X \in$ **Top**, and $\mathcal{G} \in$ **Sh**(Y). Assume that X and Y are locally compact spaces (in particular Hausdorff).*

Then for any $x \in X$ the natural morphism

$$(f_!\mathcal{G})_x \longrightarrow \Gamma_c(f^{-1}(x), \mathcal{G}_{|f^{-1}(x)})$$

is an isomorphism.

Proof. First notice that the natural morphism

$$(f_!\mathcal{G})_x \xrightarrow{\alpha} \Gamma_c(f^{-1}(x), \mathcal{G}_{|f^{-1}(x)}) \tag{5}$$

is nothing else than the restriction of the morphism (4) :

$$\begin{tikzcd}
(f_!\mathcal{G})_x \arrow[r, dashed] \arrow[d, hook] & \Gamma_c(f^{-1}(x), \mathcal{G}_{|f^{-1}(x)}) \arrow[d, hook] \\
(f_*\mathcal{G})_x \arrow[r] & \Gamma((f^{-1}(x), \mathcal{G}_{|f^{-1}(x)})
\end{tikzcd} \quad .$$

As we already showed in full generality that the morphism (4) is injective, the morphism (5) too.

Next we show that α is surjective. Let $s \in \Gamma_c(f^{-1}(x), \mathcal{G}_{|f^{-1}(x)})$ and put $K = \mathrm{supp}\,(s)$. As K is compact and Y is Hausdorff, by proposition **??**(ii) there exists an open neighbourhood U of K in Y and a section $t \in \Gamma(U, \mathcal{G})$ such that $t_{|K} = s_{|K}$. By shrinking U we may assume $t_{|U \cap f^{-1}(x)} = s_{|U \cap f^{-1}(x)}$. Let V be a relatively compact open neigbourhood of K with $\overline{V} \subset U$. Since $x \notin f(\overline{V} \cap \mathrm{supp}\,(t) \setminus V)$ there exists an open neighbourhood W of x such that $f^{-1}(W) \cap \overline{V} \cap \mathrm{supp}\,(t) \subset V$. One defines $u \in \Gamma(f^{-1}(W), \mathcal{G})$ by

$$\begin{cases} u_{|f^{-1}(W) \setminus (\mathrm{supp}\,(t) \cap \overline{V})} = 0 \ , \\ u_{|f^{-1}(W) \cap V} = t_{|f^{-1}(W) \cap V} \ . \end{cases}$$

Since $\mathrm{supp}\,(u)$ is contained in $f^{-1}(W) \cap \mathrm{supp}\,(t) \cap \overline{V}$ the map f is proper on this set. Moreover $u_{|f^{-1}(x)} = s$. ∎

Recall that a subspace of a locally compact space is locally compact for the induced topology if and only if it is locally closed, where :

Definition 4.9.8. *A subspace $Z \hookrightarrow X$ is said* locally closed *if any point $z \in Z$ has a neighbourhood U in X such that $U \cap Z$ is closed relative to U.*

Equivalently $Z = U \cap W$ with U open in X and W closed in X; or Z is open in its closure $\overline{Z}$.

Proposition 4.9.7 has the following consequence :

Corollary 4.9.9. *If X is locally compact and $j : Y \hookrightarrow X$ is a locally closed subspace then $j_!$ is exact.*

Proof. Proposition 4.9.7 implies in this case that

$$(j_!\mathcal{G})_x = \begin{cases} \mathcal{G}_x & \text{if } x \in Y \ , \\ 0 & \text{otherwise} \end{cases}$$

and the result. ∎

Notice that proposition 4 can be read as $i_x^{-1} f_! \mathcal{G} = f_{|f^{-1}(x)!} i_{f^{-1}(x)}^{-1} \mathcal{G}$ for the diagram

$$\begin{tikzcd} f^{-1}(x) \arrow[r, "f_{|f^{-1}(x)}"] \arrow[d, hook, "i_{f^{-1}(x)}"'] & \{x\} \arrow[d, hook, "i_x"] \\ Y \arrow[r, "f"'] & X \end{tikzcd}$$

We generalize it to the following :

Proposition 4.9.10 (proper base change)**.** *Let*

$$\begin{tikzcd} Y' \arrow[r, "f'"] \arrow[d, "g'"'] & X' \arrow[d, "g"] \\ Y \arrow[r, "f"'] & X \end{tikzcd}$$

be a Cartesian square of locally compact spaces. Then

$$g^{-1} \circ f_! = f'_! \circ g'^{-1} \quad .$$

Proof. We first construct a canonical morphism

$$f_! \circ g'_* \longrightarrow g_* \circ f'_! \quad .$$

Let $\mathcal{G}' \in \mathbf{Sh}(Y')$. Let $V \in \mathbf{Op}_X$. A section $t \in \Gamma(V, f_! \circ g'_*(\mathcal{G}'))$ is a section $t \in \Gamma(f^{-1}(V), g'_*\mathcal{G}')$ such that $f : \mathrm{supp}\,(t) \longrightarrow V$ is proper. Equivalently this is a section $s \in \Gamma((f \circ g')^{-1}(V), \mathcal{G})$ with $\mathrm{supp}\,(s) \subset g'^{-1}(Z)$ for $f^{-1}(V) \supset Z \overset{f \text{ proper}}{\longrightarrow} V$. But then $g'^{-1}(Z) \longrightarrow g^{-1}(V)$ is proper thus s defines a section of $g_* f'_! \mathcal{G}'$.

By adjunction :

$$\mathrm{Hom}(g^{-1} f_! \mathcal{G}, f'_! g'^{-1} \mathcal{G}) = \mathrm{Hom}(f_! \mathcal{G}, g_* f'_! g'^{-1} \mathcal{G}) \quad .$$

The morphism

$$f_! \longrightarrow f_! g'_* g'^{-1} \longrightarrow g_* f'_! g'^{-1} \quad ,$$

where the first map comes from the adjunction $1 \longrightarrow g'_* g'^{-1}$, gives the required canonical morphism

$$g^{-1} \circ f_! = f'_! \circ g'^{-1} \quad .$$

To show this is an isomorphism we compute the stalks at a point $x' \in X'$: Then

$$\begin{aligned} (g^{-1} f_! \mathcal{G})_{x'} &= (f_! \mathcal{G})_{g(x')} \\ &= \Gamma_c(f^{-1}(g(x')), \mathcal{G}) \end{aligned}$$

by proposition 4.9.7.

The map g' induces a homeomorphism $f'^{-1}(x') \simeq f^{-1}(g(x'))$ and an isomorphism

$$\Gamma_c(f^{-1}(g(x')), \mathcal{G}) \simeq \Gamma_c(f'^{-1}(x'), g'^{-1} \mathcal{G}) \simeq (f'_! g'^{-1} \mathcal{G})_{x'} \quad .$$

□

4.10. **Locally closed subspaces : the functors $(\cdot)_Z$, Γ_Z, and $j^!$.** Let $j : Z \hookrightarrow X$ a locally closed subspace of a locally compact space X. We already noticed that $j_!$ is exact in this case. Moreover the definition of $j_!$ is equivalent to :

$$\Gamma(U, j_! \mathcal{G}) = \{ s \in \Gamma(Z \cap U, \mathcal{G}) \ / \ \mathrm{supp}\,(s) \, \text{is closed relative to U} \} \quad .$$

Definition 4.10.1. *Let $\mathcal{F} \in \mathbf{Sh}(X)$. The support of $\mathcal{F}$ is*

$$\mathrm{supp}\, \mathcal{F} = \{ x \in X \ / \ \mathcal{F}_x \neq 0 \} \quad .$$

Proposition 4.10.2. *The functor $j_! : \mathbf{Sh}(Z) \longrightarrow \mathbf{Sh}(X)$ defines an equivalence of categories*

$$j_! : \mathbf{Sh}(Z) \overset{\longleftarrow}{\longrightarrow} \mathbf{Sh}_Z(X) : j^{-1} \quad ,$$

where $\mathbf{Sh}_Z(X)$ denotes the full subcategory of $\mathbf{Sh}(X)$ of sheaves with support in Z.

Proof. First notice that for any sheaf $\mathcal{G} \in \mathbf{Sh}(Z)$ then $j^{-1}j_!\mathcal{G} = \mathcal{G}$.

Next, let $\mathcal{F} \in \mathbf{Sh}_Z(X)$. One easily checks that the adjunction morphism $\mathcal{F} \longrightarrow j_*j^{-1}\mathcal{F}$ factorizes through $j_!j^{-1}\mathcal{F}$, yielding an isomorphism $\mathcal{F} \simeq j_!j^{-1}\mathcal{F}$. $\square$

4.10.1. *Restriction functor $(\cdot)_Z$.*

Definition 4.10.3. *We define the functor $(\cdot)_Z : \mathbf{Sh}(X) \longrightarrow \mathbf{Sh}(X)$ by*

$$\forall \mathcal{F} \in \mathbf{Sh}(X), \quad \mathcal{F}_Z := j_!j^{-1}\mathcal{F} \quad .$$

As both functors $j_!$ and j^{-1} are exact, the functor $(\cdot)_Z$ is exact too. If Z is closed in X then $\mathcal{F}_Z = j_*j^{-1}$ thus one has a morphism $\mathcal{F} \longrightarrow \mathcal{F}_Z$. If Z is open one easily checks that $\mathcal{F}_Z = \ker(\mathcal{F} \longrightarrow \mathcal{F}_{X\setminus Z})$ thus one has a morphism $\mathcal{F}_Z \longrightarrow \mathcal{F}$.

Proposition 4.10.4. (i) *The sheaf $\mathcal{F}_Z$ is uniquely characterized by the properties :*

$$(\mathcal{F}_Z)_{|Z} = \mathcal{F}_{|Z} \qquad \textit{and} \qquad (\mathcal{F}_Z)_{|X\setminus Z} = 0 \quad . \tag{6}$$

(ii) *For any other locally closed subspace $Z' \hookrightarrow X$ one has :*

$$(\mathcal{F}_Z)_{Z'} \simeq \mathcal{F}_{Z\cap Z'} \quad .$$

(iii) *If $Z' \subset Z$ is closed one has an exact sequence :*

$$0 \longrightarrow \mathcal{F}_{Z\setminus Z'} \longrightarrow \mathcal{F}_Z \longrightarrow \mathcal{F}_{Z'} \longrightarrow 0 \quad .$$

(iv) *If Z_1, Z_2 are two closed subsets of X the following sequence is exact :*

$$0 \longrightarrow \mathcal{F}_{Z_1\cup Z_2} \overset{(\alpha_1,\alpha_2)}{\longrightarrow} \mathcal{F}_{Z_1} \oplus \mathcal{F}_{Z_2} \overset{(\beta_1,-\beta_2)}{\longrightarrow} \mathcal{F}_{Z_1\cap Z_2} \longrightarrow 0 \quad .$$

(v) *If U_1, U_2 are two open subsets of X the following sequence is exact :*

$$0 \longrightarrow \mathcal{F}_{U_1\cap U_2} \longrightarrow \mathcal{F}_{U_1} \oplus \mathcal{F}_{U_2} \longrightarrow \mathcal{F}_{U_1\cup U_2} \longrightarrow 0 \quad .$$

4.10.2. *The functor Γ_Z.* Let $U \subset X$ be open and $Z \subset U$ be closed. One defines :

$$\Gamma_Z(U, \mathcal{F}) := \{s \in \mathcal{F}(U) \;/\; \mathrm{supp}\,(s) \subset Z\} (= \ker(\mathcal{F}(U) \longrightarrow \mathcal{F}(U \setminus Z))) \quad .$$

If $Z \subset V \subset U$ then the canonical morphism $\Gamma_Z(U, \mathcal{F}) \longrightarrow \Gamma_Z(V, F)$ is an isomorphism. Thus for Z locally closed we may define $\Gamma_Z(X, \mathcal{F})$ as $\Gamma_Z(U, \mathcal{F})$ where U is any open subset of X containing Z as a closed subset.

Definition 4.10.5. *One denotes by $\Gamma_Z(\mathcal{F})$ the sheaf $U \mapsto \Gamma_{Z\cap U}(U, \mathcal{F})$ and calls it the sheaf of sections of $\mathcal{F}$ supported by Z.*

Proposition 4.10.6. *Let $j : Z \hookrightarrow X$ be a locally closed subset of X and $\mathcal{F} \in \mathbf{Sh}(X)$. Then :*

(1) *The functors $\Gamma_Z(X, \cdot) : \mathbf{Sh}(X) \longrightarrow \mathbf{Ab}$ and $\Gamma_Z : \mathbf{Sh}(X) \longrightarrow \mathbf{Sh}(X)$ are left exact. Moreover*

$$\Gamma_Z(X, \cdot) = \Gamma(X, \cdot) \circ \Gamma_Z(\cdot) \quad .$$

(ii) *Let Z' be another locally closed subset of X. Then :*

$$\Gamma_{Z'}(\cdot) \circ \Gamma_Z(\cdot) = \Gamma_{Z\cap Z'}(\cdot) \quad .$$

(iii) *Assume that Z is open in X. Then*

$$\Gamma_Z = j_* \circ j^{-1} \quad .$$

(iv) *Let $Z' \subset Z$ be closed. Then the following sequence is exact :*

$$0 \longrightarrow \Gamma_{Z'}(\mathcal{F}) \longrightarrow \Gamma_Z(\mathcal{F}) \longrightarrow \Gamma_{Z\setminus Z'}(\mathcal{F}) \quad .$$

(v) *Let U_1, U_2 be two open subsets of X. Then the sequence*

$$0 \longrightarrow \Gamma_{U_1 \cup U_2}(\mathcal{F}) \overset{(\alpha_1,\alpha_2)}{\longrightarrow} \Gamma_{U_1}(\mathcal{F}) \oplus \Gamma_{U_2}(\mathcal{F}) \overset{(\beta_1,-\beta_2)}{\longrightarrow} \Gamma_{U_1 \cap U_2}(\mathcal{F})$$

is exact.

(vi) *Let Z_1, Z_2 be two closed subsets of X. Then the sequence*

$$0 \longrightarrow \Gamma_{Z_1 \cap Z_2}(\mathcal{F}) \overset{(\alpha_1,\alpha_2)}{\longrightarrow} \Gamma_{Z_1}(\mathcal{F}) \oplus \Gamma_{Z_2}(\mathcal{F}) \overset{(\beta_1,-\beta_2)}{\longrightarrow} \Gamma_{Z_1 \cap Z_2}(\mathcal{F})$$

is exact.

Proof. Easy exercice. □

4.10.3. *Link with $\mathcal{H}om$ and $\otimes$.* The functors $(\cdot)_Z$ and Γ_Z can be obtained using $\mathbb{Z}_Z := (\mathbb{Z}_X)_Z$:

$$\mathcal{F}_Z \simeq \mathbb{Z}_Z \otimes \mathcal{F} \tag{7}$$

$$\Gamma_Z(\mathcal{F}) \simeq \mathcal{H}om(\mathbb{Z}_Z, \mathcal{F}) \tag{8}$$

$$(\mathcal{F} \otimes \mathcal{F}')_Z \simeq \mathcal{F}_Z \otimes \mathcal{F}' \simeq \mathcal{F} \otimes \mathcal{F}'_Z \tag{9}$$

$$\Gamma_Z(\mathcal{H}om(\mathcal{F}, \mathcal{F}')) \simeq \mathcal{H}om(\mathcal{F}, \Gamma_Z(\mathcal{F}')) \simeq \mathcal{H}om(\mathcal{F}_Z, \mathcal{F}') \tag{10}$$

4.10.4. *The functor $j^!$.*

Proposition 4.10.7. *Let $j^!\mathcal{F} := j^{-1}\Gamma_Z\mathcal{F}$. Then the pair $(j_!, j^!)$ defines an adjunction :*

$$j_! : \mathbf{Sh}(Z) \overset{\longleftarrow}{\longrightarrow} \mathbf{Sh}(X) : j^! \ .$$

Proof. Let $U \in \mathbf{Op}_X$. Then

$$\begin{aligned} \Gamma(U, j_!j^!\mathcal{F}) &= \Gamma(U, (j_!j^{-1})\Gamma_Z\mathcal{F}) \\ &= \Gamma(U, \Gamma_Z\mathcal{F}) \\ &= \{s \in \Gamma(U, \mathcal{F}), \ / \operatorname{supp}(s) \subset Z\} \ . \end{aligned}$$

This formula defines a monomorphism $j_!j^! \longrightarrow 1$. Moreover it implies that any morphism $\mathcal{G} \longrightarrow \mathcal{F}$ where $\mathcal{G} \in \mathbf{Sh}_Z(X)$ factorizes through $j_!j^!\mathcal{F} \longrightarrow \mathcal{F}$. For a sheaf $\mathcal{G} \in \mathbf{Sh}(Z)$ we get accordingly

$$\mathrm{Hom}(j_!\mathcal{G}, \mathcal{F}) \simeq \mathrm{Hom}(j_!\mathcal{G}, j_!j^!\mathcal{F}) \ .$$

From proposition 4.10.2 we have an isomorphism :

$$(j_!)^{-1} : \mathrm{Hom}(j_!\mathcal{G}, j_!j^!\mathcal{F}) \simeq \mathrm{Hom}(\mathcal{G}, j^!\mathcal{F}) \ .$$

By composing the two previous isomorphisms we get the result.

□

Corollary 4.10.8. *The functor $j^! : \mathbf{Sh}(X) \longrightarrow \mathbf{Sh}(Z)$ is left exact and carries injective sheaves into injective sheaves.*

Proof. Once more it follows formally from the fact that $j^!$ is right adjoint to the exact functor $j_!$. □

Lemma 4.10.9. *If $j : Z \longrightarrow X$ is open then $j^! = j^{-1}$.*

Proof. By definition $j^!\mathcal{F} = j^{-1}\Gamma_Z\mathcal{F}$. Using the inclusion $\Gamma_Z\mathcal{F} \longrightarrow \mathcal{F}$ this yields a monomorphism $j^!\mathcal{F} \longrightarrow j^{-1}\mathcal{F}$. One checks this is an isomorphism by localization. □

Corollary 4.10.10. *The inclusion $j : U \longrightarrow X$ of an open subset transform an injective sheaf I on X into an injective sheaf $j^!I$ on U.*

5. Cohomology and derived category of sheaves

Standard definitions and constructions in homological algebra (Abelian categories, derived categories, triangulated categories, derived functors, cf. [7], [15, chap. I]) are assumed to be vaguely familiar. We recall some basic facts of interest.

Definition 5.0.11. *Let $\mathcal{A}$, $\mathcal{B}$ be Abelian categories and $F : \mathcal{A} \longrightarrow \mathcal{B}$ an (additive) left-exact functor. Recall that :*

- *an object $I \in \mathcal{A}$ is called injective if the functor* $\mathrm{Hom}(\cdot, I) : \mathcal{A} \longrightarrow \mathbf{Ab}$ *is exact, equivalently if each diagram*

$$\begin{tikzcd} A \arrow[r, hook] \arrow[dr] & B \arrow[d, dashed, "\phi"] \\ & I \end{tikzcd}$$

  in $\mathcal{A}$ admits a completion ϕ.
- *the category $\mathcal{A}$ has enough injectives if any object $A \in \mathcal{A}$ admits a monomorphism $A \hookrightarrow I$, I injective.*
- *if $\mathcal{A}$ has enough injectives then $RF(A) := F(I^\bullet)$ in the derived category $D(\mathcal{B})$, for any injective resolution $A \simeq I^\bullet$ in $\mathcal{A}$.*

Proposition 5.0.12. *(Leray) Let $\mathcal{A}$, $\mathcal{B}$ be two Abelian categories and $\mathcal{F} : \mathcal{A} \longrightarrow \mathcal{B}$ a left-exact functor. We suppose that $\mathcal{A}$ has enough injectives.*

Recall that an object $L \in \mathcal{A}$ is called F-acyclic if $R^iF(L) = 0$, $i > 0$.

For any $A \in \mathcal{A}$ the object $RF(A) \in D(\mathcal{B})$ can be computed as $F(L^\bullet)$, where $A \simeq L^\bullet$ is an F-acyclic resolution.

Proof. We show by induction on $i \geq 0$ that $R^iF(A) = H^i(F(L^\bullet))$.

We have an exact sequence

$$0 \longrightarrow A \xrightarrow{d_0} L^0 \longrightarrow B \longrightarrow 0 \quad , \tag{11}$$

where B denotes the cokernel of d_0. On the other hand one has a resolution

$$0 \longrightarrow B \xrightarrow{d_1} L^1 \longrightarrow L^2 \longrightarrow \cdots \quad . \tag{12}$$

The short exact sequence (11) gives the long exact sequence

$$\cdots \longrightarrow R^iFA \longrightarrow R^iFL^0 \longrightarrow R^iFB \longrightarrow R^{i+1}FA \longrightarrow R^{i+1}FL^0 \longrightarrow \cdots \quad ,$$

which implies, as L^0 is F-acyclic :

$$\begin{cases} R^iFB \simeq R^{i+1}FA, & i \geq 1 \\ R^1FA = \mathrm{coker}(F(L^0) \longrightarrow F(B)) \end{cases} .$$

As F is left exact, the resolution (12) implies that

$$F(B) = \ker(F(L^1) \longrightarrow F(L^2)) \quad .$$

Thus $R^1FA = H^1(F(L^\bullet))$.

The equality $R^iFB \simeq R^{i+1}FA$ and the existence of the resolution (12) gives the result by induction. □

5.1. Injectives in $\mathbf{Sh}(X)$.

Lemma 5.1.1. *The Abelian category $\mathbf{Sh}(X)$ has enough injectives.*

Proof. We first use that $\mathbf{Ab}$ has enough injectives (injectives in $\mathbf{Ab}$ are divisible groups). Let $\mathcal{F} \in \mathbf{Sh}(X)$. For any $x \in X$ choose $\mathcal{F}_x \hookrightarrow \mathcal{I}_x$ with $\mathcal{I}_x$ injective in $\mathbf{Ab}$. Define

$$\mathcal{I}^0 := \prod_{x \in X} i_{x*} I_x \quad .$$

One thus has a canonical short exact sequence $0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{I}^0$.

Let us show that $\mathcal{I}^0$ is injective in $\mathbf{Sh}(X)$. For any $\mathcal{G} \in \mathbf{Sh}(X)$ one has

$$\begin{aligned}\mathrm{Hom}_{\mathbf{Sh}(X)}(\mathcal{G}, \mathcal{I}^0) &= \prod_{x \in X} \mathrm{Hom}_{\mathbf{Sh}(X)}(\mathcal{G}, i_{x*}I_x) \\ &= \prod_{x \in X} \mathrm{Hom}_{\mathbf{Ab}}(i_x^{-1}\mathcal{G} = \mathcal{G}_x, I_x) \ .\end{aligned}$$

Completing the diagram

$$\begin{tikzcd} \mathcal{G} \arrow[r, hook] \arrow[dr] & \mathcal{G}' \arrow[d, dashed, "\phi"] \\ & \mathcal{I}^0 \end{tikzcd}$$

is thus equivalent to completing all diagrams

$$\begin{tikzcd} \mathcal{G}_x \arrow[r, hook] \arrow[dr] & \mathcal{G}'_x \arrow[d, dashed, "\phi_x"] \\ & I_x \end{tikzcd} \ .$$

As each I_x is injective in $\mathbf{Ab}$ we get that $\mathcal{I}^0$ is injective.

□

What are the injectives in $\mathbf{Sh}(X)$?

Lemma 5.1.2 (Spaltenstein ?)**.** *A sheaf $\mathcal{I} \in \mathbf{Sh}(X)$ is injective if and only if for all $U \supset V \in \mathbf{Op}_X$, $\mathcal{I}(U)$ and $\mathcal{I}(V)$ are injectives in* $\mathbf{Ab}$ *and the restriction morphism $\mathcal{I}(U) \longrightarrow \mathcal{I}(V)$ is a (split) surjection in* $\mathbf{Ab}$.

Remark 5.1.3. Such a characterization is in fact valid for sheaves in any category of modules.

Proof. Let $\mathcal{I}$ be injective. By the previous lemma one can find an injective $\mathcal{J}$ of the form $\prod_{x\in X} i_{x*}J_x$ and an injection $\mathcal{I} \hookrightarrow \mathcal{J}$.

As $\mathcal{I}$ is injective, the morphism $\mathcal{I} \hookrightarrow \mathcal{J}$ splits, thus $\mathcal{I}$ is a direct factor of $\mathcal{J}$.

Notice that the injective $\mathcal{J}$ satisfies trivially the condition of the lemma, and that this condition is stable by passing to a direct factor.

For the converse we refer to [3, 1.13]. □

5.2. **Flasque sheaves.** In practice we won't work with injective sheaves. We introduce other classes of sheaves, which will be acyclic for many functors of interest (in particular we will use them for computing the corresponding derived functors).

Definition 5.2.1. *We relax the injectivity condition in* $\mathbf{Sh}(X)$ *as follows. A sheaf $\mathcal{F} \in \mathbf{Sh}(X)$ is flasque if for any open set $U \in \mathbf{Op}_X$ the restriction map $\mathcal{F}(X) \longrightarrow \mathcal{F}(U)$ is surjective.*

Remark 5.2.2.
(a) By functoriality of the restriction, a sheaf $\mathcal{F}$ is flasque if and only if for any pair $V \subset U$ of $\mathbf{Op}_X$ the restriction $\mathcal{F}(U) \longrightarrow \mathcal{F}(V)$ is surjective.
(b) Notice that by lemma 5.1.2 a sheaf of complex vector spaces is injective if and only if it flasque.

Lemma 5.2.3. *Let $\mathcal{F} \in \mathbf{Sh}(X)$ be flasque. Then :*
(i) *for any continuous map $f : X \longrightarrow E$ the sheaf $f_*\mathcal{F} \in \mathbf{Sh}(E)$ is flasque.*
(ii) *for any locally closed $Z \hookrightarrow X$ the sheaf $\Gamma_Z\mathcal{F} \in \mathbf{Sh}(X)$ is flasque.*

Proof. The first statement is clear by definition of f_*.

For the second one, let $U \in \mathbf{Op}_X$ such that Z is a closed subspace of U. As $\Gamma_Z\mathcal{F} = i_{U*}\Gamma_Z\mathcal{F}_{|U}$ one can thus assume that $Z \hookrightarrow X$ is closed.

We have to prove that for any $U \in \mathbf{Op}_X$ the restriction morphism

$$\Gamma_Z(X, \mathcal{F}) \longrightarrow \Gamma_Z(U, \mathcal{F})$$

is surjective. Let $s \in \Gamma_Z(U, \mathcal{F}) = \Gamma_{Z\cap U}(U, \mathcal{F}) = \ker(\Gamma(U, \mathcal{F}) \longrightarrow \Gamma(U \setminus U \cap Z, \mathcal{F})$. We first extend s by 0 to a section $t \in \Gamma((X \setminus Z) \cup U, \mathcal{F})$. As $\mathcal{F}$ is flasque we can then extend t to a section of $\Gamma(X, \mathcal{F})$, necessarily in $\Gamma_Z(X, \mathcal{F})$. □

Proposition 5.2.4. *Let $\mathcal{F} \in \mathbf{Sh}(X)$ be flasque. If $Z \hookrightarrow X$ is locally closed, $Z' \subset Z$ is closed, Z_1, Z_2 are closed in X and U_1, U_2 are open in X, then the sequences*

$$0 \longrightarrow \Gamma_{Z'}(\mathcal{F}) \longrightarrow \Gamma_Z(\mathcal{F}) \longrightarrow \Gamma_{Z\setminus Z'}(\mathcal{F}) \tag{13}$$

$$0 \longrightarrow \Gamma_{U_1\cup U_2}(\mathcal{F}) \xrightarrow{(\alpha_1,\alpha_2)} \Gamma_{U_1}(\mathcal{F}) \oplus \Gamma_{U_2}(\mathcal{F}) \xrightarrow{(\beta_1,-\beta_2)} \Gamma_{U_1\cap U_2}(\mathcal{F}) \tag{14}$$

$$0 \longrightarrow \Gamma_{Z_1\cap Z_2}(\mathcal{F}) \xrightarrow{(\alpha_1,\alpha_2)} \Gamma_{Z_1}(\mathcal{F}) \oplus \Gamma_{Z_2}(\mathcal{F}) \xrightarrow{(\beta_1,-\beta_2)} \Gamma_{Z_1\cup Z_2}(\mathcal{F}) \tag{15}$$

are also surjective on the right.

Proof. For any open set $U \in \mathbf{Op}_X$, $\Gamma(U, \Gamma_Z(\mathcal{F})) \longrightarrow \Gamma(U, \Gamma_{Z\setminus Z'}(\mathcal{F})) \simeq \Gamma(U \setminus Z', \Gamma_Z(\mathcal{F}))$ is surjective as $\Gamma_Z(\mathcal{F})$ is flasque by the previous lemma.

To show that $\Gamma_{U_1}(\mathcal{F}) \oplus \Gamma_{U_2}(\mathcal{F}) \xrightarrow{(\beta_1,-\beta_2)} \Gamma_{U_1\cap U_2}(\mathcal{F})$ is surjective, notice that the first factor $\Gamma_{U_1}(\mathcal{F}) \longrightarrow \Gamma_{U_1\cap U_2}(\mathcal{F})$ is already surjective as $\mathcal{F}$ is flasque.

To show that $\Gamma_{Z_1}(\mathcal{F}) \oplus \Gamma_{Z_2}(\mathcal{F}) \xrightarrow{(\beta_1,-\beta_2)} \Gamma_{Z_1\cap Z_2}(\mathcal{F})$ is surjective let $s \in \Gamma_{Z_1\cap Z_2}(X, \mathcal{F})$. We may find $s_i \in \Gamma_{Z_i}(X \setminus Z_1 \cap Z_2, \mathcal{F})$ $(i = 1, 2)$, such that $s = s_1 - s_2$ on $X \setminus Z_1 \cap Z_2$. We extend s_1 and s_2 to all X as $\mathcal{F}$ is flasque. Then $s_1 - s_2 = s + s'$ for some $s' \in \Gamma_{Z_1\cap Z_2}(X, \mathcal{F})$. Thus $(s_1 - s') - s_2 = s$. □

Proposition 5.2.5. *Let $0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$ be an exact sequence in $\mathbf{Sh}(X)$. Let $Z \hookrightarrow X$ be locally closed.*

Assume that $\mathcal{F}'$ is flasque. Then the exact sequences

$$0 \longrightarrow \Gamma_Z(X, \mathcal{F}') \longrightarrow \Gamma_Z(X, \mathcal{F}) \longrightarrow \Gamma_Z(X, \mathcal{F}'')$$

and

$$0 \longrightarrow \Gamma_Z(\mathcal{F}') \longrightarrow \Gamma_Z(\mathcal{F}) \longrightarrow \Gamma_Z(\mathcal{F}'')$$

are also surjective on the right.

Proof. We first prove the surjectivity on the right of the first sequence in the case $Z = X$.

Let $s'' \in \Gamma(X, \mathcal{F}'')$.

Let Σ be the set of pairs (U, s) with $U \in \mathbf{Op}_X$ and $s \in \Gamma(U, \mathcal{F})$ mapping to $s''_{|U}$. The set Σ can be ordered by setting $(U, s) \leq (V, t)$ if $U \subset V$ and $t_{|U} = s$. Clearly any totally ordered set in Σ admits an upper-bound. By Zorn's lemma Σ admits a maximal element (U, s).

Suppose $U \neq X$. Let $x \in X \setminus U$. Then there exists an open neighbourhood V of x and a section $t \in \Gamma(V, \mathcal{F})$ such that t is mapped to $s''_{|V}$. On $U \cap V$ the difference $s - t$ belongs to $\Gamma(U \cap V, \mathcal{F}')$. As $\mathcal{F}'$ is flasque one can extend $s - t$ to a section $r \in \Gamma(X, \mathcal{F}')$. Replacing t by $t - r$ we may assume $t = s$ on $U \cap V$. Hence s can be extended to $U \cup V$, contradiction to the maximality of (U, s).

For a general locally closed Z choose $U \in \mathbf{Op}_X$ containing Z as a closed subset. One has the commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \Gamma_{Z\cap U}(U,\mathcal{F}') \arrow[r] \arrow[d] & \Gamma_{Z\cap U}(U,\mathcal{F}) \arrow[r] \arrow[d] & \Gamma_{Z\cap U}(U,\mathcal{F}'') \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Gamma(U,\mathcal{F}') \arrow[r] \arrow[d] & \Gamma(U,\mathcal{F}) \arrow[r] \arrow[d] & \Gamma(U,\mathcal{F}'') \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Gamma(U\setminus Z,\mathcal{F}') \arrow[r] \arrow[d] & \Gamma(U\setminus Z,\mathcal{F}) \arrow[r] \arrow[d] & \Gamma(U\setminus Z,\mathcal{F}'') \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

By the previous case the second and third rows are exact. All the columns are exact by definition of $\Gamma_{Z\cap U}$. It implies that the first row is exact too. This proof the first statement of the lemma.

The second follows immediately. □

Corollary 5.2.6. *Let $0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$ be an exact sequence in $\mathbf{Sh}(X)$. Suppose that $\mathcal{F}'$ and $\mathcal{F}$ are flasque. Then $\mathcal{F}''$ is flasque.*

Proof. In the commutative diagram

$$\begin{tikzcd}
\Gamma(X,\mathcal{F}) \arrow[r] \arrow[d, "\alpha"] & \Gamma(X,\mathcal{F}'') \arrow[d, "\gamma"'] \\
\Gamma(U,\mathcal{F}) \arrow[r, "\beta"'] & \Gamma(U,\mathcal{F}'')
\end{tikzcd}$$

the maps α is surjective as $\mathcal{F}$ is flasque and β is surjective as $\mathcal{F}'$ is flasque (previous proposition).

Thus γ is surjective. □

Definition 5.2.7. *Let $\mathcal{A}, \mathcal{A}'$ be Abelian categories and $F : \mathcal{A} \longrightarrow \mathcal{A}'$ an additive functor. Recall that a full additive subcategory $\mathcal{J} \subset \mathcal{A}$ is F-injective if :*

(a) *for any $X \in \mathcal{A}$ there exists $X' \in \mathcal{J}$ and an exact sequence $0 \longrightarrow X \longrightarrow X'$.*
(b) *if $0 \longrightarrow X' \longrightarrow X \longrightarrow X'' \longrightarrow 0$ is exact in $\mathcal{A}$ and $X', X \in \mathcal{J}$ then $X'' \in \mathcal{J}$.*
(c) *if $0 \longrightarrow X' \longrightarrow X \longrightarrow X'' \longrightarrow 0$ is a sequence in $\mathcal{J}$, exact in $\mathcal{A}$ then $0 \longrightarrow F(X') \longrightarrow F(X) \longrightarrow F(X'') \longrightarrow 0$ is exact.*

Corollary 5.2.8. *Let $Z \hookrightarrow X$ be locally closed. Then the full subcategory of flasque sheaves in $\mathbf{Sh}(X)$ is injective with respect to the functors f_*, $\Gamma_Z(\cdot)$ and $\Gamma_Z(X,\cdot)$.*

Proof. With what we already proved it is enough to show that flasque sheaves are acyclic for any of these functors. Let $\mathcal{F}$ be a flasque sheaf. Let $0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{I} \longrightarrow \mathcal{F}'' \longrightarrow 0$ be an exact sequence, with $\mathcal{I}$ injective in $\mathbf{Sh}(X)$. As $\mathcal{I}$ is injective it is flasque. By the previous proposition the sheaf $\mathcal{F}''$ is thus flasque.

Let F be any of the functors f_*, $\Gamma_Z(\cdot)$ or $\Gamma_Z(X,\cdot)$. As $F(\mathcal{I}) \longrightarrow F(\mathcal{F}'')$ is surjective the previous short exact sequence gives the long exact sequence

$$0 \longrightarrow R^1F(\mathcal{F}) \longrightarrow R^1F(\mathcal{I}) = 0 \longrightarrow R^1F(\mathcal{F}'') \longrightarrow R^2F(\mathcal{F}) \longrightarrow R^2F(\mathcal{I}) = 0 \longrightarrow \cdots \quad .$$

Hence $R^1F(\mathcal{F}) = 0$. As $\mathcal{F}''$ is also flasque $R^1F(\mathcal{F}'') = 0$ and then $R^2F(\mathcal{F}) = 0$ etc... □

Corollary 5.2.9. *Let $\mathcal{F}^\bullet \in D^+(\mathbf{Sh}(X))$. If $Z \hookrightarrow X$ is locally closed, $Z' \subset Z$ is closed, Z_1, Z_2 are closed in X and U_1, U_2 are open in X, then we have the following exact triangles :*

$$R\Gamma_{U_1\cup U_2}(\mathcal{F}^\bullet) \longrightarrow R\Gamma_{U_1}(\mathcal{F}^\bullet) \oplus R\Gamma_{U_2}(\mathcal{F}^\bullet) \longrightarrow R\Gamma_{U_1\cap U_2}(\mathcal{F}^\bullet) \xrightarrow{+1} \tag{16}$$

$$R\Gamma_{Z_1\cap Z_2}(\mathcal{F}^\bullet) \longrightarrow R\Gamma_{Z_1}(\mathcal{F}^\bullet) \oplus R\Gamma_{Z_2}(\mathcal{F}^\bullet) \longrightarrow R\Gamma_{Z_1\cup Z_2}(\mathcal{F}^\bullet) \xrightarrow{+1} \tag{17}$$

$$\mathcal{F}^\bullet_{U_1\cap U_2} \longrightarrow \mathcal{F}^\bullet_{U_1} \oplus \mathcal{F}^\bullet_{U_2} \longrightarrow \mathcal{F}^\bullet_{U_1\cup U_2} \xrightarrow{+1} \tag{18}$$

$$R\Gamma_{Z'}(\mathcal{F}^\bullet) \longrightarrow R\Gamma_{Z}(\mathcal{F}^\bullet) \longrightarrow R\Gamma_{Z\setminus Z'}(\mathcal{F}^\bullet) \xrightarrow{+1} \quad . \tag{19}$$

Proof. This follows immediately from the previous corollary and the following classical homological result applied to $\mathcal{J}$ the full subcategory of flasque sheaves of $\mathbf{Sh}(X)$:

Proposition 5.2.10. *Let $\mathcal{A}$ and $\mathcal{A}'$ be Abelian categories and $F \longrightarrow F' \longrightarrow F''$ three additive functors from $\mathcal{A}$ to $\mathcal{A}'$.*

Assume there exists a full subcategory $\mathcal{J} \subset \mathcal{A}$ injective with respect to F, F' and F'' and such that for any object $X \in \mathcal{J}$ the sequence

$$0 \longrightarrow F'(X) \longrightarrow F(X) \longrightarrow F''(X) \longrightarrow 0$$

is exact. Then there exists a morphism of functors $\nu : RF'' \longrightarrow RF'[1]$ such that for any $X^\bullet \in D^+(\mathcal{A})$ one has an exact triangle :

$$RF'(X^\bullet) \longrightarrow RF(X^\bullet) \longrightarrow RF''(X^\bullet) \xrightarrow{+1} \quad .$$

Proof. We replace $X^\bullet$ by a quasi-isomorphic object $J^\bullet$ composed of objects of $\mathcal{J}$. We thus obtain an exact sequence of complexes

$$0 \longrightarrow F'(J^\bullet) \longrightarrow F(J^\bullet) \longrightarrow F''(J^\bullet) \longrightarrow 0 \quad .$$

But any such exact sequence of complexes defined an exact triangle in $D^+(\mathcal{A})$. One easily checks that this triangle is independent of the choice of $J^\bullet$. □

□

5.3. **The fundamental exact triangle.** We will apply the previous reminder to the Abelian category $\mathbf{Sh}(X)$.

Definition 5.3.1. *We denote by $\mathbf{DGSh}(X)$ the category of complexes of $\mathbb{Z}_X$-modules (i.e. sheaves of Abelian groups) on X and by $D(\mathbb{Z}_X)$ the derived category of $\mathbf{Sh}(X)$.*

5.3.1. *The general locally closed case.*

Proposition 5.3.2. *Let $S \overset{i_S}{\hookrightarrow} X$ be locally closed. Let $A := X \setminus S \overset{i_A}{\hookrightarrow} X$. Let $\mathcal{F} \in \mathbf{Sh}(X)$.*

Then the sequence (where all morphisms are given by adjunction)

$$0 \longrightarrow i_{S!}{i_S}^!\mathcal{F} \longrightarrow \mathcal{F} \longrightarrow i_{A*}{i_A}^{-1}\mathcal{F} \tag{20}$$

is exact.

If moreover $\mathcal{F}$ is flasque then this sequence is also surjective on the right.

Remark 5.3.3. Notice that in general A is not locally closed in X : consider for example $S =]0,1[$ embedded in $X = \mathbb{R}^2$. Then A is not open in its closure X.

Proof. Let U be any open set in X containing S as a closed subset. If $x \in X \setminus U$ then the sequence (20) gives at the level of stalks :

$$0 \longrightarrow 0 \longrightarrow \mathcal{F}_x \longrightarrow \mathcal{F}_x$$

which is exact (and even surjective on the right).

To show the result for the stalks at $x \in U$, notice that we can replace $\mathcal{F}$ by $\mathcal{F}_{|U}$. Thus we are reduced to the case where S is closed and A is open.

In this case the sequence for stalks reads :

$$\begin{cases} 0 \longrightarrow {i_S}^! \mathcal{F}_x \longrightarrow \mathcal{F}_x \longrightarrow \mathrm{colim}_{V \ni x} \mathcal{F}(A \cap V) & \text{if } x \in S \\ 0 \longrightarrow 0 \longrightarrow \mathcal{F}_x \longrightarrow \mathcal{F}_x & \text{if } x \in A \end{cases}$$

This sequence is exact as $\Gamma_S(U, \mathcal{F}) = \ker(\Gamma(U, \mathcal{F}) \longrightarrow \Gamma(A \cap U, \mathcal{F}))$ by definition of Γ_S.

When $\mathcal{F}$ is flasque the same reduction to the case S closed makes the surjectivity on the right trivial. □

Corollary 5.3.4. *Let $S \overset{i_S}{\hookrightarrow} X$ be locally closed. Let $A := X \setminus S \overset{i_A}{\hookrightarrow} X$. Let $\mathcal{F}^\bullet \in D^+(\mathbf{Sh}(X))$.*

Then the triangle

$$\Delta(S, X, A; \mathcal{F}^\bullet) := (i_{S!} R{i_S}^! \mathcal{F}^\bullet \longrightarrow \mathcal{F}^\bullet \longrightarrow Ri_{A*}(\mathcal{F}^\bullet_{|A}) \xrightarrow{+1})$$

is exact.

5.3.2. *The open or closed case.* From now on we are in the following situation

$$Z \underset{i_Z}{\overset{/}{\hookrightarrow}} X \underset{i_U}{\overset{o}{\hookleftarrow}} U \quad .$$

The triangle $\Delta(U, X, Z, \mathcal{F}^\bullet)$ is just the short exact sequence of complexes of sheaves

$$\Delta(U, X, Z, \mathcal{F}^\bullet) = (0 \longrightarrow i_{U!} \mathcal{F}^\bullet_{|U} \longrightarrow \mathcal{F}^\bullet \longrightarrow i_{Z*} \mathcal{F}^\bullet_{|Z} \longrightarrow 0) \quad .$$

By applying $R\Gamma(X, \cdot)$ to this triangle we obtain the long exact sequence :

$$\cdots \longrightarrow \mathbb{H}^m(X, i_{U!}\mathcal{F}^\bullet_{|U}) \longrightarrow \mathbb{H}^m(X, \mathcal{F}^\bullet) \longrightarrow \mathbb{H}^m(Z, \mathcal{F}^\bullet_{|Z}) \longrightarrow \cdots \quad .$$

By applying $R\Gamma_c(X, \cdot)$ we obtain

$$\cdots \longrightarrow \mathbb{H}^m_c(U, \mathcal{F}^\bullet_{|U}) \longrightarrow \mathbb{H}^m_c(X, \mathcal{F}^\bullet) \longrightarrow \mathbb{H}^m_c(Z, \mathcal{F}^\bullet_{|Z}) \longrightarrow \cdots \quad .$$

The triangle $\Delta(Z, X, U, \mathcal{F}^\bullet)$ is

$$\Delta(Z, X, U, \mathcal{F}^\bullet) = (i_{Z!} R{i_Z}^! \mathcal{F}^\bullet) \longrightarrow \mathcal{F}^\bullet \longrightarrow Ri_{U*} \mathcal{F}^\bullet_{|U} \xrightarrow{+1}) \quad .$$

Applying $R\Gamma(X, \cdot)$ we obtain

$$\cdots \longrightarrow \mathbb{H}^m_Z(X, \mathcal{F}^\bullet) \longrightarrow \mathbb{H}^m(X, \mathcal{F}^\bullet) \longrightarrow \mathbb{H}^m(U, \mathcal{F}^\bullet_{|U}) \longrightarrow \cdots \quad .$$

5.4. **c-soft sheaves.** In the previous section we defined the notion of flasque sheaves in order to study Rf_* and $R\Gamma_Z$. We do the same here for the functors $Rf_!$ and $R(\cdot)_Z$, the corresponding notion being the notion of c-soft sheaf.

In this section X is locally compact.

Definition 5.4.1. *A sheaf $\mathcal{F} \in \mathbf{Sh}(X)$ is said to be soft (resp. c-soft) if for any closed (resp. compact) subspace $K \subset X$ the restriction*

$$\Gamma(X, \mathcal{F}) \longrightarrow \Gamma(K, \mathcal{F})$$

is surjective.

Lemma 5.4.2. *Suppose moreover that X is paracompact. Then injective $\Longrightarrow$ flasque $\Longrightarrow$ soft $\Longrightarrow$ c-soft.*

Proof. This follows immediately from the fact we already proved that

$$\mathrm{colim}_{U \subset K} \mathcal{F}(U) \simeq \Gamma(K, \mathcal{F})$$

under our topological assumptions. □

Lemma 5.4.3. *A sheaf $\mathcal{F} \in \mathbf{Sh}(X)$ is c-soft if and only if for any closed $Z \hookrightarrow X$ the restriction $\Gamma_c(X, \mathcal{F}) \longrightarrow \Gamma_c(Z, \mathcal{F}_{|Z})$ is surjective.*

Proof. If K is compact then $\Gamma(K, \mathcal{F}) = \Gamma_c(K, \mathcal{F})$. This proves the sufficiency of the condition.

Conversely suppose $\mathcal{F}$ is c-soft. Let $s \in \Gamma_c(Z, \mathcal{F}_{|Z})$ with compact support K. Let U be a relatively compact open neighbourhood of K in X. Define $\tilde{s} \in \Gamma(\partial U \cup (Z \cap \overline{U}), \mathcal{F})$ by setting $\tilde{s}_{|Z \cap \overline{U}} = s$, $\tilde{s}_{\partial U} = 0$, and extend $\tilde{s}$ to a section $t \in \Gamma(X, \mathcal{F})$ as $\mathcal{F}$ is c-soft and $\partial U \cup (Z \cap \overline{U})$ is compact. Since $t = 0$ on a neighbourhood of ∂U we may assume that t is supported by $\overline{U}$. □

Proposition 5.4.4. *Let $\mathcal{F} \in \mathbf{Sh}(X)$ be c-soft. Then :*

(i) *Let $Z \hookrightarrow X$ be locally closed and $f : X \longrightarrow Y$ a continuous map. Then $f_! \mathcal{F}$, $F_{|Z}$ and F_Z are c-soft.*

(ii) *If $0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$ is an exact sequence in $\mathbf{Sh}(X)$ and assume that $\mathcal{F}'$ is c-soft. Then the sequence*

$$0 \longrightarrow f_! \mathcal{F}' \longrightarrow f_! \mathcal{F} \longrightarrow f_! \mathcal{F}'' \longrightarrow 0$$

is exact in $\mathbf{Sh}(X)$. In particular the sequence

$$0 \longrightarrow \Gamma_c(X, \mathcal{F}') \longrightarrow \Gamma_c(X, \mathcal{F}) \longrightarrow \Gamma_c(X, \mathcal{F}'') \longrightarrow 0$$

is exact.

Proof. Let us prove that $F_{|Z}$ is c-soft. If Z is open this is clear by definition. Thus we can assume that Z is closed. The result follows then from proposition 5.4.3.

For $f_! \mathcal{F}$: if K is a compact subset of Y then $\Gamma(K, f_! \mathcal{F}) = \Gamma_c(f^{-1}(K), \mathcal{F})$. Since $\Gamma_c(Y, f_! \mathcal{F}) = \Gamma_c(X, \mathcal{F})$ the result follows from proposition 5.4.3.

For F_Z : the result follows from the two previous cases as $F_Z = f_!(F_{|Z})$.

For (ii) : it is enough to show that for all $y \in Y$:

$$0 \longrightarrow (f_! \mathcal{F}')_y \longrightarrow (f_! \mathcal{F})_y \longrightarrow (f_! \mathcal{F}'')_y \longrightarrow 0$$

is exact. As $(f_! \mathcal{G})_y = (f_!(\mathcal{G}_{|f^{-1}(y)}))_y$ we can assume $f = p_X : X \longrightarrow *$.

Let $s'' \in \Gamma_c(X, \mathcal{F}'')$ and let U a relatively compact open neighbourhood of $\operatorname{supp} s''$. We will show that s'' is in the image off $\Gamma_c(X, \mathcal{F}) \longrightarrow \Gamma_c(X, \mathcal{F}'')$. By replacing $\mathcal{G}$ by $\mathcal{G}_{|U}$ (for $\mathcal{G} = \mathcal{F}$, $\mathcal{F}'$ or $\mathcal{F}''$) and then X by $\overline{U}$ one may assume that X is compact.

For $s'' \in \Gamma(X, \mathcal{F}'')$ let $\{K_i\}_{1 \leq i \leq n}$ be a finite covering of X by compact subsets such that there exists $s_i \in \Gamma(K_i, \mathcal{F})$ whose image is $s''_{|K_i}$. We argue by induction on n. For $n \geq 2$, on $K_1 \cap K_2$, $s_1 - s_2$ defines an element of $\Gamma(K_1 \cap K_2, \mathcal{F}')$ hence extend to $s' \in \Gamma(X, \mathcal{F}')$. Replacing s_2 by $s_2 + s'$ we may assume $s_{1|K_1 \cap K_2} = s_{2|K_1 \cap K_2}$. Therefore there exist $t \in \Gamma(K_1 \cup K_2)$ such that $t_{|K_i} = s_i$, $i = 1, 2$. Thus the induction proceeds. □

Similarly to the flasque case one proves the following :

Corollary 5.4.5. *Let $0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$ an exact sequence in $\mathbf{Sh}(X)$. If $\mathcal{F}'$ and $\mathcal{F}$ are c-soft then $\mathcal{F}''$ is c-soft.*

Corollary 5.4.6. *The full subcategory of c-soft sheaves is injective with respect to the functor $f_!$, thus in particular also with respect to $(\cdot)_Z$, $\Gamma_c(X, \cdot)$ and $\Gamma(K, \cdot)$ (K compact).*

Corollary 5.4.7. *Let $\mathcal{F}^\bullet \in D^+(\mathbb{Z}_X)$. If $Z \hookrightarrow X$ is locally closed, $Z' \subset Z$ is closed, Z_1, Z_2 are closed in X then we have the following exact triangles :*

$$\mathcal{F}^\bullet_{Z_1 \cup Z_2} \longrightarrow \mathcal{F}^\bullet_{Z_1} \oplus \mathcal{F}^\bullet_{Z_2} \longrightarrow \mathcal{F}^\bullet_{Z_1 \cap Z_2} \xrightarrow{+1} \tag{21}$$

$$\mathcal{F}^\bullet_{Z \setminus Z'} \longrightarrow \mathcal{F}^\bullet_Z \longrightarrow \mathcal{F}^\bullet_{Z'} \xrightarrow{+1} \tag{22}$$

The following result is also of interest :

Proposition 5.4.8. *Let X be locally compact and countable at infinity. Then the category of c-soft sheaves is injective with respect to the functor $\Gamma(X, \cdot)$.*

Proof. Let $0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$ be an exact sequence in $\mathbf{Sh}(X)$ with $\mathcal{F}'$ c-soft. Let $(K_n)_{n \in \mathbb{N}}$ be an increasing sequence of compact subspaces of X such that $X = \cup_{n \in \mathbb{N}} K_n$ and K_n is contained in the interior of K_{n+1} for all n. As $\mathcal{F}'$ is c-soft all the sequences

$$0 \longrightarrow \Gamma(K_n, \mathcal{F}') \longrightarrow \Gamma(K, \mathcal{F}) \longrightarrow \Gamma(K, \mathcal{F}'') \longrightarrow 0$$

are exact. By taking the limit over $n \in \mathbb{N}$ and as $\Gamma(K_{n+1}, \mathcal{F}') \longrightarrow \Gamma(K_n, \mathcal{F}')$ is surjective for all n, the limit sequence is still exact :

$$0 \longrightarrow \lim_n \Gamma(K_n, \mathcal{F}') \longrightarrow \lim_n \Gamma(K_n, \mathcal{F}) \longrightarrow \lim_n \Gamma(K_n, \mathcal{F}'') \longrightarrow 0 \ .$$

Since for any sheaf $\mathcal{G} \in \mathbf{Sh}(X)$ one has $\Gamma(X, \mathcal{G}) \simeq \lim \Gamma(K_n, \mathcal{G})$ (the sheaf condition !) one gets the result. $\square$

6. Sheaf theoretic intersection cohomology

6.1. Sheafification of intersection homology.

Definition 6.1.1. *Let X be a PL pseudomanifold of dimension n with stratification*

$$\mathcal{X}: \quad X = X_n \supset X_{n-2} \supset \cdots \supset X_0 \supset X_{-1} = \emptyset \ .$$

Let p be a classical perversity and let $\mathcal{E}$ be a local system on $U_2 := X \setminus X_{n-2}$. We define the complex of presheaves $I^p\mathcal{C}^\bullet(X, \mathcal{E})$ by

$$\Gamma(U, I^p\mathcal{C}^\bullet(X, \mathcal{E})) := I^pC_{n-\bullet}(U, \mathcal{E}) \ .$$

Remark 6.1.2. In order to get restriction maps and thus a presheaf it is crucial to use *unbounded* chains.

Lemma 6.1.3. *The complex $I^p\mathcal{C}^\bullet(X, \mathcal{E}) \in \mathbf{DGSh}(X)$ is a complex of c-soft sheaves.*

Proof. One easily checks that this is a complex of sheaves. For the c-soft property : let $K \subset X$ be compact and $s \in \Gamma(K, I^p\mathcal{C}^{n-i}(X, \mathcal{E}))$. We want to show that s can be represented by a section over X. As $\Gamma(K, I^p\mathcal{C}^\bullet(X, \mathcal{E})) = \mathrm{colim}_{U\supset K}\Gamma(U, I^p\mathcal{C}^\bullet(X, \mathcal{E}))$ there exists $U \supset K$ open in X and $\xi \in I^pC_i(U, \mathcal{E})$ defining s. The problem to see ξ as an element of $I^pC_i(X, \mathcal{E})$ is that ξ might not have *closed* support in X.

We choose a triangulation of U fine enough such that there is a closed PL-neighbourhood N of K entirely in U. Then $\xi \cap N \in \Gamma(X, I^p\mathcal{C}^{n-i}(X, \mathcal{E}))$ still represents s. □

Definition 6.1.4.

$$I^pH^i(X, \mathcal{E}) := \mathbb{H}^i(X, I^p\mathcal{C}^\bullet(X, \mathcal{E}))\ ; \quad I^pH^i_c(X, \mathcal{E}) := \mathbb{H}^i_c(X, I^p\mathcal{C}^\bullet(X, \mathcal{E})) \ .$$

As c-soft sheaves are Γ_c-acyclic and also Γ-acyclic because X is locally compact paracompact, we obtain :

Corollary 6.1.5. *$I^pH^i(X, \mathcal{E}) = I^pH_{n-i}(X, \mathcal{E})$ and $I^pH^i_c(X, \mathcal{E}) = I^pH^c_{n-i}(X, \mathcal{E})$, where both terms on the right were defined simplicially.*

Remark 6.1.6. Notice that although every $x \in X$ has a system of contractible neighbourhoods, the stalks

$$\mathcal{H}^i(I^p\mathcal{C}^\bullet(X, \mathcal{E}))_x := \mathrm{colim}_{U\ni x} I^pH_{n-i}(U, \mathcal{E})$$

are in general non-trivial as I^pH_l is not homotopy invariant. In particular the complex $I^p\mathcal{C}^\bullet(X, |E)$ is not in general q.i. to a sheaf concentrated in a single degree. This has to be compared to the classical case $\mathcal{C}^\bullet(X) = \mathcal{C}_{n-\bullet}(X) \stackrel{q.i.}{\simeq} \mathcal{O}r_X$.

6.2. First axiomatic characterization of the intersection cohomology sheaf.

Given

$$\mathcal{X}: \qquad X = X_n \supset X_{n-2} \supset \cdots \supset X_0$$

one defines for $2 \leq q \leq n+1$:

- $S^k = X_{n-k} \setminus X_{n-k-1}$ the union of codimension k strata.
- $U_k = X \setminus X_{n-k}$ open in X, one has $U_2 \subset U_3 \subset \cdots \subset U_{n+1} = X$.

We have the fundamental triangle

$$U_k \xhookrightarrow[j_k]{o} U_{k+1} \xhookleftarrow[i_k]{/} S^k \quad .$$

Definition 6.2.1. *Given $\mathcal{F}^\bullet \in \mathbf{DGSh}(X)$ we define $\mathcal{F}^\bullet_k := \mathcal{F}^\bullet_{|U_k}$.*

The attachment map is $\alpha_k : \mathcal{F}^\bullet_{k+1} \longrightarrow j_{k*}\mathcal{F}^\bullet_k \longrightarrow Rj_{k*}\mathcal{F}^\bullet_k$.

Definition 6.2.2. *Given a stratification $\mathcal{X}$ of X, p a perversity and $\mathcal{E}$ a local system on U_2 we denote by $(AX1)_{\mathcal{X},\mathcal{E},p}$ the following set of axioms for $\mathcal{F}^\bullet \in \mathbf{DGSh}(X)$:*

(1) *$\mathcal{H}^i\mathcal{F}^\bullet = 0$ for $i < 0$; $\mathcal{F}^\bullet$ is bounded below ; $\mathcal{F}^\bullet_2 \simeq \mathcal{E}$.*
(ii) *if $x \in S^k$ then $H^i(\mathcal{F}^\bullet_x) = 0$ for $i > p(k)$.*
(iii) *$\alpha_k : \mathcal{F}^\bullet_{k+1} \longrightarrow Rj_{k*}\mathcal{F}^\bullet_k$ is a quasi-isomorphism in degree $\leq p(k)$.*

Remark 6.2.3. Given $\mathcal{F}^\bullet \in \mathbf{DGSh}(X)$ and $k \in \mathbb{Z}$ one defines

$$(\tau_{\leq k}\mathcal{F}^\bullet)^i = \begin{cases} \mathcal{F}^i & \text{if } i < k, \\ \ker(d^i : \mathcal{F}^i \longrightarrow \mathcal{F}^{i+1}) & \text{if } i = k, \\ 0 & \text{if } i > k \quad . \end{cases}$$

Then the natural inclusion $\tau_{\leq k}\mathcal{F}^\bullet \longrightarrow \mathcal{F}^\bullet$ induces a quasi-isomorphism in degree $\leq k$.

Proposition 6.2.4. *The complex $I^p\mathcal{C}^\bullet(X, \mathcal{E})$ satisfies $(AX1)_{\mathcal{X}, \mathcal{E}\otimes \mathcal{O}r_{U_2}, p}$.*

Proof. Let $\mathcal{F}^\bullet := I^p\mathcal{C}^\bullet(X, \mathcal{E})$.

For (i) : obviously $\mathcal{F}^\bullet$ is bounded below and its cohomology sheaves vanish in negative degrees. Moreover :

$$\mathcal{F}_2^\bullet = \mathcal{C}_{n-\bullet}(\mathcal{E}) \simeq \mathcal{E} \otimes \mathcal{O}r_{U_2} \quad ,$$

where the first equality follows from the fact that the perversity conditions are empty on U_2.

For (ii) : By definition $(\mathcal{H}^i I^p\mathcal{C}_{n-\bullet}(X, \mathcal{E}))_x = \mathrm{colim}_{U \ni x} I^pH_{n-i}(U, \mathcal{E})$. If $x \in S^k$ let $U \simeq \mathbb{R}^{n-k} \otimes C(L)$ be a distinguished neighbourhood of x, where C is the open cone and L is a PL-pseudomanifold of dimension $k-1$.

One easily checks the following Künneth type result : the product

$$\mathbb{R} \times \cdot : C_i(X, \mathcal{E}) \longrightarrow C_{i+1}(X, \mathcal{E})$$

induces an isomorphism

$$I^pH_\bullet(X, \mathcal{E}) \simeq I^pH_{\bullet+1}(\mathbb{R} \times X, p_2^*\mathcal{E}) \quad .$$

Thus

$$I^pH_{n-i}(U, \mathcal{E}) \simeq I^pH_{k-i}(C(L), \mathcal{E}) \quad .$$

Recall the crucial cone formula :

$$I^pH_l(C(L), \mathcal{E}) = \begin{cases} 0 & \text{if } l < k - p(k), \\ I^pH_{l-1}(L, \mathcal{E}) & \text{if } l \geq k - p(k) \quad . \end{cases}$$

It follows (in cohomological notations) that

$$I^pH^i(U, \mathcal{E}) = I^pH^i(C(L), \mathcal{E}) = \begin{cases} 0 & \text{if } \ i > p(k), \\ I^pH^{i-1}(L, \mathcal{E}) & \text{if } \ i \leq p(k) \quad . \end{cases}$$

This implies (ii).

For (iii) : consider

$$U_k \overset{o}{\underset{j_k}{\hookrightarrow}} U_{k+1} \overset{/}{\underset{i_k}{\hookleftarrow}} S^k \quad .$$

We want to show that the map α_k in the following diagram is a quasi-isomorphism up to degree $p(k)$:

$$\begin{tikzcd} I^p\mathcal{C}_{n-\bullet}(X, \mathcal{E})_{k+1} \arrow[r] \arrow[dr, "\alpha_k"'] & j_{k*}I^p\mathcal{C}_{n-\bullet}(X, \mathcal{E})_k \arrow[d] \\ & Rj_{k*}I^p\mathcal{C}_{n-\bullet}(X, \mathcal{E})_k \end{tikzcd}$$

As $I^p\mathcal{C}_{n-\bullet}(X, \mathcal{E})$ is a complex of c-soft sheaves the map $j_{k*}I^p\mathcal{C}_{n-\bullet}(X, \mathcal{E})_k \longrightarrow Rj_{k*}I^p\mathcal{C}_{n-\bullet}(X, \mathcal{E})_k$ is a quasi-isomorphism hence it is enough to show that

$$I^p\mathcal{C}_{n-\bullet}(X, \mathcal{E})_{k+1} \longrightarrow j_{k*}I^p\mathcal{C}_{n-\bullet}(X, \mathcal{E})_k$$

is a quasi-isomorphism up to degree $p(k)$.

Let us check it at the stalk level. This is obvious if x belongs to U_k. On the other hand any $x \in S^k$ admits a system of compatible neighbourhoods in U_{k+1} of the form

$$\begin{tikzcd} V & \mathbb{R}^{n-k} \times C(L) \arrow[l, "\sim"'] \\ V \cap U_k \arrow[u, hook] & \mathbb{R}^{n-k} \times C(L)^* \arrow[l, "\sim"'] \arrow[u, hook] \end{tikzcd}$$

Notice that as a PL-manifold $C(L)^* \simeq \mathbb{R} \times L$. Thus we have the commutative diagram :

$$\begin{tikzcd} \Gamma(V, I^p\mathcal{C}^\bullet(X,\mathcal{E})) \arrow[d, "\alpha_k(V)"'] & I^pC^\bullet(C(L),\mathcal{E}) \arrow[l, "q.i"', "\sim"] \arrow[d] \arrow[r, equal] & \tau_{\leq p(k)} I^pC^\bullet(L,\mathcal{E}) \\ \Gamma(V \cap U_k, I^p\mathcal{C}^\bullet(X,\mathcal{E})) & I^pC^\bullet(L,\mathcal{E}) \arrow[l, "\simeq"'] & \end{tikzcd}$$

which gives

$$\Gamma(V, I^p\mathcal{C}^\bullet(X,\mathcal{E})) \overset{\alpha_k(V)}{\simeq} \Gamma(V, \tau_{\leq p(k)} j_{k*} I^p\mathcal{C}^\bullet(X,\mathcal{E}))$$

and the result. □

Using $(AX1)_{\mathcal{X},\mathcal{E},p}$ we will find a complex quasi-isomorphic to $I^p\mathcal{C}^\bullet(X,\mathcal{E})$ whose definition does not depend on any simplicial structure.

Lemma 6.2.5. *Assume $\mathcal{F}^\bullet$ satisfies $(AX1)_{\mathcal{X},\mathcal{E},p}$. Then $\mathcal{F}^\bullet_{k+1} \simeq \tau_{\leq p(k)} Rj_{k*}\mathcal{F}^\bullet_k$.*

Proof. Consider the commutative diagram

$$\begin{tikzcd} \mathcal{F}^\bullet_{k+1} \arrow[r, "\alpha_k"] & Rj_{k*}\mathcal{F}^\bullet_k \\ \tau_{\leq p(k)}\mathcal{F}^\bullet_{k+1} \arrow[u, "\beta_{k+1}"] \arrow[r, "\alpha'_k"'] & \tau_{\leq p(k)} Rj_{k*}\mathcal{F}^\bullet_k \end{tikzcd}$$

By the axiom (ii), the map β_{k+1} is a quasi-isomorphism.

By the axiom (iii) the map α'_k is a quasi-isomorphism. This concludes the proof. □

6.3. Deligne's extension.

Definition 6.3.1. *Given a stratified pseudomanifold $(X,\mathcal{X})$ and $\mathcal{E}$ a local system on U_2 we define inductively*

$$\mathcal{P}^\bullet(\mathcal{E})_2 = \mathcal{E} \qquad \textit{and} \quad \mathcal{P}^\bullet(\mathcal{E})_{k+1} := \tau_{\leq p(k)} Rj_{k*}\mathcal{P}^\bullet(\mathcal{E})_k \ .$$

Hence

$$\mathcal{P}^\bullet(\mathcal{E}) := \tau_{\leq p(n)} Rj_{n*}\tau_{\leq p(n-1)} Rj_{n-1*} \cdots \tau_{\leq p(2)} Rj_{2*}\mathcal{E} \ .$$

Theorem 6.3.2. (a) *$\mathcal{P}^\bullet(\mathcal{E})$ satisfies $(AX1)_{\mathcal{X},\mathcal{E},p}$.*
(b) *Any $\mathcal{F}^\bullet \in$ **DGSh**(X) satisfying $(AX1)_{\mathcal{X},\mathcal{E},p}$ is quasi-isomorphic to $\mathcal{P}(\mathcal{E})^\bullet$.*

Proof. The first assertion is clear.

For (2) : Suppose $\mathcal{F}^\bullet$ satisfies $(AX1)_{\mathcal{X},\mathcal{E},p}$. Thus $\mathcal{F}^\bullet_2 \simeq \mathcal{E} \simeq \mathcal{P}(\mathcal{E})^\bullet_2$. By induction assume $\mathcal{F}^\bullet_k$ and $\mathcal{P}(\mathcal{E})^\bullet_k$ are quasi-isomorphic for some k. Thus :

$$\mathcal{F}^\bullet_{k+1} = \tau_{\leq p(k)} Rj_{k*}\mathcal{F}^\bullet_k = \tau_{\leq p(k)} Rj_{k*}\mathcal{P}(\mathcal{E})^\bullet_k \simeq \mathcal{P}(\mathcal{E})^\bullet_{k+1} \ .$$

□

Corollary 6.3.3. $I^pH_{n-i}(X,\mathcal{E}) = \mathbb{H}^i(X, \mathcal{P}^\bullet(\mathcal{E} \otimes \mathcal{O}r_{U_2}))$

Remark 6.3.4. The previous theorem enables us to extend the definition of intersection homology from PL pseudomanifold to any pseudomanifold.

We can now prove some results we stated for simplicial intersection homology.

Proposition 6.3.5. *Let $(X, \mathcal{X})$ be a stratified pseudomanifold. Let $p = 0$ be the zero perversity. Let $\mathcal{E}$ be a local system on $U_2 \overset{j}{\hookrightarrow} X$. Then $j_*\mathcal{E}$ satisfies $(AX1)_{\mathcal{X},\mathcal{E},0}$. In particular*

$$I^0H^\bullet(X, \mathcal{E}) = H^\bullet(X, j_*(\mathcal{E} \otimes \mathcal{O}r_{U_2})) \ .$$

If X is normal oriented then $j_*\mathbb{Z}_{U_2} = \mathbb{Z}_X$ and $I^0H^\bullet(X, \mathbb{Z}_{U_2}) = H^\bullet(X, \mathbb{Z}_X)$.

Proof. The sheaf $j_*\mathcal{E}$ always satisfies (i). As $j_*\mathcal{E}$ is in degree 0 it also satisfies (ii) trivially.

For (iii) let $j_{2,k} : U_2 \hookrightarrow U_k$. If $\mathcal{A} \in \mathbf{Sh}(U_2)$ and $\mathcal{B} \in \mathbf{Sh}(U_k)$ then

$$(j_*\mathcal{A})_{k+1} = j_{2,k+1_*}\mathcal{A} = j_{k_*}j_{2,k_*}\mathcal{A} \tag{23}$$

$$\tau_{\leq 0}Rj_{k_*}\mathcal{B} = j_{k_*}\mathcal{B} \ . \tag{24}$$

hence $j_*\mathcal{E}$ satisfies (iii).

If X is normal any $x \in S^k$ has a fundamental system of neighbourhoods U such that $U \setminus X_{n-2}$ is connected. Hence $j_*\mathbb{Z}_{U_2} = \mathbb{Z}_X$. □

Similarly a good exercice is to prove :

Theorem 6.3.6. *Assume X to be normal and let t be the maximal perversity. Then $\mathcal{C}_{n-\bullet}$ satisfies $(AX1)_{\mathcal{X},\mathcal{O}r_{U_2},t}$. In particular*

$$I^tH^i(X, \mathcal{O}r_{U_2}) = \mathbb{H}^i(X, \mathcal{C}_{n-\bullet}) = H_{n-i}(X, \mathbb{Z}_X) \ .$$

Proof. See [**?**, p.66] for details. □

7. Cohomological dimension

Definition 7.0.7. *Let $X \in \mathbf{Top}$. One denotes by $\dim_{coh} X$ the smallest $n \in \mathbb{N} \cup \{\infty\}$ such that*

$$\forall U \in \mathbf{Op}_X, i > n, \mathcal{F} \in \mathbf{Sh}(X), \quad H^i_c(U, \mathcal{F}) = 0 \ .$$

Proposition 7.0.8. *Let X be locally compact and countable at infinity. Then the following are equivalent :*

(i) $\dim_{coh} X \leq n$.
(ii) $H^{n+1}_c(X, \mathcal{F}) = 0$ *for all* $\mathcal{F} \in \mathbf{Sh}(X)$.
(iii) $H^{n+1}_c(U, \mathbb{Z}_U) = 0$ *for all* $U \in \mathbf{Op}_X$.
(iv) *Any* $\mathcal{F} \in \mathbf{Sh}(X)$ *admits a c-soft resolution of length at most* n.

Moreover in this case X satisfies the following property : any $\mathcal{F} \in \mathbf{Sh}(X)$ admits a flasque resolution of length at most $n + 1$.

Proof. Clearly (iv) admits (i) as the restriction to U of a c-soft sheaf on X is c-soft. Conversely let us show $(i) \Rightarrow (iv)$. Let $0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}^0 \longrightarrow \cdots$ be a c-soft resolution. We thus have an exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}^0 \longrightarrow \cdots \longrightarrow \mathcal{F}^{n-1} \longrightarrow \mathcal{G}^n \longrightarrow 0$$

where $\mathcal{G}^n := \mathrm{Im}\ (\mathcal{F}^{n-1} \longrightarrow \mathcal{F}^n)$. As the $\mathcal{F}^i$ are c-soft one immediately obtains

$$\forall i \geq 0, \quad H^i_c(U, \mathcal{G}^n) \simeq H^{i+n}_c(U, \mathcal{F}) \ .$$

Hence $H^1_c(U, \mathcal{G}_n) = H^{n+1}_c(U, \mathcal{F}) = 0$, i.e. $\mathcal{G}^n$ is c-soft.

Clearly (i) implies (ii).

For $(ii) \Rightarrow (iii)$: $H^{n+1}_c(U, \mathbb{Z}_U) = H^{n+1}_c(X, j_!\mathbb{Z}_U)$.

For $(iii) \Rightarrow (ii)$: arguing as above one obtains that $\mathbb{Z}_U$ admits a c-soft resolution of length at most n. Thus $H^i_c(U, \mathbb{Z}_U) = 0$ for $i > n$. Let M be the family of all sheaves $\mathcal{F}$ such that $H^{n+1}_c(X, \mathcal{F}) = 0$. Thus M contains ideals $n\mathbb{Z}_U$, satisfies the property "2 out of 3" for exact sequences and is stable by colimits as H^i_c commutes with colimits. This implies that $M = \mathbf{Sh}(X)$.

For $(ii) \Rightarrow (i)$: $H^i_c(U, \mathcal{F}) = H^i_c(X, j_!\mathcal{F})$.

Which finishes the proof of the different equivalences.

Suppose now that $0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}^0 \longrightarrow F^1 \longrightarrow \cdots$ is flasque resolution of $\mathcal{F}$. Using the same notation $\mathcal{G}^n$ as before we obtain a short exact sequence

$$0 \longrightarrow \mathcal{G}^n \longrightarrow \mathcal{F}^n \longrightarrow \mathcal{B} \longrightarrow 0 \ .$$

As the F^i's are flasque they are c-soft. Arguing as before one obtains that $\mathcal{G}^n$ is c-soft. Hence we have the commutative diagram :

$$\begin{tikzcd} \mathcal{F}^n(X) \arrow[r] \arrow[d, "\alpha"'] & \mathcal{B}(X) \arrow[d] & \\ \mathcal{F}^n(U) \arrow[r, "\beta"'] & \mathcal{B}(U) \arrow[r] & H^1(U, \mathcal{G}^n) \end{tikzcd} .$$

The map α is surjective as $\mathcal{F}^n$ is flasque. On the other hand for U paracompact $H^1(U, \mathcal{G}^n) = 0$ as $\mathcal{G}^n$ is c-soft thus Γ-acyclic, hence β is surjective. As α and β are surjective the map $\mathcal{B}(X) \longrightarrow \mathcal{B}(U)$ is surjective.

As X is countable at infinity every point $x \in X$ has a neighbourhood U whose open subsets are paracompact, hence $\mathcal{B}_{|U}$ is flasque.

Being flasque is a local property thus $\mathcal{B}$ is flasque. $\square$

Proposition 7.0.9. *Let M be an n-dimensional topological manifold (thus paracompact). Then* $\dim_{coh} M = n$.

Proof. Nice exercice (cf. [14, p.195]). $\square$

8. Constructibility

Definition 8.0.10. *Let $X \in$ **Top** and $\mathcal{F}^\bullet \in$ **DGSh**(X).*

(i) *The complex $\mathcal{F}^\bullet$ is said* cohomologically locally constant *(clc) if $\mathcal{H}^\bullet\mathcal{F}^\bullet$ is locally constant.*

(ii) *Suppose $\mathcal{X} : X = X_n \supset X_{n-1} \supset \cdots \supset X_0 \supset X_{-1} = \emptyset$ is a filtration by closed subsets. The DG-sheaf $\mathcal{F}^\bullet$ is $\mathcal{X}$-*clc *if $\mathcal{H}^\bullet\mathcal{F}^\bullet$ is locally constant on each stratum $X_i \setminus X_{i-1}$. It is said $\mathcal{X}$-*cohomologically constructible *($\mathcal{X}$-cc) if it is $\mathcal{X}$-clc and the stalks $\mathcal{H}^\bullet\mathcal{F}^\bullet_x$ are finitely generated.*

(iii) *$\mathcal{F}^\bullet$ is* cohomologically constructible *(cc) if :*
   - *for $x \in X$ and $m \in \mathbb{Z}$ the inverse system $\mathbb{H}^m_c(U_x, \mathcal{F}^\bullet)$ (over all open neighbourhoods of x) is essentially constant and its limit is finitely generated.*
   - *for $x \in X$ and $m \in \mathbb{Z}$ the direct system $\mathbb{H}^m(U_x, \mathcal{F}^\bullet)$ (over all open neighbourhoods of x) is essentially constant and its limit is finitely generated.*

Remark 8.0.11. Recall that an inverse system $(A_i)_{i\in I}$ is said to be essentially constant if for each $i \in I$ there exists $i' \in I$ such that $\mathrm{Im}\ (A_{i'} \longrightarrow A_i) = \mathrm{Im}\ (\lim A_j \longrightarrow A_i)$ and moreover there exists $i_0 \in I$ such that $\lim A_j \longrightarrow A_{i_0}$ is injective. Dually for direct systems.

The goal of this section is to show :

Proposition 8.0.12. *Let $(X, \mathcal{X})$ be a stratified pseudo-manifold and $\mathcal{F}^\bullet \in$ **DGSh**(X) a $\mathcal{X}$-cc complex. Then $\mathcal{F}^\bullet$ is cohomologically constructible.*

We start with the following

Lemma 8.0.13. *Let M be a an n-dimensional manifold and $\mathcal{F}^\bullet \in$ **DGSh**(M) which is cohomologically locally constant. Let $x \in M$. Then :*

(1) *The inverse system $\mathbb{H}^i_c(U_x, \mathcal{F}^\bullet)$ is constant on the set of neighbourhoods of x homeomorphic to open balls, equal to $\mathbb{H}^{n-i}(\mathcal{F}^\bullet_x)$. The direct system $\mathbb{H}^i(U, \mathcal{F}^\bullet)$ is constant on this set, equal to $\mathbb{H}^i\mathcal{F}^\bullet_x$.*

(2) *$i^!_x\mathcal{F}^\bullet = i^*_x\mathcal{F}^\bullet[-n]$. In particular $H^i(i^!_x\mathcal{F}^\bullet) = H^{i-n}(\mathcal{F}^\bullet_x)$. Moreover if $\mathcal{H}^\bullet\mathcal{F}^\bullet$ has finitely generated stalks then $\mathcal{F}^\bullet$ is cc.*

Proof. We concentrate on $i^!_x$, everything for i^*_x is similar and easier.

By hypothesis on $\mathcal{F}^\bullet$ the complex $\mathcal{H}^\bullet\mathcal{F}^\bullet$ is constant on sufficiently small U neighbourhoods of x, therefore the hypercohomology spectral sequence for $\mathbb{H}_c(U, \mathcal{F}^\bullet)$ collapses. We have :

$$\mathbb{H}^i_c(U, \mathcal{H}^\bullet\mathcal{F}^\bullet) = \begin{cases} \mathcal{H}^\bullet\mathcal{F}^\bullet_x & \text{if } i = n \\ 0 & \text{otherwise} \end{cases}$$

thus

$$\mathbb{H}^j_c(U, \mathcal{F}^\bullet) = H^{j-n}(\mathcal{F}^\bullet_x) \ \ ,$$

hence (1).

For (2) : let U be an open neighbourhood of x whose closure is homeomorphic to a closed ball and let $\mathcal{J}^\bullet$ be an injective resolution of $\mathcal{F}^\bullet$. We have natural maps

$$i^!_x\mathcal{F}^\bullet = \Gamma_{\{x\}}(U, \mathcal{J}^\bullet) \xrightarrow{\beta} \Gamma_c(U, \mathcal{J}^\bullet) \xleftarrow{\alpha} \Gamma(U, \mathcal{J}^\bullet)[-n]$$

where the map α is the cup product with the fundamental class $[U]_c$.

Using the hypercohomology spectral sequence and the fact that

$$H^i(U, E) \xrightarrow{\cup[U]_c} H^{i+n}_c(U, E)$$

is an isomorphism for any constant system E (both terms vanish for $i \neq 0$ and are equal to E for $i = 0$), the map α is an isomorphism.

As we know that $\Gamma(U, \mathcal{J}^\bullet) \longrightarrow \mathcal{J}^\bullet_x \longleftarrow \mathcal{F}^\bullet_x$ is a quasi-isomorphism, to conclude that $i^!_x\mathcal{F}^\bullet = i^*_x\mathcal{F}^\bullet[-n]$ we are reduced to show that

$$\beta : \Gamma_{\{x\}}(U, \mathcal{J}^\bullet) \xrightarrow{\beta} \Gamma_c(U, \mathcal{J}^\bullet)$$

is a quasi-isomorphism.

We consider the commutative diagram

$$\begin{tikzcd} \cdots \arrow[r] & H^i(i_x^! \mathcal{F}^\bullet) \arrow[r] \arrow[d, "\beta_i"'] & \mathbb{H}^i(\overline{U}, \mathcal{F}^\bullet) \arrow[r] \arrow[d, equal] & \mathbb{H}^i(\overline{U} \setminus \{x\}, \mathcal{F}^\bullet) \arrow[r] \arrow[d, "\gamma_i"'] & \cdots \\ \cdots \arrow[r] & \mathbb{H}^i_c(U, \mathcal{F}^\bullet) \arrow[r] & \mathbb{H}^i(\overline{U}, \mathcal{F}^\bullet) \arrow[r] & \mathbb{H}^i(\overline{U} \setminus U, \mathcal{F}^\bullet) \arrow[r] & \cdots \end{tikzcd}$$

We want to show that β_i is an isomorphism. It is enough to show that the restriction map γ_i is an isomorphism for all i's. It is enough to show that the restriction map induces an isomorphism of the E_2-terms of the corresponding hypercohomology spectral sequences, i.e.

$$\mathbb{H}^\bullet(\overline{U} \setminus \{x\}, \mathcal{H}^\bullet \mathcal{F}^\bullet) \longrightarrow \mathbb{H}^\bullet(\overline{U} \setminus U, \mathcal{H}^\bullet \mathcal{F}^\bullet)$$

is an isomorphism.

We may write

$$\overline{U} \setminus \{x\} = S^{n-1} \times (0,1], \quad \overline{U} \setminus U = S^{n-1} \times \{1\}$$

with the obvious inclusion map. Hence the result as $\mathcal{H}^\bullet \mathcal{F}^\bullet$ is constant on $\overline{U}$.

□

We then have the following refined version of proposition 8.0.12 :

Proposition 8.0.14. *Let $(X, \mathcal{X})$ be a stratified pseudomanifold. Let $\mathcal{F}^\bullet \in \mathbf{DGSh}(X)$ be $\mathcal{X}$-clc. Let $x \in X$. Then :*

1. *The inverse system $\mathbb{H}^i_c(U_x, \mathcal{F}^\bullet)$ is constant on distinguished neighbourhoods of x and its limit equals $H^i(i_x^! \mathcal{F}^\bullet)$.*
2. *The direct system $\mathbb{H}^i(U, \mathcal{F}^\bullet)$ is constant on this set and its colimit is $H^i(\mathcal{F}^\bullet_x)$.*
3. *For any stratum S_α the sheaf ${i_{S_\alpha}}^! \mathcal{F}^\bullet$ is clc on S_α.*
4. *if $\mathcal{F}^\bullet$ is $\mathcal{X}$-cc then $\mathcal{F}^\bullet$ is cc.*

Proof. For (1) : let $x \in S^k$. Let $U = B^{n-k} \times C(L)$ a distinguished neighbourhood of x in X. Let $Z := U \cap S^k \simeq B^{n-k}$. We have an exact sequence

$$\cdots \longrightarrow \mathbb{H}^j_c(U - Z, \mathcal{F}^\bullet) \longrightarrow \mathbb{H}^j_c(U, \mathcal{F}^\bullet) \longrightarrow \mathbb{H}^j_c(Z, \mathcal{F}^\bullet) \longrightarrow \cdots \quad .$$

As $\mathcal{F}^\bullet_{|S^k}$ is clc and S^k is a manifold lemma 8.0.13 implies that $H^j_c(Z, \mathcal{F}^\bullet)$ is a constant inverse system. Since $U \setminus Z = B^{n-k} \times C(L)^* \simeq B^{n-k+1} \times L$ we obtain

$$\mathbb{H}^j_c(U \setminus Z, \mathcal{F}^\bullet) = \mathbb{H}^{j-(n-k+1)}(L, \mathcal{F}^\bullet)$$

(see [**?**, V lemma 3.8.b)] for a rigorous proof of this intuitive statement). It follows that $H^j_c(U \setminus Z, \mathcal{F}^\bullet)$ is also a constant inverse system.

By the 5-lemma the inverse system $\mathbb{H}^j_c(U, \mathcal{F}^\bullet)$ is constant over distinguished neighbourhoods of x.

Arguing essentially as in the smooth case this shows that $H^i(i_x^! \mathcal{F}^\bullet) = \lim \mathbb{H}^i_c(U_x, \mathcal{F}^\bullet)$, which finishes the proof of (1).

For (3) : One considers the long exact sequence in $\mathbf{Sh}(S^k)$

$$\cdots \longrightarrow \mathcal{H}^i(i_k^! \mathcal{F}^\bullet_{k+1}) \longrightarrow \mathcal{H}^i(i_k^* \mathcal{F}^\bullet_{k+1}) \longrightarrow \mathcal{H}^i(i_k^* Rj_{k*} \mathcal{F}^\bullet_k) \longrightarrow \cdots \quad .$$

The second term is locally constant as $\mathcal{F}^\bullet$ is $\mathcal{X}$-clc. To prove that the first is, it is enough to show that the third one is, i.e. that $Rj_{k*} \mathcal{F}^\bullet_k$ is $\mathcal{X}$-clc. One computes

$$\mathbb{H}^\bullet(U, Rj_{k*} \mathcal{F}^\bullet_k) = \mathbb{H}^\bullet(U \cap U_k, \mathcal{F}^\bullet_k) = \mathbb{H}^\bullet(L, \mathcal{F}^\bullet) \tag{25}$$

for a distinguished $U = B \times C(L)$ as $U \cap U_k = B \times C(L)^* = B \times \mathbb{R} \times L$.

For (2) : the same computation shows that the third term of the long exact sequence

$$\cdots \longrightarrow \mathbb{H}^i(B, i_k^! \mathcal{F}^\bullet) \longrightarrow \mathbb{H}^i(U, \mathcal{F}^\bullet) \longrightarrow \mathbb{H}^i(U, Rj_{k*} \mathcal{F}^\bullet_k) \longrightarrow \cdots$$

is constant. The first term is constant by (3) and the case of manifolds. Thus the middle term is constant by the 5-lemma.

For (4) : One has to follows the finite generation in all steps. □

The following corollary to proposition 8.0.12 will be of importance for us :

Proposition 8.0.15. *Let $(X, \mathcal{X})$ be a pseudomanifold with stratification $\mathcal{X}$. Let $\mathcal{F}^\bullet \in \mathbf{DGSh}(X)$ satisfy axioms $(AX1)_{\mathcal{X},\mathcal{E},p}$ for some local system $\mathcal{E}$ on U_2. Then $\mathcal{F}^\bullet$ is $\mathcal{X}$-cc and cc.*

Proof. In view of proposition 8.0.12 it suffices to show that Deligne's sheaf $\mathcal{P}^\bullet(\mathcal{E})$ is $\mathcal{X}$-cc. The local system $\mathcal{E}$ is $\mathcal{X}$-cc on U_2 and $\mathcal{P}^\bullet(\mathcal{E})$ is constructed by successive applications of operations Rj_{k*} and $\tau_{\leq p(k)}$. The truncation trivially preserves $\mathcal{X}$-constructibility, while we already showed (cf. equation (25)) that Rj_{k*} does. □

9. Second characterization of Deligne's extension and topological invariance of intersection homology

9.1. **Changing the axioms.** Let $(X, \mathcal{X})$ be a pseudomanifold of dimension n with a given stratification $\mathcal{X}$ and let $\mathcal{F}^\bullet \in \mathbf{DGSh}(X)$. Recall that we introduced the set of axioms $(AX1)_{\mathcal{X},\mathcal{E},p}$:

(i) $\mathcal{F}^\bullet$ is bounded below, $\mathcal{H}^i\mathcal{F}^\bullet = 0$ for $i < 0$ and $\mathcal{F}_2^\bullet = \mathcal{E}$.
(ii) if $x \in S^k$ then $H^j(\mathcal{F}_x^\bullet) = 0$ for $j > p(k)$.
(iii) the attachment map $\alpha_k : \mathcal{F}_{k+1}^\bullet \longrightarrow Rj_{k*}\mathcal{F}_k^\bullet$ is a quasi-isomorphism in degree up to $p(k)$.

Proposition 9.1.1. *The DG-sheaf $\mathcal{F}^\bullet$ satisfies $(AX1)_{\mathcal{X},\mathcal{E},p}$ if and only if it satisfies the axioms $(AX1)'_{\mathcal{X},\mathcal{E},p}$:*

(i) *$\mathcal{F}^\bullet$ is bounded below, $\mathcal{H}^i\mathcal{F}^\bullet = 0$ for $i < 0$, $\mathcal{F}_2^\bullet = \mathcal{E}$ and $\mathcal{F}^\bullet$ is $\mathcal{X}$-clc.*
(ii) *if $x \in S^k$ then $H^j(\mathcal{F}_x^\bullet) = 0$ for $j > p(k)$.*
(iii) *if $x \in S^k$ then $H^j(i_x^!\mathcal{F}^\bullet) = 0$ for $j < n - q(k)$.*

Proof. First by proposition 8.0.15 $\mathcal{F}^\bullet$ is $\mathcal{X}$-clc if $\mathcal{F}$ satisfies $(AX1)_{\mathcal{X},\mathcal{E},p}$.

Thus it is enough to show that the two conditions (iii) are equivalent under the other ones. We have the following long exact sequence :

$$\cdots \longrightarrow H^j(i_k^!\mathcal{F}^\bullet)_x \longrightarrow H^j(\mathcal{F}_x^\bullet) \xrightarrow{\alpha_k} H^j(Rj_{k*}\mathcal{F}_k^\bullet)_x \longrightarrow \cdots \ .$$

It implies that the attachment map $\alpha_k : \mathcal{F}_{k+1}^\bullet \longrightarrow Rj_{k*}\mathcal{F}_k^\bullet$ is a quasi-isomorphism in degree up to $p(k)$ if and only if $H^j(i_k^!\mathcal{F}^\bullet)_x = 0$ for $j \leq p(k)+1$ (using that $H^{p(k)+1}(\mathcal{F}_x^\bullet) = 0$).

Consider the factorisation

$$\begin{tikzcd} \{x\} \arrow[r, "l_x"] \arrow[dr, "i_x"'] & S^k \arrow[d, "i_k"] \\ & X \end{tikzcd} \ .$$

Thus $i_x^! = l_x^! \circ i_k^!$ and $H^j(i_k^!\mathcal{F}^\bullet) = H^j(l_x^!(i_k^!\mathcal{F}^\bullet))$. By proposition 8.0.14 the sheaf $i_k^!\mathcal{F}^\bullet$ is clc. As S^k is a manifold of dimension $n-k$ lemma 8.0.13 implies that

$$H^j(l_x^!(i_k^!\mathcal{F}^\bullet)) = H^{j-(n-k)}(i_k^!\mathcal{F}^\bullet)_x \ .$$

As $j < n - q(k)$ if and only if $j - (n-k) \leq p(k) + 1$ we obtain the result. □

Proposition 9.1.2. *The DG-sheaf $\mathcal{F}^\bullet$ satisfies $(AX1)_{\mathcal{X},\mathcal{E},p}$ if and only if it satisfies the axioms $(AX2)_{\mathcal{X},\mathcal{E},p}$:*

(i) *$\mathcal{F}^\bullet$ is bounded below, $\mathcal{H}^i\mathcal{F}^\bullet = 0$ for $i < 0$, $\mathcal{F}_2^\bullet = \mathcal{E}$ and $\mathcal{F}^\bullet$ is $\mathcal{X}$-clc.*
(ii) *$\dim \operatorname{supp}(\mathcal{H}^j\mathcal{F}^\bullet)(= \dim\{x \in X \ / \ H^j(i_x^*\mathcal{F}^\bullet) \neq 0\}) \leq n - p^{-1}(j)$ for all $j > 0$.*
(iii) *$\dim\{x \in X \ / \ H^j(i_x^!\mathcal{F}^\bullet) \neq 0\}) \leq n - q^{-1}(n-j)$ for all $j < n$.*

In particular $(AX2)_{\mathcal{X},\mathcal{E},p}$ caracterizes $\mathcal{F}^\bullet$ uniquely up to quasi-isomorphism.

Remark 9.1.3. Notice that now only the first condition depends on $\mathcal{X}$.

Proof. First notice that

$$j \leq p(k) \Longleftrightarrow n - k \leq n - p^{-1}(j) \ , \tag{26}$$

$$j \geq n - q(k) \Longleftrightarrow n - k \leq n - q^{-1}(n-j) \ . \tag{27}$$

Let us show that $(AX1)_{\mathcal{X},\mathcal{E},p}(ii)$ implies $(AX2)_{\mathcal{X},\mathcal{E},p}(ii)$. If $x \in S^k$ and $H^j(\mathcal{F}_x^\bullet) \neq 0$ then $j \leq p(k)$. Hence $\dim S^k \leq n - k \leq n - p^{-1}(j)$ and the result.

Conversely if $x \in S^k$ and $H^j(\mathcal{F}^\bullet_x) \neq 0$ then $H^j(\mathcal{F}^\bullet_y) \neq 0$ for y in some neighbourhood of x in S^k since $\mathcal{F}^\bullet$ is $\mathcal{X}$-cc. Therefore $n-k \leq n-p^{-1}(j)$ and thus $j \leq p(k)$.

We prove that $(AX1)_{\mathcal{X},\mathcal{E},p}(iii)$ implies $(AX2)_{\mathcal{X},\mathcal{E},p}(iii)$ in the same way as for (ii). Conversely if $x \in S^k$ then $i_x = i_k \circ l_x$ where $l_x : \{x\} \hookrightarrow S^k$ and $i^!_x = {l_x}^! \circ {i_k}^!$. In particular ${i_x}^!\mathcal{F}^\bullet = {l_x}^!({i_k}^!\mathcal{F}^\bullet)$. It follows from lemma 8.0.13 that $H^j({i_y}^!\mathcal{F}^\bullet) = H^j({i_x}^!\mathcal{F}^\bullet)$ for y in some neighbourhood of x in S^k. We can then proceed as for (ii). $\square$

9.2. **Extension of local systems.**

Lemma 9.2.1. *Let M be a manifold of dimension n and $U \subset M$ a dense open subset whose complement has codimension at least 2. If $\mathcal{E}$ is a local system on U then there exists a largest open subset $V \supset U$ of M over which $\mathcal{E}$ extends to a local system.*

Proof. We can assume that M is connected, then so is U by the codimension 2 condition.

First notice that if $\mathcal{F}$ and $\mathcal{F}'$ are two local systems on M and $f : \mathcal{F}_{|U} \longrightarrow \mathcal{F}'_{|U}$ is a morphism then there exists a unique morphism $g : \mathcal{F} \longrightarrow \mathcal{F}'$ extending f. This follows from the fact that $\pi_1(U) \twoheadrightarrow \pi_1(X)$. Moreover g is an isomorphism if f is.

Using this remark one can glue together the local systems $\mathcal{E}'$ extending $\mathcal{E}$ over some $U' \supset U$. $\square$

Definition 9.2.2. *Let $\mathcal{E}$ be a local system on some open dense submanifold of X whose complement has codimension at least 2. A stratification $\mathcal{X}$ of X is said to be adapted to $\mathcal{E}$ if its open stratum is contained in the largest open submanifold of X over which $\mathcal{E}$ can be extended.*

9.3. **Second characterization of Deligne's extension.** We now define a new set of conditions on $\mathcal{F}^\bullet \in \mathbf{DGSh}(X)$.

Definition 9.3.1. *Let $\mathcal{E}$ be a local system on some open dense submanifold of X whose complement has codimension at least 2. We denote by $(AX2)_{\mathcal{E},p}$ the following set of conditions on $\mathcal{F}^\bullet \in \mathbf{DGSh}(X)$:*

(i) *$\mathcal{H}^i\mathcal{F}^\bullet = 0$ for $i < 0$; $\mathcal{F}^\bullet$ is $\mathcal{X}$-clc for some PL-pseudomanifold stratification $\mathcal{X}$ of X ; there exists an open dense subset submanifold U of X whose complement has codimension at least 2 on which $\mathcal{E}$ is defined such that $\mathcal{F}^\bullet_{|U} \simeq \mathcal{E}_{|U}$.*

(ii) *$\dim \operatorname{supp} \mathcal{H}^j\mathcal{F}^\bullet \le n - p^{-1}(j)$ for all $j > 0$.*

(iii) *$\dim \{x \in X \ / \ H^j(i_x^!\mathcal{F}^\bullet) \neq 0\} \le n - q^{-1}(n-j)$ for all $j < n$, where $q = t - p$ is the dual perversity.*

Notice that in axioms $(AX2)_{\mathcal{E},p}$ the stratification $\mathcal{X}$ and the open submanifold U are not related. We can however assume that $U \subset U_2$. As $\mathcal{H}^i(\mathcal{F}^\bullet)$ is locally constant on U_2 and has restriction $\mathcal{E}$ or 0 on U depending whether $i = 0$ or not, $\mathcal{E}$ can be extended to U_2 and $\mathcal{F}^\bullet_{|U_2}$ is quasi-isomorphic to $\mathcal{E}$. In particular $\mathcal{F}^\bullet$ satisfies $(AX2)_{\mathcal{E},p}$ if and only if it satisfies $(AX2)_{\mathcal{X},\mathcal{E},p}$ for some stratification $\mathcal{X}$ adapted to $\mathcal{E}$.

The main result of this section is the following :

Theorem 9.3.2. *Let $\mathcal{E}$ be a local system on some open dense submanifold of X whose complement has codimension at least 2 and let $\tilde{U}_2$ be the largest open submanifold of X over which $\mathcal{E}$ extends to a local system.*

Assume that there exists a stratification $\mathcal{X}$ of X which is adapted to $\mathcal{E}$.

Then there exists $\tilde{\mathcal{P}}^\bullet \in \mathbf{DGSh}(X)$ satisfying $(AX2)_{\mathcal{E},p}$ with $\tilde{\mathcal{P}}^\bullet_{|\tilde{U}_2} = \mathcal{E}$ and also $(AX2)_{\mathcal{X},\mathcal{E},p}$ for every stratification $\mathcal{X}$ adapted to $\mathcal{E}$.

Proof. Notice that Deligne's construction is well defined for any *filtration* by closed subsets (not necessarily a stratification of the pseudomanifold X) and for $\mathcal{E}$ a local system on the open stratum (not necessarily a manifold). We construct $\tilde{\mathcal{P}}^\bullet$ as the Deligne sheaf associated to $\mathcal{E}$ and some well-chosen filtration $\tilde{\mathcal{X}}$ of X by closed subsets satisfying $\tilde{\mathcal{X}}_{n-2} := X \setminus \tilde{U}_2$.

The main difficulty is to construct $\tilde{\mathcal{X}}$. We proceed by induction on k. With the usual notations we require, for $2 \le k \le n$:

(I_k) (a_k) $\tilde{S}^k$ is a manifold of dimension $n-k$ or is empty.
(b_k) $\tilde{i}_k^*\tilde{\mathcal{P}}^\bullet_{k+1}$ is clc.
(c_k) $\tilde{i}_k^{\,!}\tilde{\mathcal{P}}^\bullet_{k+1}$ is clc.

(II_k) for every stratification $\mathcal{X}$ of X which is adapted to $\mathcal{E}$, $\tilde{S}^k$ is a union of strata of $\mathcal{X}$ and $U_{k+1} \subset \tilde{U}_{k+1}$.

Suppose that $\tilde{\mathcal{X}}$ has these properties. As we are assuming there exists at least one stratification $\mathcal{X}$ of X which is adapted to $\mathcal{E}$, by (II_n) we have $\tilde{U}_{n+1} \supset U_{n+1} = X$. Let $\tilde{\mathcal{P}}^\bullet = \tilde{\mathcal{P}}^\bullet_{n+1} \in \mathbf{DGSh}(X)$. One easily checks that $\tilde{\mathcal{P}}^\bullet$ satisfies $(AX1)_{\tilde{X},\mathcal{E},p}$ (even if $\tilde{\mathcal{X}}$ is only a filtration). One can also check that the conditions (I_k) on the filtration $\tilde{\mathcal{X}}$ are enough to ensure the equivalence of the axioms $(AX1)_{\tilde{\mathcal{X}},\mathcal{E},p}$ and $(AX2)_{\tilde{\mathcal{X}},\mathcal{E},p}$. Thus $\tilde{\mathcal{P}}^\bullet$ satisfies $(AX2)_{\tilde{\mathcal{X}},\mathcal{E},p}$. If $\mathcal{X}$ is a stratification adapted to X we find by (II) that $\tilde{\mathcal{P}}^\bullet$ is $\mathcal{X}$-clc. Therefore it satisfies $(AX2)_{\mathcal{X},\mathcal{E},p}$. As there exists at least one such stratification $\mathcal{X}$, $\tilde{\mathcal{P}}^\bullet$ satisfies $(AX2)_{\mathcal{E},p}$ and has the required properties.

We are reduced to construct a filtration $\tilde{\mathcal{X}}$ satisfying the properties (I_k) and (II_k) for all $k \ge 2$.

The dense stratum $\tilde{U}_2$ is already defined and $\tilde{\mathcal{P}}^\bullet_2 = \mathcal{E}$. Let $\mathcal{X}$ be a stratification of X adapted to $\mathcal{E}$. Suppose by induction that $\tilde{U}_i$, $2 \le i \le k$, are already defined and that (I_i) and (II_i), $2 \le i < k$, hold. Let $\tilde{\mathcal{P}}^\bullet_{k+1} := \tau_{\le p(k)} R\tilde{j}_{k*}\tilde{\mathcal{P}}^\bullet_k$, where $\tilde{j}_k : U_k \hookrightarrow X$.

Let $\tilde{S}^k_1$ be the largest submanifold of $\tilde{X}_{n-k} = X \setminus \tilde{U}_k$ of dimension $n-k$. Let $\tilde{S}^k_2$ (resp. $\tilde{S}^k_3$) be the largest open subset of $\tilde{X}_{n-k}$ over which $\tilde{i}_k^*\tilde{\mathcal{P}}^\bullet_{k+1}$ (resp. $\tilde{i}_k^!\tilde{\mathcal{P}}^\bullet_{k+1}$) is clc. We take

$$\tilde{S}^k := \tilde{S}^k_1 \cap \tilde{S}^k_2 \cap \tilde{S}^k_3, \qquad \tilde{U}_{k+1} = \tilde{U}_k \cup \tilde{S}^k \quad .$$

Clearly $\tilde{U}_{k+1}$ is open and satisfies (I_k). It remains to check it satisfies (II_k). We refer to [**?**, p.93-94] for details. $\square$

Corollary 9.3.3. *Let $\mathcal{F}^\bullet$ satisfy $(AX2)_{\mathcal{E},p}$. Then $\mathcal{F}^\bullet$ is quasi-isomorphic to $\tilde{\mathcal{P}}^\bullet$.*

Proof. By the remark before the statement of theorem 9.3.2 we know that $\mathcal{F}^\bullet$ satisfies $(AX2)_{\mathcal{X},\mathcal{E},p}$ for some stratification $\mathcal{X}$ adapted to $\mathcal{E}$. The DG-sheaf $\tilde{P}^\bullet$ also satisfies $(AX2)_{\mathcal{X},\mathcal{E},p}$ by the theorem. Thus $\mathcal{F}^\bullet$ is qi to $\tilde{P}^\bullet$ by proposition 9.1.2. $\square$

In particular, as every stratification is adapted to the orientation sheaf, one obtains :

Corollary 9.3.4. *Let X be a PL pseudo-manifold. Then the intersection homology groups $I^pH_\bullet(X)$ are independant of the PL-stratification used in their definition.*

10. Poincaré-Verdier duality

Our next goal is to prove Poincaré duality for intersection cohomology. It will appear as a special case of the Poincaré-Verdier duality.

Let $f : X \longrightarrow Y \in$ **Top**. We defined three functors :

$$f^{-1} : \qquad \mathcal{D}^+(\mathbb{Z}_Y) \rightleftarrows \mathcal{D}^+(\mathbb{Z}_X) \qquad : Rf_*$$

and

$$Rf_! : \mathcal{D}^+(\mathbb{Z}_X) \longrightarrow \mathcal{D}^+(\mathbb{Z}_Y) \ .$$

In this section we show that, under certain assumptions, the functor $Rf_!$ admits a right adjoint $f^!$, called the "exceptional inverse image".

Theorem 10.0.5. *Let X, Y be two locally compact spaces and $f : X \longrightarrow Y$ a continuous map such that $f_!$ has finite cohomological dimension (i.e. there exists $n \in \mathbb{N}$ such that for any $F \in \mathcal{D}^+(\mathbb{Z}_X)$, $R^if_!(F) = 0$ for $i > n$). Then*

$$Rf_! : \mathcal{D}^+(\mathbb{Z}_X) \longrightarrow \mathcal{D}^+(\mathbb{Z}_Y)$$

admits a right adjoint

$$f^! : \mathcal{D}^+(\mathbb{Z}_Y) \longrightarrow \mathcal{D}^+(\mathbb{Z}_X) \ ,$$

i.e. there exist two morphisms of functors $\mathrm{id} \longrightarrow f^!Rf_!$ *and* $Rf_!f^! \longrightarrow \mathrm{id}$ *inducing :*

$$\forall F \in \mathcal{D}^+(\mathbb{Z}_X),\ G \in \mathcal{D}^+(\mathbb{Z}_Y), \qquad \mathrm{Hom}_{\mathcal{D}^+(\mathbb{Z}_X)}(F, f^!G) \simeq \mathrm{Hom}_{\mathcal{D}^+(\mathbb{Z}_Y)}(Rf_!F, G) \ .$$

Remark 10.0.6. We already constructed $f^! : \mathbf{Sh}(Y) \longrightarrow \mathbf{Sh}(X)$ when $f : X \longrightarrow Y$ is a locally closed subspace. In this case $f^! : \mathcal{D}^+(\mathbb{Z}_Y) \longrightarrow \mathcal{D}^+(\mathbb{Z}_X)$ is nothing else than our previous $Rf^!$. For a general f however, $f^!$ exists only at the level of derived categories.

Before proving theorem 10.0.5 we first deduce some corollaries. As $(f \circ g)_! = f_! \circ g_!$ and $g_!$ maps c-soft sheaves to c-soft sheaves (which are acyclic for any $h_!$) one has $R(f \circ g)_! = Rf_! \circ Rg_!$. By adjunction one obtains :

$$(f \circ g)^! \simeq g^! \circ f^! \ .$$

Corollary 10.0.7. *Let $f : X \longrightarrow Y$ such that $f_!$ has finite cohomological dimension. Let $F \in \mathcal{D}^b(\mathbb{Z}_X)$, $G \in \mathcal{D}^+(\mathbb{Z}_Y)$ and $G' \in \mathcal{D}^b(\mathbb{Z}_Y)$. Then :*

$$Rf_*\ R\mathcal{H}om_{\mathbb{Z}_X}(F, f^!G) \simeq R\mathcal{H}om_{\mathbb{Z}_Y}(Rf_!F, G), \tag{28}$$

$$R\,\mathrm{Hom}_{\mathbb{Z}_X}(F, f^!G) \simeq R\,\mathrm{Hom}_{\mathbb{Z}_Y}(Rf_!F, G), \tag{29}$$

$$f^!\ R\mathcal{H}om_{\mathbb{Z}_Y}(G', G) \simeq R\mathcal{H}om_{\mathbb{Z}_X}(f^{-1}G', f^!G). \tag{30}$$

Proof. Note that (2) follows from (1) by applying $R\Gamma(Y, \cdot)$.

For (1) : using adjunction and the projection formula we obtain for all $H \in \mathcal{D}^+(\mathbb{Z}_Y)$:

$$\begin{aligned}\mathrm{Hom}_{\mathcal{D}^+(\mathbb{Z}_Y)}(H, Rf_* \, R\mathcal{H}om_{\mathbb{Z}_X}(F, f^! G)) &\simeq \mathrm{Hom}_{\mathcal{D}^+(\mathbb{Z}_X)}(f^{-1}H, R\mathcal{H}om_{\mathbb{Z}_X}(F, f^! G)) \\ &\simeq \mathrm{Hom}_{\mathcal{D}^+(\mathbb{Z}_X)}(f^{-1}H \otimes F, f^! G) \\ &\simeq \mathrm{Hom}_{\mathcal{D}^+(\mathbb{Z}_Y)}(Rf_!(f^{-1}H \otimes F), G) \\ &\simeq \mathrm{Hom}_{\mathcal{D}^+(\mathbb{Z}_Y)}(H \otimes Rf_! F, G) \\ &\simeq \mathrm{Hom}_{\mathcal{D}^+(\mathbb{Z}_Y)}(H, R\mathcal{H}om_{\mathbb{Z}_Y}(Rf_! F, G)) \ .\end{aligned}$$

For (3) : similarly for all $H \in \mathcal{D}^+(\mathbb{Z}_X))$:

$$\begin{aligned}\mathrm{Hom}_{\mathcal{D}^+(\mathbb{Z}_X)}(H, f^! \, R\mathcal{H}om_{\mathbb{Z}_Y}(G', G)) &\simeq \mathrm{Hom}_{\mathcal{D}^+(\mathbb{Z}_Y)}(Rf_! H, R\mathcal{H}om_{\mathbb{Z}_Y}(G', G)) \\ &\simeq \mathrm{Hom}_{\mathcal{D}^+(\mathbb{Z}_Y)}(Rf_! H \otimes G', G) \\ &\simeq \mathrm{Hom}_{\mathcal{D}^+(\mathbb{Z}_Y)}(Rf_!(H \otimes f^{-1}G'), G) \\ &\simeq \mathrm{Hom}_{\mathcal{D}^+(\mathbb{Z}_X)}(H \otimes f^{-1}G', f^! G) \\ &\simeq \mathrm{Hom}_{\mathcal{D}^+(\mathbb{Z}_X)}(H, R\mathcal{H}om_{\mathbb{Z}_Y}(f^{-1}G', f^! G)) \ .\end{aligned}$$

□

Definition 10.0.8. *Given $f : X \longrightarrow Y$ we define the* relative dualizing complex

$$\omega_{X/Y} := f^! \mathbb{Z}_Y \in \mathcal{D}^+(\mathbb{Z}_X) \ .$$

If $Y = \{*\}$ we write $\omega_X := f^! \mathbb{Z}$ and call it the dualizing complex of X.
If $F \in \mathcal{D}^b(\mathbb{Z}_X)$ the dual of F *is $D_X F := R\mathcal{H}om_{\mathbb{Z}_X}(F, \omega_X) \in \mathcal{D}^+(\mathbb{Z}_X)$.*

By definition $f^! \omega_Y = \omega_X$ thus the previous corollary implies :

$$Rf_* \, D_X F \simeq D_Y(Rf_! F) \ , \tag{31}$$

$$f^!(D_Y G) \simeq D_X(f^{-1}G) \ . \tag{32}$$

Proposition 10.0.9. *Let X be a (topological) n-manifold. Then*

$$\omega_X \simeq \mathcal{O}r_X[n] \ .$$

Proof. Let x be a point of X and U a neighbourhood of x homeomorphic to a n-ball. Let $p_X : X \longrightarrow \{*\}$ be the canonical projection. Then

$$\begin{aligned}R\Gamma(U, \omega_X) &= R\,\mathrm{Hom}_{\mathbb{Z}_X}(\mathbb{Z}_U, p_X^! \mathbb{Z}) \\ &= R\,\mathrm{Hom}(Rp_{X!}\mathbb{Z}_U, \mathbb{Z}) \\ &= R\,\mathrm{Hom}(R\Gamma_c(U, \mathbb{Z}_U), \mathbb{Z}) \ .\end{aligned}$$

The choice of a local orientation of X at x gives a canonical isomorphism $R\Gamma_c(U, \mathbb{Z}_U) = \mathbb{Z}[n]$ thus the result. □

Corollary 10.0.10 (weak Poincaré duality for manifolds)**.** *Let X be a (topological) n-manifold. Then*

$$H_c^i(X, \mathbb{Q}_X)^* \simeq H^{n-i}(X, \mathcal{O}r_{X,\mathbb{Q}}) \ .$$

Proof.

$$\begin{aligned}H^{n-i}(X, \mathcal{O}r_{X,\mathbb{Q}}) &= H^{-i} R\,\mathrm{Hom}(\mathbb{Q}_X, {p_X}^! \mathbb{Q}) \\ &= H^{-i} R\,\mathrm{Hom}(Rp_{X!}\mathbb{Q}_X, \mathbb{Q}) \\ &= (H^i R\Gamma_c(X, \mathbb{Q}_X))^* \ ,\end{aligned}$$

where we used that the derived category of $\mathbb{Q}$-vector spaces is nothing else than the sum indexed by $\mathbb{Z}$ of copies of the category of $\mathbb{Q}$-vector spaces. □

10.1. **Proof of theorem 10.0.5.** We first introduce a relative version of the notion of c-soft sheaves.

Definition 10.1.1. *Let $f : X \longrightarrow Y \in \mathbf{Top}$. A sheaf $F \in \mathbf{Sh}(X)$ is f-soft if $H^i R f_! F = 0$ for all $i \neq 0$.*

Lemma 10.1.2. *$F \in \mathbf{Sh}(X)$ is f-soft if and only if for all $y \in Y$, the sheaf F_{X_y} is c-soft.*

Proof. We proved that $(R^i f_! F)_y = R^i \Gamma_c(X_y, F_{X_y})$. Hence the result. □

Remark 10.1.3. By the lemma above any flasque sheaf is f-soft.

Definition 10.1.4. *A sheaf $\mathcal{F} \in \mathbf{Sh}(X)$ is flat if for any monomorphism $\mathcal{E} \overset{i}{\hookrightarrow} \mathcal{F} \in \mathbf{Sh}(X)$ the morphism $\mathcal{E} \otimes \mathcal{F} \xrightarrow{i \otimes 1} \mathcal{G} \otimes \mathcal{F}$ is also injective.*

Thus a sheaf $\mathcal{F} \in \mathbf{Sh}(X)$ is flat if and only if $\mathcal{F}_x$ is a flat $\mathbb{Z}$-module for all $x \in X$.

Proposition 10.1.5. *Let X, Y be two locally compact spaces and $f : X \longrightarrow Y$ a continuous map such that $f_!$ has finite cohomological dimension r. Then $\mathbb{Z}_X$ admits a flat and f-soft finite resolution :*

$$0 \longrightarrow \mathbb{Z}_X \longrightarrow K^0 \longrightarrow \cdots \longrightarrow K^r \longrightarrow 0 \ .$$

Proof. Let X^δ denotes the space X with the discrete topology and let $i : X^\delta \longrightarrow X$ be the canonical continuous map.

The sheaf $i^{-1}\mathbb{Z}_X \in \mathbf{Sh}(X^\delta)$ is flasque thus also $K^0 := i_* i^{-1}\mathbb{Z}_X$. Moreover one has the canonical injection $K^{-1} := \mathbb{Z}_X \hookrightarrow K^0 = i_* i^{-1}\mathbb{Z}_X$. By induction we define, for $1 \leq j \leq r-1$, $K^j = i_* i^{-1}\mathrm{coker}(K^{j-2} \longrightarrow K^{j-1})$. The sheaves $K^j \in \mathbf{Sh}(X)$, $0 \leq j \leq r-1$, are flasque, thus f-soft.

Let $K^r := \mathrm{coker}(K^{r-2} \longrightarrow K^{r-1})$. As the K^j, $0 \leq j \leq r-1$, are f-soft, one has $R^i f_! K^r = R^{r+i} f_! \mathbb{Z}_X$. This last group vanishes because $f_!$ has cohomological dimension r. Hence K^r is f-soft. Finally we obtain a f-soft resolution :

$$0 \longrightarrow \mathbb{Z}_X \longrightarrow K^0 \longrightarrow \cdots \longrightarrow K^r \longrightarrow 0 \ .$$

Let us show that the K^j's, $0 \leq j \leq r$, are flat. By induction if is enough to show that if $\mathcal{F} \in \mathbf{Sh}(X)$ is flat then $i_* i^{-1}\mathcal{F}$ and $\mathrm{coker}(\mathcal{F} \longrightarrow i_* i^{-1}\mathcal{F})$ are also flat. As flatness can be checked on germs it is enough to check that $(i_* i^{-1}\mathcal{F})_x$ and $(i_* i^{-1}\mathcal{F}/\mathcal{F})_x$ are flat for all $x \in X$. One computes :

$$(i_* i^{-1}\mathcal{F})_x = \mathrm{colim}_{U \ni x} \prod_{z \in U} \mathcal{F}_z, \tag{33}$$

$$(i_* i^{-1}\mathcal{F}/\mathcal{F})_x = \mathrm{colim}_{U \ni x} \prod_{z \in U \setminus \{x\}} \mathcal{F}_z \tag{34}$$

and the result. □

We will construct $f^!$ using the following heuristics. Suppose that $f^!$ exists. Then for any $U \in \mathbf{Op}_X$ we have :

$$R\Gamma(U, f^! G) \simeq R\,\mathrm{Hom}(\mathbb{Z}_U, f^! G) \simeq R\,\mathrm{Hom}(R f_! \mathbb{Z}_U, G) \ .$$

Hence if $K^\bullet$ is a f-soft resolution of $\mathbb{Z}_X$ (thus by lemma 10.1.2 $K^\bullet_U$ is a f-soft resolution of $\mathbb{Z}_U$) and $I^\bullet$ is an injective resolution of G one has

$$R\Gamma(U, f^! G) = \mathrm{Hom}^\bullet(f_! K^\bullet_U, I^\bullet) \tag{35}$$

Definition 10.1.6. *Let $K \in \mathbf{Sh}(X)$ be f-soft and $G \in \mathbf{Sh}(Y)$. One defines a presheaf $f_K^! G \in PSh(X)$ by :*
$$f_K^! G(U) := \mathrm{Hom}_{\mathbb{Z}_Y}(f_! K_U, G) \ .$$

Proposition 10.1.7. *Let $K \in \mathbf{Sh}(X)$ be f-soft and flat. Let $G \in \mathbf{Sh}(Y)$ be injective.*

(i) *if $F \in \mathbf{Sh}(X)$ then $F \otimes K$ is f-soft.*
(ii) *the functor $F \mapsto f_!(F \otimes K)$ from $\mathbf{Sh}(X)$ to $\mathbf{Sh}(Y)$ is exact.*
(iii) *$f_K^! G$ is an injective sheaf.*
(iv) *for any $F \in \mathbf{Sh}(X)$ one has*
$$\mathrm{Hom}_{\mathbb{Z}_Y}(f_!(F \otimes K), G) \simeq \mathrm{Hom}_{\mathbb{Z}_X}(F, f_K^! G) \ .$$

Proof. For (i) : first notice that any $F \in \mathbf{Sh}(X)$ admits a resolution
$$\cdots \longrightarrow F^{-l} \longrightarrow \cdots \longrightarrow F^0 \longrightarrow F \longrightarrow 0 \ ,$$
with $F^i = \oplus_{U \in I_i} \mathbb{Z}_U$. For example take $F^0 = \oplus_{U \in \mathbf{Op}_X} \oplus_{s \in F(U)} \mathbb{Z}_U$ and $d^0 : F^0 \longrightarrow F$ the obvious map sum of $1 \mapsto s$. Construct F^i inductively, replacing F by $\ker d^0$, etc ...

As K is f-soft, $\mathbb{Z}_U \otimes K = K_U$ is f-soft by lemma 10.1.2. Hence $F^i \otimes K$ is f-soft. As K is flat the sequence
$$\cdots \longrightarrow F^{-l} \otimes K \longrightarrow \cdots \longrightarrow F^0 \otimes K \longrightarrow F \otimes K \longrightarrow 0$$
is exact.

As $f_!$ has finite cohomological dimension this implies that $F \otimes K$ is also f-soft.

For (ii) : it follows immediately from (i).

For (iii) : first we show that $f_K^! G$ is a sheaf. Let (U_i) be a family of open sets in X with union U. We have to show that the sequence :
$$0 \longrightarrow (f_K^! G)(U) \longrightarrow \prod (f_K^! G)(U_i) \longrightarrow \prod (f_K^! G)(U_i \cap U_j)$$
is exact.

As $(f_K^! G)(V) = \mathrm{Hom}_{\mathbb{Z}_Y}(f_! K_V, G)$ and $\mathrm{Hom}_{\mathbb{Z}_Y}(\cdot, G)$ is exact as G in injective, it it enough to show that the sequence of sheaves
$$f_!(\oplus K_{U_i \cap U_j}) \longrightarrow f_!(\oplus K_{U_i}) \longrightarrow f_!(K_U) \longrightarrow 0$$
is exact.

This follows from (ii) and the fact that the sequence
$$\oplus \mathbb{Z}_{U_i \cap U_j} \longrightarrow \oplus \mathbb{Z}_{U_i} \longrightarrow \mathbb{Z}_U \longrightarrow 0$$
is exact.

The fact that the sheaf $f_K^! G$ is injective follows from (iv), (ii) and the fact that G is injective.

For (iv) : we first define a morphism
$$\alpha(F, G) : \mathrm{Hom}_{\mathbb{Z}_Y}(f_!(F \otimes K), G) \longrightarrow \mathrm{Hom}_{\mathbb{Z}_X}(F, f_K^! G) \ .$$
Let $\phi \in \mathrm{Hom}_{\mathbb{Z}_Y}(f_!(F \otimes K), G)$. We define $\alpha(\phi)$ as the morphism of presheaves defined by the collection
$$\alpha(\phi)(U) : F(U) = \mathrm{Hom}_{\mathbb{Z}_X}(\mathbb{Z}_U, F) \xrightarrow{f_!(\cdot \otimes K)} \mathrm{Hom}_{\mathbb{Z}_Y}(f_!(\mathbb{Z}_U \otimes K = K_U), f_!(F \otimes K))$$
$$\xrightarrow{\phi} \mathrm{Hom}_{\mathbb{Z}_Y}(f_! K_U, G) =: f_K^! G(U) \ .$$
We leave as an exercise the fact that $\alpha(\phi)$ is really a morphism of (pre)sheaves and that $\alpha(F, G)$ is functorial in F and G.

Let us prove that α is an isomorphism. First this is true for $F = \mathbb{Z}_U$. Indeed in this case the left term is $f_k^! G(U)$ by definition, the right one is also $f_K^! G(U)$ (and one easily checks that α is the identity). By additivity if follows that α is an isomorphism for any $F = \oplus \mathbb{Z}_{U_i}$. For a general F we can find an exact sequence
$$0 \longrightarrow F'' \longrightarrow F' = \oplus \mathbb{Z}_{U_i} \longrightarrow F \longrightarrow 0 \ .$$

As α is functorial and $f_!(\cdot \otimes K)$ is exact we have a commutative diagram of exact sequences :

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & \mathrm{Hom}(f_!(F\otimes K), G) \arrow[r] \arrow[d, "\alpha(F,G)"'] & \mathrm{Hom}(f_!(F'\otimes K), G) \arrow[r] \arrow[d, "\alpha(F',G)"'] & \mathrm{Hom}(f_!(F''\otimes K), G) \arrow[d, "\alpha(F'',G)"'] \\
0 \arrow[r] & \mathrm{Hom}(F, f_K^! G) \arrow[r] & \mathrm{Hom}(F', f_K^! G) \arrow[r] & \mathrm{Hom}(F'', f_K^! G)
\end{tikzcd}$$

As $\alpha(F', G)$ is an isomorphism the morphism $\alpha(F, G)$ is injective for any F. In particular $\alpha(F'', G)$ is injective. This forces $\alpha(F', G)$ to be an isomorphism. □

Let us finish the proof of theorem 10.0.5. It is enough to define $f^!$ on complexes of injectives objects in $\mathcal{D}^+(\mathbb{Z}_Y)$. Let $\mathcal{I}_Y \subset \mathbf{Sh}(Y)$ denote the full subcategory of injective sheaves.

Fix $0 \longrightarrow \mathbb{Z}_X \longrightarrow K^0 \longrightarrow K^1 \longrightarrow \cdots \longrightarrow K^r \longrightarrow 0$ a flat f-soft resolution of $\mathbb{Z}_X$. For $G^\bullet \in C^+(\mathcal{I}_Y)$ let $f^!_{K^\bullet} G^\bullet$ be the total complex of the double complex $C^{i,j} = f^!_{K^{-i}} G^j$. By proposition 10.1.7[(iii)] this is a complex of injective sheaves. One easily checks that $f^!_{K^\bullet}(\cdot)$ send morphisms homotopic to zero to morphisms homotopic to zero and thus defines a functor

$$f^!_{K^\bullet}(\cdot) : \mathbf{K}^+(\mathcal{I}_Y) \longrightarrow \mathbf{K}^+(\mathcal{I}_X) \ .$$

By proposition 10.1.7[(iv)] one has for any $F \in \mathbf{K}^+(\mathcal{I}_X)$ and $G \in \mathbf{K}^+(\mathcal{I}_Y)$ an isomorphism :

$$\mathrm{Hom}_{\mathbf{K}^+(\mathcal{I}_Y)}(f_!(F \otimes K^\bullet), G) \simeq \mathrm{Hom}_{\mathbf{K}^+(\mathcal{I}_X)}(F, f^!_{K^\bullet} G) \ .$$

As the K^i are flat one has $F \otimes K^\bullet \simeq F \otimes \mathbb{Z}_X \simeq F$ in $\mathcal{D}^+(\mathbb{Z}_X)$. As $F \otimes K^\bullet$ is a complex of f-soft sheaves one has $f_!(F \otimes K^\bullet) \simeq Rf_! F$ in $\mathcal{D}^+(\mathbb{Z}_Y)$.

This proves that $f^!_{K^\bullet}$ is a right adjoint to $Rf_!$. Such an adjoint is unique thus the functor $f^!_{K^\bullet}$ is independent of the choice of the resolution $K^\bullet$ and is the required $f^!$.

Remarks 10.1.8.

(1) Notice that ω_X belongs to $D^b(X)$. Indeed as a complex $\omega_X^\bullet$ can be defined by

$$\omega_X^i(U) := \oplus_{j\in\mathbb{Z}} \mathrm{Hom}(\Gamma_c(K_U^j), I^{i+j}) \ ,$$

where $\mathbb{Z}_X \simeq K^\bullet$ is a c-soft flat resolution and $\mathbb{Z} \simeq I^\bullet$ is an injective resolution. Thus $K^i = 0$ for $i < 0, i > r$ where r is the cohomological dimension of X and $I^i = 0$ for $i < 0$, $i > 1$ (or more generally $i > d$ if we replace $\mathbb{Z}$ by a ring R of finite dimension d). Thus $\omega_X^i \neq 0$ only for $i \in [-r, 1]$.

(2) By proposition 10.1.7 the DG-sheaf $\omega_X^\bullet$ is injective. Hence :

$$\mathbb{H}^i(U, \omega_X^\bullet) = H^i(\Gamma(U, \omega_X^\bullet)) = \mathrm{Ext}^i(\Gamma_c(K_U^\bullet, \mathbb{Z})) \ .$$

The hypercohomology spectral sequence with

$$E_2^{p,q} = \mathrm{Ext}^p(H_c^{-1}(U, \mathbb{Z}), \mathbb{Z})$$

gives then (this would still hold replacing $\mathbb{Z}$ by any ring R of dimension 1) :

$$0 \longrightarrow \mathrm{Ext}^1(H_c^{i+1}(U, \mathbb{Z}), \mathbb{Z}) \longrightarrow \mathbb{H}^{-i}(U, \omega_X^\bullet) \longrightarrow \mathrm{Hom}(H_c^i(U, \mathbb{Z}), \mathbb{Z}) \longrightarrow 0 \ .$$

(3) As the complex $\omega_X^\bullet$ is injective the functor $F^\bullet \mapsto D_X F^\bullet$ from $C^+(X)$ to itself is exact.

11. Verdier duality and constructibility

The goal of this section is to prove the following :

Theorem 11.0.9. *Let $(X, \mathcal{X})$ be a stratified pseudomanifold. Let $F \in D^+(X)$ be $\mathcal{X}$-clc (resp. $\mathcal{X}$-cc). Then $D_X F \in D^+(X)$ is $\mathcal{X}$-clc (resp. $\mathcal{X}$-cc).*

As $D_X F = R\mathcal{H}om(F, \omega_X)$ it is enough to show the two following results :

Proposition 11.0.10. *Let $(X, \mathcal{X})$ be a stratified pseudomanifold. Then ω_X is $\mathcal{X}$-cc.*

Theorem 11.0.11. *Let $(X, \mathcal{X})$ be a stratified pseudomanifold. Let $F, G \in D^+(X)$ be $\mathcal{X}$-clc (resp. $\mathcal{X}$-cc). Then $R\mathcal{H}om(F, G) \in D^+(X)$ is $\mathcal{X}$-clc (resp. $\mathcal{X}$-cc).*

Proof of proposition 11.0.10. :

First notice that
$$(\omega_X)_k := j^*_{U_k \to X}\omega_X = j^!_{U_k \to X}\omega_X = \omega_{U_k} \quad .$$
We will prove by induction on $k \geq 2$ that ω_{U_k} is $\mathcal{X}$-cc.

For $k = 2$: $\omega_{U_2} \simeq \mathcal{O}r_{U_2}[n]$ and we are done.

Assume ω_{U_k} is $\mathcal{X}$-cc for some $k \geq 2$. Consider the exact triangle :
$$i_{k!}{i_k}^!(\omega_{U_{k+1}}) \longrightarrow \omega_{U_{k+1}} \longrightarrow Rj_{k*}\omega_{U_k} \xrightarrow{+1} \quad .$$
To show that $\omega_{U_{k+1}}$ is $\mathcal{X}$-cc it is enough to show the $\mathcal{X}$-constructibility of the other two terms in the triangle.

As ω_{U_k} is $\mathcal{X}$-cc by induction hypothesis, and Rj_{k*} maps $\mathcal{X}$-cc on U_k to $\mathcal{X}$-cc on U_{k+1}, we obtain that $Rj_{k*}\omega_{U_k}$ is $\mathcal{X}$-cc on U_{k+1}.

On the other hand :
$${i_k}^!\omega_{U_{k+1}} = {i_k}^! D_{U_{k+1}}\mathbb{Z}_{U_{k+1}} = D_{S^k}(i_k^{-1}\mathbb{Z}_{U_k}) = \omega_{S^k} \quad .$$
As ω_{S^k} is constructible on S^k it follows that $i_{k!}{i_k}^!\omega_{U_{k+1}} = i_{k!}\omega_{S^k}$ is $\mathcal{X}$-cc. □

Before proving theorem 11.0.11 we recall some classical facts :

First let $Y \in \mathbf{Top}$ and $F, G \in \mathbf{Sh}(Y)$. For any point $x \in Y$ one has a natural map
$$e_x : \mathcal{H}om(F,G)_x \longrightarrow \mathrm{Hom}(F_x, G_x)$$
which extends to a map
$$e_x^\bullet : R\mathcal{H}om(F,G)_x \longrightarrow R\,\mathrm{Hom}(F_x, G_x) \quad .$$
This map is neither injective nor surjective in general. However if F is locally constant with finitely generated stalks we can compute $R\mathcal{H}om(F,G)$ using locally a left resolution of F by finitely generated free sheaves (instead of a right injective resolution of G). As $\mathcal{H}om(\mathbb{Z}_X, G) = G$ we obtain that $e_x^\bullet$ is an isomorphism in this case.

Second we have the following useful lemma :

Lemma 11.0.12. *Let Y be locally compact and locally contractible. Then for $F \in C(X)$ which is clc, every point $y \in Y$ admits a neighbourhood U on which F is quasi-isomorphic to a complex of constant sheaves.*

Proof. Let $U \ni y$ be a contractible neighbourhhod. Thus $\mathcal{H}^\bullet F$ is constant on U. The hypercohomology spectral sequence starting at
$$E_2^{p,q} = H^p(U, \mathcal{H}^q(F)) = \begin{cases} 0 & \text{if } p \neq 0 \\ H^q(F_y) & \text{if } p = 0 \end{cases}$$
degenerates, giving
$$H^q(U,F) = E_2^{0,q} = \mathcal{H}^q F_y \quad .$$
Choose $F \simeq I^\bullet$ an injective resolution and let T^p be the constant sheaf $T^p_y = \Gamma(U, I^p)$. Then $T^\bullet \simeq I$ and the result. □

Proof of theorem 11.0.11. We start with the following proposition, which implies theorem 11.0.11 for X a manifold with trivial stratification.

Proposition 11.0.13. *Suppose Y locally compact and locally contractible. Let $F, G \in C(Y)$ be clc (resp. cc).*

Then for any $x \in Y$, $R\mathcal{H}om(F,G)_x \simeq R\,\mathrm{Hom}(F_x, G_x)$ and $R\mathcal{H}om(F,G)$ is clc (resp. cc).

Proof. The statement is local thus by the previous lemma we can assume that F and G are complexes of constant sheaves. Then F has a left resolution $C^\bullet \longrightarrow F$ by a complex $C^\bullet$ of constant sheaves with free stalks. As $R\mathcal{H}om(F,G) = \mathcal{H}om(C^\bullet, G)$ and $\mathcal{H}om(C^\bullet, G)_x = \mathrm{Hom}(C^\bullet_x, G_x)$ by the first fact we recalled, we are done. □

For $(X, \mathcal{X})$ a stratified pseudo-manifold and $F, G \in C(X)$ which are clc we prove by induction on k that $R\mathcal{H}om(F,G)_k$ is $\mathcal{X}$-clc.

For $k = 2$ it follows from proposition 11.0.13. Suppose the statement holds for some $k \geq 2$. Consider the exact triangle

$$j_{k!}{j_k}^! F_{k+1} \longrightarrow F_{k+1} \longrightarrow i_{k*}{i_k}^* F^{k+1} \xrightarrow{+1} \quad .$$

Applying $R\mathcal{H}om(\cdot, G_{k+1})$ we obtain the exact triangle :

$$R\mathcal{H}om(j_{k!}F_k, G_{k+1}) \longrightarrow R\mathcal{H}om(F_{k+1}, G_{k+1}) \longrightarrow R\mathcal{H}om(i_{k*}{i_k}^*F_{k+1}, G_{k+1}) \xrightarrow{+1} \quad .$$

Hence it is enough to show that both $R\mathcal{H}om(j_{k!}F_k, G_{k+1})$ and $R\mathcal{H}om(i_{k*}{i_k}^*F_{k+1}, G_{k+1})$ are $\mathcal{X}$-clc on U_{k+1}.

First, $R\mathcal{H}om(j_{k!}F_k, G_{k+1}) = Rj_{k*}R\mathcal{H}om(F_k, G_k)$. As $R\mathcal{H}om(F_k, G_k)$ is $\mathcal{X}$-clc on U_k by induction hypothesis and Rj_{k*} maps $\mathcal{X}$-clc on U_k to $\mathcal{X}$-clc on U_{k+1} one obtains that $R\mathcal{H}om(j_{k!}F_k, G_{k+1})$ is $\mathcal{X}$-clc on U_{k+1}.

Second, $R\mathcal{H}om(i_{k*}{i_k}^*F_{k+1}, G_{k+1}) = Ri_{k*}R\mathcal{H}om({i_k}^*F_{k+1}, {i_k}^!G_{k+1})$. As G_{k+1} is $\mathcal{X}$-clc on U_{k+1} the complex ${i_k}^!G_{k+1}$ is clc on S^k. By proposition 11.0.13 $R\mathcal{H}om({i_k}^*F_{k+1}, {i_k}^!G_{k+1})$ is thus clc on S^k. Hence its extension by zero $Ri_{k*}R\mathcal{H}om({i_k}^*F_{k+1}, {i_k}^!G_{k+1})$ is $\mathcal{X}$-clc on U_{k+1}. □

12. Biduality

Given $(X, \mathcal{X})$ a stratified pseudo-manifold we first define a morphism of functors

$$BD_X : \mathrm{id} \longrightarrow D_X \circ D_X$$

on the bounded derived category of $\mathcal{X}$-constructible sheaves $D^b(\mathcal{X})$.

We proceed as follows. For any space X, given $S, T \in \mathbf{Sh}(X)$ one can define

$$e \in \mathrm{Hom}(S, \mathcal{H}om(\mathcal{H}om(S,T), T)) \simeq \mathrm{Hom}(S \otimes \mathcal{H}om(S,T), T)$$

in the obvious way. This construction extends to graded sheaves. For DG-sheaves one has to be careful about signs (we leave the details to the reader). Applying this contruction to $T = \omega_X^\bullet$ and as $D_X F = \mathcal{H}om(F, \omega_X^\bullet)$ we obtain the required biduality map $BD_X : \mathrm{id} \longrightarrow D_X \circ D_X$.

Theorem 12.0.14. *Let $(X, \mathcal{X})$ be a stratified pseudo-manifold. Then for any $F \in D^b(X)$ the biduality map*

$$BD_X : F \longrightarrow D_X D_X F$$

is an isomorphism.

Proof. First assume that X is a manifold with trivial stratification. Then F is clc with finitely generated stalk cohomology. The statement is local hence we can assume $X = B^n$ (in particular oriented). Thus $\omega_X^\bullet[n]$ is an injective resolution of $\mathbb{Z}_X$. Moreover $\mathcal{H}^\bullet F$ is constant on X. By lemma 11.0.12 we can assume F is a complex of constant sheaves.

If F is just one constant sheaf E, then $D_X E$ is clc.

If moreover E is free with finitely generated stalks $E^{**} = E$, $D_X E[-n] = E^*$ and $E \simeq D_X D_X E$ thus we are done.

If E_x is finitely generated not free we choose $C^\bullet \longrightarrow E$ a bounded left resolution by constant free sheaves with finitely generated stalks. Applying BD_X to $C^\bullet \longrightarrow E$ and using that $D_X D_X$ is exact and the previous case for $BD_X : C^i \longrightarrow D_X D_X C^i$ we conclude that $BD_X : E \longrightarrow D_X D_X E$ is an isomorphism.

This proves the result for $\mathcal{H}^\bullet F$ concentrated in one degree. We then proceed by induction on the length of $\mathcal{H}^\bullet F$. Let b the greatest integer such that $\mathcal{H}^b F \neq 0$. Consider the short exact sequence :

$$0 \longrightarrow \tau_{<b} F \longrightarrow F \longrightarrow \tau^{\geq b} F \longrightarrow 0 \quad .$$

As $\mathcal{H}^\bullet(\tau^{\geq b}F)$ is concentrated in one degree and as $D_X D_X$ is exact we easily conclude. This finishes the proof in the case X is a manifold.

For $(X, \mathcal{X})$ a general stratified pseudomanifold we proceed by induction on k : let us prove that for any $k \geq 2$ the biduality map $F_k \xrightarrow{BD_X} D_X D_X F_k$ is an isomorphism. For $k = 2$ this follows from the previous case. Assume this is true for some $k \geq 2$. Consider the exact triangle :

$$i_{k!} i_k^! F_{k+1} \longrightarrow F_{k+1} \longrightarrow Rj_{k*} j_k^* F_{k+1} \xrightarrow{+1} \quad .$$

Applying $D_{U_{k+1}} D_{U_{k+1}}$ we obtain the map of exact triangles in $D^b(\mathcal{X})$:

$$\begin{tikzcd}
i_{k!} i_k^! F_{k+1} \arrow[r] \arrow[d, "\alpha"] & F_{k+1} \arrow[r] \arrow[d, "\beta"] & Rj_{k*} j_k^* F_{k+1} \arrow[r, "+1"] \arrow[d, "\gamma"] & {} \\
D_{U_{k+1}} D_{U_{k+1}} i_{k!} i_k^! F_{k+1} \arrow[r] & D_{U_{k+1}} D_{U_{k+1}} F_{k+1} \arrow[r] & D_{U_{k+1}} D_{U_{k+1}} Rj_{k*} j_k^* F_{k+1} \arrow[r, "+1"] & {}
\end{tikzcd}$$

We want to show that β is an isomorphism in $D^b(\mathcal{X})$. It is enough to show that α and γ are. As

$$D_{U_{k+1}} D_{U_{k+1}} i_{k!} i_k^! F_{k+1} = i_{k!} D_{S^k} D_{S^k} i_k^! F_{k+1} = i_{k!} i_k^! F_{k+1}$$

(where the last equality follows from by the manifold case) the map α is an isomorphism. As

$$D_{U_{k+1}} D_{U_{k+1}} Rj_{k*} F_k = D_{U_{k+1}} j_{k!} D_{U_k} F_k = Rj_{k*} D_{U_k} D_{U_k} F_k$$

we conclude by the induction hypothesis that γ is an isomorphism. $\square$

12.1. **Third (and last) characterization of Deligne' s extension.** Let $\mathcal{F}^\bullet \in D^+(\mathbb{Z}_X)$. Then for $U \in \mathbf{Op}_X$ and $i \in \mathbb{Z}$:

$$\begin{aligned} R\operatorname{Hom}(\mathbb{Z}_U, D_X\mathcal{F}^\bullet) &= R\operatorname{Hom}(\mathbb{Z}_U, R\mathcal{H}om(F, \omega_X)) \\ &= R\operatorname{Hom}(\mathbb{Z}_U \otimes^L \mathcal{F}^\bullet, p_X^!\mathbb{Z}) \\ &= R\operatorname{Hom}(Rp_{X!}\mathcal{F}^\bullet_U, \mathbb{Z}) \quad . \end{aligned} \tag{36}$$

In particular :

$$\mathbb{H}^i(U, D_X\mathcal{F}^\bullet[-n]) = R^{i-n}\operatorname{Hom}(\mathbb{H}^\bullet_c(U, \mathcal{F}^\bullet), \mathbb{Z}) \quad . \tag{37}$$

Suppose now that $\mathcal{F}^\bullet \in D^+(\mathbb{Q}_X)$ (here $\mathbb{Q}$ could be replaced by any field). Then :

$$\begin{aligned} \mathbb{H}^i(U, D_X\mathcal{F}^\bullet[-n]) &= \operatorname{Hom}^{i-n}(\mathbb{H}^\bullet_c(U, \mathcal{F}^\bullet), \mathbb{Q}) \\ &= \mathbb{H}^{n-i}_c(U, \mathcal{F}^\bullet)^* \quad , \end{aligned} \tag{38}$$

where we used that $D(\mathbb{Q}-Vec)$ is the sum indexed by $\mathbb{Z}$ of copies of $\mathbb{Q}-Vec$, and where $*$ denotes the dual in $\mathbb{Q}-Vec$.

This yields :

$$\forall x \in X, \forall i \in \mathbb{Z}, \quad H^i(D_X\mathcal{F}^\bullet[-n])_x \simeq H^{n-i}(i_x^!\mathcal{F}^\bullet)^* \quad .$$

The following corollary follows immediately :

Corollary 12.1.1. *Let $\mathcal{E}$ be a local system on some open dense submanifold of X whose complement has dimension at least 2. The set of axioms $(AX2)_{\mathcal{E},p}$ is equivalent to the following set $(AX3)_{\mathcal{E},p}$ for $\mathcal{F}^\bullet \in \mathbf{DGSh}(X)$:*

(i) *$\mathcal{H}^i\mathcal{F}^\bullet = 0$ for $i < 0$; $\mathcal{F}^\bullet$ is $\mathcal{X}$-clc for some PL-pseudomanifold stratification $\mathcal{X}$ of X ; there exists an open dense subset submanifold U of X of codimension at least 2 on which $\mathcal{E}$ is defined such that $\mathcal{F}^\bullet_{|U} \simeq \mathcal{E}_U$.*

(ii) $\dim\operatorname{supp}\mathcal{H}^j\mathcal{F}^\bullet \leq n - p^{-1}(j)$ *for all* $j > 0$.

(iii) $\dim\operatorname{supp}\mathcal{H}^jD_X\mathcal{F}^\bullet[-n] \leq n - q^{-1}(j)$ *for all* $j > 0$.

Theorem 12.1.2. (1) *The set of conditions $(AX3)_{\mathcal{E},p}$ determines $\mathcal{F}^\bullet$ in $D^b(\mathbb{Z}_X)$ uniquely and is satisfied by $\mathcal{P}_p(\mathcal{E})$.*

(2) *If moreover we work in $D(\mathbb{Q}_X)$ then $\mathcal{F}^\bullet$ satisfies $(AX3)_{\mathcal{E},p}$ if and only if $D_X\mathcal{F}^\bullet[-n]$ satisfies $(AX3)_{\mathcal{E}^*\otimes\mathcal{O}r,q}$ where $\mathcal{E}^*$ denotes the local system* $\operatorname{Hom}(\mathcal{E}, \mathbb{Q}_{U_2})$ *and q is the perversity dual to p.*

Proof. The first part follows from the analogous result for $(AX2)_{\mathcal{E},p}$.

For the second statement , let $\mathcal{G}^\bullet = D_X\mathcal{F}^\bullet[-n]$. Then $D_X\mathcal{G}^\bullet[-n] \simeq D_XD_X\mathcal{F}^\bullet \simeq F^\bullet$ by biduality. Thus $(AX3)_{\mathcal{E},p}(ii)$ for $\mathcal{F}^\bullet$ is equivalent to $(AX3)_{\mathcal{E}^*\otimes\mathcal{O}r,q}(iii)$ for $\mathcal{G}^\bullet$ and similarly exchanging (ii) and (iii). Hence it remains to show that $\mathcal{G}^\bullet$ satisfies $(AX3)_{\mathcal{E}^*\otimes\mathcal{O}r,q}(i)$. But :

$$\mathcal{H}^i\mathcal{G}^\bullet_x = H^i(D_X\mathcal{F}^\bullet[-n])_x = H^{n-i}(i_x^!\mathcal{F}^\bullet)^* = 0 \quad \text{for } i < 0 \quad .$$

Moreover $\mathcal{G}^\bullet$ is $\mathcal{X}$-clc if $\mathcal{F}^\bullet$ is $\mathcal{X}$-clc by theorem 11.0.9. Finally :

$$\mathcal{G}^\bullet_{|U_2} = D_X\mathcal{F}^\bullet[-n]_{|U_2} = D_{U_2}\mathcal{F}^\bullet_{|U_2}\mathcal{E}[-n] \simeq \mathcal{E}^* \otimes \mathcal{O}r_{U_2} \quad .$$

□

Corollary 12.1.3 (Poincaré duality, cf. theorem 2.9.4)**.** *Let $\mathcal{E}$ be a $\mathbb{Q}$-local system on a dense open submanifold of X with complement of codimension at least 2. Then*

$$I^qH^i(X, \mathcal{E}^* \otimes \mathcal{O}r) = I^pH^{n-i}_c(X, \mathcal{E})^* \quad .$$

Proof.

$$\begin{aligned} I^qH^i(X, \mathcal{E}^* \otimes \mathcal{O}r) &= \mathbb{H}^i(X, \mathcal{P}^\bullet_q((\mathcal{E}^* \otimes \mathcal{O}r) \otimes \mathcal{O}r)) = \mathbb{H}^i(X, \mathcal{P}^\bullet_q((\mathcal{E} \otimes \mathcal{O}r)^* \otimes \mathcal{O}r)) \\ &= \mathbb{H}^i(X, D_X\mathcal{P}^\bullet_p(\mathcal{E} \otimes \mathcal{O}r)[-n]) \\ &= \mathbb{H}^{n-i}_c(X, \mathcal{P}^\bullet_p(\mathcal{E} \otimes \mathcal{O}r))^* \quad \text{by equation (38)} \\ &= I^pH^{n-i}_c(X, \mathcal{E})^* \quad . \end{aligned}$$

□

Examples 12.1.4. In this section we assume that X is normal.

Taking $\mathcal{E} = \mathcal{O}r$, $p = t$ and $q = 0$ we obtain :

$$\begin{aligned} H^i(X,\mathbb{Q}_X) &= I^0H^i(X,\mathbb{Q}_X) = I^qH^i(X,\mathcal{O}r^* \otimes \mathcal{O}r) \\ &= I^pH_c^{n-i}(X,\mathcal{O}r)^* = I^tH_c^{n-i}(X,\mathcal{O}r)^* = H_i^c(X,\mathbb{Q}_X)^* \ . \end{aligned}$$

Taking $\mathcal{E} = \mathbb{Q}_X$, $p = 0$ and $q = t$ we get :

$$\begin{aligned} H_{n-i}(X,\mathbb{Q}_X) &= I^tH^i(X,\mathcal{O}r) = I^qH^i(X,\mathbb{Q}_X^* \otimes \mathcal{O}r) \\ &= I^pH_c^{n-i}(X,\mathbb{Q}_X)^* = I^0H_c^{n-i}(X,\mathbb{Q}_X)^* = H_c^{n-i}(X,\mathbb{Q}_X) \ . \end{aligned}$$

12.2. **Pairings.** Let us first consider the functoriality of Deligne's extension.

Proposition 12.2.1. *Let $f_2 : \mathcal{E} \longrightarrow \mathcal{F}$ be a morphism of local sytems on U_2. Let p, q be perversities satisfying $p \leq q$. Then f_2 extends in a unique way to a morphism in $D(\mathbb{Z}_X)$:*

$$f : \mathcal{P}_p^\bullet(\mathcal{E}) \longrightarrow \mathcal{P}_q^\bullet(\mathcal{F}) \ .$$

Proof. Let $L^\bullet := \mathcal{P}_p^\bullet(\mathcal{E})$ and $M^\bullet := \mathcal{P}_q^\bullet(\mathcal{F})$. By induction it is enough to show that $f_k : L_k^\bullet \longrightarrow M_k^\bullet \in D(\mathbb{Z}_{U_k})$ extends uniquely to $f_{k+1} : L_{k+1}^\bullet \longrightarrow M_{k+1}^\bullet \in D(\mathbb{Z}_{U_{k+1}})$ $(k \geq 2)$. By definition of $M_{k+1}^\bullet$ one has

$$\mathrm{Hom}_{D(\mathbb{Z}_{U_{k+1}})}(L_{k+1}^\bullet, M_{k+1}^\bullet) = \mathrm{Hom}_{D(\mathbb{Z}_{U_{k+1}})}(L_{k+1}^\bullet, \tau_{\leq q(k)}Rj_{k*}M_k^\bullet) \ .$$

On the other hand one has a natural sequence of homomorphisms :

$$\mathrm{Hom}_{D(\mathbb{Z}_{U_{k+1}})}(L_{k+1}^\bullet, \tau_{\leq q(k)}Rj_{k*}M_k^\bullet) \xrightarrow{\phi} \mathrm{Hom}_{D(\mathbb{Z}_{U_{k+1}})}(L_{k+1}^\bullet, Rj_{k*}M_k^\bullet) \simeq \mathrm{Hom}_{D(\mathbb{Z}_{U_k})}(L_k^\bullet, M_k^\bullet) \ ,$$

where the last isomorphism is given by adjunction.

Note that $\tau_{\leq q(k)}L_{k+1}^\bullet = L_{k+1}^\bullet$ as $q(k) \geq p(k)$ and $L_{k+1}^\bullet := \tau_{\leq p(k)}Rj_{k*}L_k^\bullet$. The following easy lemma (exercice) applied to $\mathcal{C} = D(\mathbb{Z}_{U_{k+1}})$, $A = L_{k+1}^\bullet$, $B = Rj_{k*}M_k^\bullet$ and $m = q(k)$, implies that ϕ is an isomorphism. The result follows.

Lemma 12.2.2. *Let $\mathcal{C}$ be a triangulated category, $A \in \mathcal{C}$ and $m \in \mathbb{Z}$. Suppose that the natural morphism $\tau_{\leq m}A \longrightarrow A$ is an isomorphism in $\mathcal{C}$. Then for any $B \in \mathcal{C}$ the natural homomorphism*

$$\mathrm{Hom}_{\mathcal{C}}(A, \tau_{\leq m}B) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(A, B)$$

is an isomorphism.

□

Proposition 12.2.3. *Let $\mu_2 : \mathcal{E} \otimes \mathcal{F} \longrightarrow G$ be a pairing of local systems on U_2 and let p, q, r be perversities such that $p + q \leq r$. Then there exists a unique morphism in $D^b(\mathbb{Z}_X)$*

$$\mu : \mathcal{P}_p^\bullet(\mathcal{E}) \otimes^L \mathcal{P}_q^\bullet(\mathcal{F}) \longrightarrow \mathcal{P}_r^\bullet(\mathcal{G})$$

which coïncide with μ_2 on U_2.

Proof. First notice the following lemma, left as an exercice :

Lemma 12.2.4. *Let $\mathcal{A}$ be an Abelian category and $D(A)$ its derived category. Let $A, B \in D(\mathcal{A})$ satisfying $\mathcal{H}^i(A) = 0$ for $i > 0$ and $\mathcal{H}^i(B) = 0$ for $i < 0$. Then the natural homomorphism*

$$\mathrm{Hom}_{D(\mathcal{A})}(A, B) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(\mathcal{H}^0(\mathcal{A}), \mathcal{H}^0(B))$$

is an isomorphism.

We apply this lemma to $\mathcal{A} = \mathbf{DGSh}(X)$, noticing that $\mathcal{H}^i(\mathcal{E} \otimes^L \mathcal{F}) = 0$ for $i > 0$ and $\mathcal{H}^0(\mathcal{E} \otimes^L \mathcal{F}) \simeq \mathcal{E} \otimes \mathcal{F}$. Thus μ_2 can be seen as $\mu_2 : \mathcal{E} \otimes^L \mathcal{F} \longrightarrow \mathcal{G}$ in $D^b(U_2)$ and the statement of the proposition makes sense.

Once more we proceed by induction. Let $L^\bullet := \mathcal{P}_p^\bullet(\mathcal{E})$, $M^\bullet := \mathcal{P}_q^\bullet(\mathcal{F})$, $N^\bullet := L^\bullet \otimes^L M^\bullet$ and $Q^\bullet := \mathcal{P}_r^\bullet(\mathcal{G})$. We prove by induction the extension of $\mu_k : N_k^\bullet \longrightarrow R_k^\bullet \in D^b(\mathbb{Z}_{U_k})$ to $\mu_{k+1} \in D^b(\mathbb{Z}_{U_{k+1}})$.

We claim that $\tau_{\leq r(k)} N^\bullet_{k+1} \longrightarrow N^\bullet_{k+1}$ is a quasi-isomorphism. Applying lemma 12.2.2 it then follows that

$$\mathrm{Hom}_{D(\mathbb{Z}_{U_{k+1}})}(N^\bullet_{k+1}, Q^\bullet_{k+1} = \tau_{\leq r(k)} Rj_{k*} Q^\bullet_k) \longrightarrow \mathrm{Hom}_{D(\mathbb{Z}_{U_k})}(N^\bullet_k, Q^\bullet_k)$$

is an isomorphism. Thus μ_k uniquely extends to μ_{k+1}.

To show the claim, notice that $N^\bullet_{k+1}$ is quasi-isomorphic to $L^\bullet_{k+1} \otimes^L M^\bullet_{k+1}$. As $\tau_{\leq p(k)} L^\bullet_{k+1} \longrightarrow L^\bullet_{k+1}$ and $\tau_{\leq q(k)} M^\bullet_{k+1} \longrightarrow M^\bullet_{k+1}$ are quasi-isomorphism we can choose for computing $L^\bullet_{k+1} \otimes^L M^\bullet_{k+1}$ flat resolutions of $L^\bullet_{k+1}$ and $M^\bullet_{k+1}$ vanishing in degree higher than $p(k)$ and $q(k)$ respectively. The total complex formed from these resolutions vanishes in degree larger than $r(k) \geq p(k) + q(k)$.

□

From proposition 12.2.3 we obtain various pairings in hypercohomology. In particular we get a map

$$\mathbb{H}^i_c(X, \mathcal{P}^\bullet_p(\mathcal{E})) \otimes \mathbb{H}^j(X, \mathcal{P}^\bullet_q(\mathcal{F})) \longrightarrow \mathbb{H}^{i+j}_c(X, \mathcal{P}^\bullet_r(\mathcal{G})) \ .$$

Taking $\mathcal{F} = \mathcal{E}^* \otimes \mathcal{O}r$, $\mathcal{G} = \mathcal{O}r$ and $\mu_2 : \mathcal{E} \otimes (\mathcal{E}^* \otimes \mathcal{O}r) \longrightarrow \mathcal{O}r$ the canonical pairing we obtain the pairing

$$I^p H^i_c(X, \mathcal{E} \otimes \mathcal{O}r) \otimes I^q H^j(X, \mathcal{E}^*) \longrightarrow I^r H^{i+j}_c(X, \mathbb{Q}_X)$$

described in theorem 2.9.3.

13. Perverse sheaves

13.1. **Summary of what we did.** Let X be a pseudomanifold of dimension n. For simplicity we suppose X oriented. Let p be a Goreski-MacPherson perversity. Given $\mathcal{E}$ a local system on $U \subset X$ an open dense submanifold whose complementary has dimension at least 2, we defined Deligne's extension $\mathcal{P}_p^\bullet(\mathcal{E})$ as the unique differential graded sheaf $\mathcal{F}^\bullet$ (up to quasi-isomorphism) satisfying the set of axioms $(AX3)_{\mathcal{E},p}$:

(i) $\mathcal{F}^\bullet$ is constructible ; $\mathcal{H}^i\mathcal{F}^\bullet = 0$ for $i < 0$; there exists an open dense subset submanifold U of X of codimension at least 2 on which $\mathcal{E}$ is defined such that $\mathcal{F}^\bullet_{|U} \simeq \mathcal{E}_U$.
(ii) $\dim \operatorname{supp} \mathcal{H}^j\mathcal{F}^\bullet \leq n - p^{-1}(j)$ for all $j > 0$.
(iii) $\dim \operatorname{supp} \mathcal{H}^j(D_X\mathcal{F}^\bullet[-n]) \leq n - q^{-1}(j)$ for all $j > 0$.

Remark 13.1.1. Recall that for $\mathcal{X}$ any stratification admissible for $\mathcal{E}$ then $(AX3)(ii)$ and (iii) are equivalent respectively to :

(ii') $\forall x \in S^k, \quad H^j(i_x^*\mathcal{F}^\bullet) = 0$ for $j > p(k)$,
(iii)' $\forall x \in S^k, \quad H^j(i_x^!\mathcal{F}^\bullet) = 0$ for $j < n - q(k)$,

where q denotes the dual perversity.

From now on we will assume for simplicity the following :

- X and all its strata have even dimensions. We will denote by $\dim_{\mathbb{C}}$ the complex dimension $\dim_{\mathbb{R}}/2$. We write $r = \dim_{\mathbb{C}} X$. This will be satisfied if we work in the category of complex analytic spaces with analytic Whitney stratifications or in the category of complex algebraic varieties with algebraic stratifications.
- we restrict ourselves to the case where $p = m$ is the middle perversity (thus $m(2k) = m(2k+1) = k-1$).

The axioms $(AX3)_{\mathcal{E},m}$ become :

(i) idem.
(ii) $\dim_{\mathbb{C}} \operatorname{supp} \mathcal{H}^j\mathcal{F}^\bullet \leq r - j - 1$ for all $j > 0$.
(iii) $\dim_{\mathbb{C}} \operatorname{supp} \mathcal{H}^j(D_X\mathcal{F}^\bullet[-n]) \leq r - j - 1$ for all $j > 0$.

Under these assumptions it will be convenient to shift our sheaves : rather than working with $\mathcal{F}^\bullet$ concentrated in degree $[0, 2r]$ we will work with $\mathcal{S} := \mathcal{F}^\bullet[r]$ concentrated in degree $[-r, r]$.

Our main result up to now can be stated as follows :

Theorem 13.1.2. *Let $\mathcal{S}^\bullet \in D^b(X)$. Suppose that :*

(i) *$\mathcal{S} \in D^b_c(X)$; $\mathcal{H}^i\mathcal{S}^\bullet = 0$ for $i < -r$; there exists an open dense subset submanifold U of X of codimension at least 2 on which $\mathcal{E}$ is defined and such that $\mathcal{F}^\bullet_{|U} \simeq \mathcal{E}_{|U}[r]$.*
(ii) *$\dim_{\mathbb{C}} \operatorname{supp} \mathcal{H}^j\mathcal{S}^\bullet < -j$ for all $j > -r$.*
(iii) *$\dim_{\mathbb{C}} \operatorname{supp} \mathcal{H}^j(D_X\mathcal{S}^\bullet) < -j$ for all $j > -r$.*

Then $\mathcal{S}^\bullet[-r] \simeq \mathcal{P}^\bullet(\mathcal{E})$.

Remark 13.1.3. As above one can equivalently replace (ii) and (iii) by : for any stratum S of X and any $x \in S$:

(ii) $H^j(i_x^*\mathcal{S}^\bullet) = 0$ for $j \geq -\dim_{\mathbb{C}} S$,
(iii) $H^j(i_x^!\mathcal{S}^\bullet) = 0$ for $j \leq \dim_{\mathbb{C}} S$.

13.2. **Perverse sheaves : definition and first main result.**

Definition 13.2.1. *$\mathcal{S}^\bullet \in D^b_c(X)$ is called a perverse sheaf if it satisfies the following* relaxed *version of conditions (ii) and (iii) above :*

- $\dim_{\mathbb{C}} \operatorname{supp} \mathcal{H}^j\mathcal{S}^\bullet \leq -j$ *for all $j > -r$.*
- $\dim_{\mathbb{C}} \operatorname{supp} \mathcal{H}^j(D_X\mathcal{S}^\bullet) \leq -j$ *for all $j > -r$.*

We denote by $\operatorname{Perv}(X) \subset D^b_c(X)$ *the full subcategory of perverse sheaves.*

Theorem 13.2.2 (BBDG). (i) *The category* $\operatorname{Perv}(X)$ *is Abelian (this is true more generally for any perversity p).*

(ii) *Suppose we are in the algebraic case with middle perversity. Then :*
   (a) *The category* $\mathrm{Perv}(X)$ *is stable under Verdier duality, Noetherian and Artinian. In particular any perverse sheaf has finite length.*
   (b) *Let* $U \subset Z$ *be a Zariski-open subset of a closed irreducible subvariety* $Z \overset{i}{\hookrightarrow} X$ *and* $\mathcal{E}$ *an irreducible local system on* U*. Then* $i_*(\mathcal{P}^\bullet(Z, \mathcal{E})[\dim_{\mathbb{C}} Z])$ *is a simple object of* $\mathrm{Perv}(X)$ *and any simple object is of this form.*

13.3. **Some remarks on what we did.**

13.3.1. To study our space X we started with the category Loc_X of $\mathbb{Q}$-local systems (of finite rank) on X, equivalently the category of $\mathbb{Q}[\pi_1(X)]$-modules of finite $\mathbb{Q}$-rank over X. This is an Abelian, Noetherian, Artinian category with simple objects the irreducible local systems.

However this category is much too small (especially if X is simply connected...). Notice moreover that it does not contain sufficiently many injectives (we have to enlarge the category to all $\mathbb{Q}[\pi_1(X)]$-modules).

Remark 13.3.1. The natural functor $\mathrm{Loc}_X \longrightarrow \mathbf{Sh}(X)$ is exact thus induces $D(\mathrm{Loc}_X) \longrightarrow D(\mathbb{Q}_X)$. Can we describe the essential image of this functor ?

13.3.2. One replaced Loc_X by the full subcategory $\mathbf{Sh}_c(X) \subset \mathbf{Sh}(X)$ of constructible sheaves (i.e. $\mathcal{F} \in \mathbf{Sh}_c(X)$ if there exists a filtration in the adequate category such that $\mathcal{F}$ in restriction to every stratum is isomorphic to a local system on this stratum).

We proved that $\mathbf{Sh}_c(X)$ is an Abelian category. Notice that $\mathbf{Sh}_c(X)$ is really an extension of Loc_X in the following sense :

Lemma 13.3.2. *Let* $\mathcal{F} \in \mathbf{Sh}_{c,\mathcal{X}}(X)$*. For any* $x \in X$ *there exists a neighbourhood* V *of* x *and a finite filtration*

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_m = \mathcal{F}_{|V}$$

such that the $\mathcal{F}_k$*'s are* $\mathcal{X}$*-constructible and the* $\mathcal{F}_{k+1}/F_k$*'s are of the form* $i_!\mathcal{L}$ *where* $i : S \cap V \hookrightarrow V$ *is the inclusion of the stratum* S *and* $\mathcal{L}$ *is a local system on* $S \cap V$*.*

Moreover in the algebraic case one can take $V = X$*.*

Proof. By induction on the number of strata of $\mathcal{X}$.

This is clear if $|\mathcal{X}| = 1$.

Otherwise let

$$U \overset{j}{\hookrightarrow} X \overset{i}{\hookleftarrow} Y$$

where U denotes the finite union of open strata (in a sufficiently small neighbourhood V of a given point x in general, globally if X is algebraic). Consider the exact sequence

$$0 \longrightarrow j_!j^*\mathcal{F} \longrightarrow \mathcal{F} \overset{u}{\longrightarrow} i_*i^*\mathcal{F} \longrightarrow 0 \ \ .$$

Notice that $j^*\mathcal{F}$ is a local system on U and $\mathcal{G} := i^*\mathcal{F}$ is $\mathcal{X}^0$-constructible on Y, where $\mathcal{X}^0$ denotes the filtration of Y induced by $\mathcal{X}$. As $|\mathcal{X}^0| < |\mathcal{X}|$ there exists by induction hypothesis a filtration

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_m = \mathcal{G}$$

of $\mathcal{G}$ on $V \cap Y$ and the $\mathcal{F}_{k+1}/\mathcal{F}_k$'s have the required form. One then define $\mathcal{F}'_0 = 0$, $\mathcal{F}'_k := u^{-1}i_*\mathcal{F}^{k-1}$ which is $\mathcal{X}$-constructible. The $\mathcal{F}'_{k+1}/\mathcal{F}'_k$ still have the required form as the sequence

$$0 \longrightarrow j_!j^*\mathcal{F} \longrightarrow \mathcal{F}'_k \longrightarrow i_*\mathcal{F}_{k-1} \longrightarrow 0$$

is exact. ☐

The full-subcategory $\mathbf{Sh}_c(X) \subset D^b_c(X)$ is Abelian but does not have very nice functorial properties (in particular is not stable under Verdier duality). Notice also that we have to enlarge it to ensure it has enough injectives.

The natural embedding $\mathbf{Sh}_c(X) \subset D^b_c(X)$ induces a natural functor

$$D^b(\mathbf{Sh}_c(X)) \hookrightarrow D^b_c(X) \ \ .$$

Theorem 13.3.3 (Beilinson-Nori)**.** *In the algebraic case the natural functor $D^b(\mathbf{Sh}_c(X)) \hookrightarrow D^b_c(X)$ is an equivalence of categories.*

On the other hand our Abelian category $\mathrm{Perv}(X) \subset D^b_c(X)$ will have good functorial properties. Moreover one still has :

Theorem 13.3.4 (Beilinson)**.** *In the algebraic case the natural functor $D^b(\mathrm{Perv}(X)) \longrightarrow D^b_c(X)$ is an equivalence of categories.*

14. t-structures

Definition 14.0.5. *Let $\mathcal{D}$ be a triangulated category. A t-structure on $\mathcal{D}$ is a pair $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ of two stricly full subcategories satisfying :*

(1) *$\mathcal{D}^{\leq -1} \subset \mathcal{D}^{\leq 0}$ and $\mathcal{D}^{\geq 1} \subset \mathcal{D}^{\geq 0}$, where for any integer $a \in \mathbb{Z}$ one defines $\mathcal{D}^{\geq a} := \mathcal{D}^{\geq 0}[-a]$ and $D^{\leq a} := \mathcal{D}^{\leq 0}[-a]$.*

(2) *$\mathrm{Hom}_{\mathcal{D}}(\mathcal{D}^{<0}, \mathcal{D}^{\geq 0}) = 0$, where $\mathcal{D}^{<0}$ is another notation for $\mathcal{D}^{\leq -1}$.*

(3) *For any $X \in \mathcal{D}$ there exists a exact triangle*

$$A \longrightarrow X \longrightarrow B \xrightarrow{+1}$$

where $A \in \mathcal{D}^{\leq 0}$ and $B \in \mathcal{D}^{>0}$.

Example 14.0.6. The example to keep in mind is $\mathcal{D} = D(\mathcal{A})$ the derived category of an Abelian category $\mathcal{A}$ with

$$\mathcal{D}^{\leq 0} := \{X \in \mathcal{D} \ / \ H^i X = 0 \text{ for all } i > 0\}$$

and

$$\mathcal{D}^{\geq 0} := \{X \in \mathcal{D} \ / \ H^i X = 0 \text{ for all } i < 0\} \ .$$

The axiom (3) reduces to the existence of truncation functors $\tau_{\leq 0} : \mathcal{D} \longrightarrow \mathcal{D}^{\leq 0}$ and $\tau_{>0} : \mathcal{D} \longrightarrow \mathcal{D}^{>0}$ such that any $X \in \mathcal{D}$ lies in a exact triangle

$$\tau_{\leq 0} X \longrightarrow X \longrightarrow \tau_{>0} X \xrightarrow{+1} \quad .$$

Definition 14.0.7. *The core of the t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ is the full subcategory $\mathcal{C} := \mathcal{D}^{\geq 0} \cap \mathcal{D}^{\leq 0}$ of $\mathcal{D}$.*

Our first goal in this section will be to prove that the core $\mathcal{C}$ is an Abelian subcategory of $\mathcal{D}$.

14.1. **Truncation functors.** First we show that the choice of A and B in axiom (3) is in fact functorial :

Proposition 14.1.1. *There exist functors $\tau_{\leq i} : \mathcal{D} \longrightarrow \mathcal{D}^{\leq i}$ and $\tau_{>i} : \mathcal{D} \longrightarrow \mathcal{D}^{>i}$ such that any object $X \in \mathcal{D}$ lies in a exact triangle*

$$\tau_{\leq i} X \longrightarrow X \longrightarrow \tau_{>i} X \xrightarrow{+1} \quad .$$

The functor $\tau_{\leq i}$ is right adjoint to the inclusion $\mathcal{D}^{\leq i} \subset \mathcal{D}$ and $\tau_{>i}$ is left adjoint to the inclusion $\mathcal{D}^{>i} \subset \mathcal{D}$.

Proof. Let $A \longrightarrow X \longrightarrow B$ be as in (3). Given any $A' \in \mathcal{D}^{\leq 0}$ and any morphism $A' \longrightarrow X$ there exists a unique factorisation

$$\begin{tikzcd} A \arrow[r] & X \\ & A' \arrow[u] \arrow[ul, dashed] \end{tikzcd} \quad .$$

Indeed the exact triangle $B[-1] \longrightarrow A \longrightarrow X \xrightarrow{+1}$ induces the long exact sequence :

$$\mathrm{Hom}(A', B[-1]) \longrightarrow \mathrm{Hom}(A', A) \longrightarrow \mathrm{Hom}(A, X) \longrightarrow \mathrm{Hom}(A, B) \ .$$

As $B \in \mathcal{D}^{\geq 1}$ (hence $B[-1] \in \mathcal{D}^{\geq 2}$) and $A' \in \mathcal{D}^{\leq 0}$ axiom (2) gives

$$\mathrm{Hom}(A', B[-1]) = \mathrm{Hom}(A, B) = 0$$

and the result.

Hence A is unique up to unique isomorphism and we define $\tau_{\leq 0} X := A$ (axiom of choice).

Similarly for $\tau_{\geq 1}X$.

The adjunction properties follow easily. $\square$

We now proof that the completion

$$\tau_{\leq 0}X \longrightarrow X \longrightarrow \tau_{\geq 1}X \dashrightarrow \tau_{\leq 0}X[1]$$

is functorial :

Proposition 14.1.2. *Given $X \in \mathcal{D}$ there exists a unique morphism*

$$d(X) : \tau_{\geq 1}X \longrightarrow \tau_{\leq 0}X[1]$$

such that the triangle $\tau_{\leq 0}X \longrightarrow X \longrightarrow \tau_{\geq 1}X \longrightarrow \tau_{\leq 0}X[1]$ *is exact. Moreover d is a morphism of functors.*

Proof. The existence of $d(X)$ follows from axiom (3). Let us prove uniqueness. Suppose $h_i : \tau_{\geq 1}X \longrightarrow \tau_{\leq 0}X[1]$, $i = 1, 2$, are two such morphisms. By TR_4 there exists a commutative diagram

$$\begin{tikzcd}
\tau_{\leq 0}X \arrow[r] \arrow[d, equal] & X \arrow[r, "g"] \arrow[d, equal] & \tau_{\geq 1}X \arrow[r, "h_1"] \arrow[d, dashed, "\phi"'] & \tau_{\leq 0}X[1] \arrow[d, equal] \arrow[dl, dotted, "\psi"'] \\
\tau_{\leq 0}X \arrow[r] & X \arrow[r, "g"] & \tau_{\geq 1}X \arrow[r, "h_2"] & \tau_{\leq 0}X[1]
\end{tikzcd}$$

hence $(\mathrm{id}_{\tau_{\geq 1}}X - \phi) \circ g = 0$. Thus there exists $\psi : \tau_{\leq 0}X[1] \longrightarrow \tau_{\geq 1}X$ such that

$$\mathrm{id}_{\tau_{\geq 1}X} - \phi = \psi \circ h_1 \quad .$$

As $\tau_{\leq 0}X[1] \in \mathcal{D}^{\leq -1}$ and $\tau_{\geq 1}X \in \mathcal{D}^{\geq 1}$ necessarily $\psi = 0$ by axiom (2), hence $\phi = \mathrm{id}_{\tau_{\geq 1}}X$ and $h_1 = h_2$.

The fact that d is a morphism of functors follows formally. $\square$

Definition 14.1.3. *One defines $\tau_{\leq n}X = (\tau_{\leq 0}(X[n]))[-n]$ and $\tau_{\geq n}X = (\tau_{\geq 0}(X[n]))[-n]$.*

Lemma 14.1.4. *$\mathcal{D}^{\leq n} = {}^{\perp}D^{>n}$ and $\mathcal{D}^{>n} = \mathcal{D}^{\leq n\perp}$.*

Proof. Follows from axiom (3). $\square$

Lemma 14.1.5. *Let $X \longrightarrow Y \longrightarrow Z \stackrel{+1}{\longrightarrow}$ be an exact triangle. If $X, Z \in \mathcal{D}^{\leq n}$ then $Y \in \mathcal{D}^{\leq n}$. Similarly replacing $\mathcal{D}^{\leq n}$ by $\mathcal{D}^{>n}$.*

Proof. Suppose $X, Z \in \mathcal{D}^{>0}$. We have to show that $\tau_{\leq -1}Y = 0$. Applying $\mathrm{Hom}(\tau_{\leq -1}Y, \cdot)$ to our triangle we obtain the exact sequence

$$\mathrm{Hom}(\tau_{\leq -1}Y, X) \longrightarrow \mathrm{Hom}(\tau_{\leq -1}Y, Y) \longrightarrow \mathrm{Hom}(\tau_{\leq -1}Y, Z) \quad .$$

As $X, Z \in \mathcal{D}^{>0}$ and $\tau_{\leq -1}Y \in \mathcal{D}^{\leq -1}$ one has $\mathrm{Hom}(\tau_{\leq -1}Y, X) = \mathrm{Hom}(\tau_{\leq -1}Y, Z) = 0$. Hence $\mathrm{Hom}(\tau_{\leq -1}Y, Y) = 0$. However $\mathrm{Hom}(\tau_{\leq -1}Y, Y) = \mathrm{Hom}(\tau_{\leq -1}Y, \tau_{\leq -1}Y)$ by adjunction, hence $\tau_{\leq -1}Y = 0$.

The other case is similar. $\square$

Of course for $a \leq b$ one has $\tau_{\leq a}\tau_{\leq b} = \tau_{\leq a}$. More interestingly one has :

Lemma 14.1.6. *If $a \leq b$ one has $\tau_{\geq a}\tau_{\leq b} \simeq \tau_{\leq b}\tau_{\geq a}$.*

Proof. Consider the diagram given by the octahedron axiom :

$$\begin{tikzcd}
\tau_{<a}\tau_{\leq b}X = \tau_{<a}X \arrow[r] \arrow[dr] & \tau_{\leq b}X \arrow[r] \arrow[d] & \tau_{\geq a}\tau_{\leq b}X \arrow[dd, dashed] \\
 & X \arrow[dr] \arrow[ddr] & \\
 & & \tau_{\geq a}X \arrow[d] \\
 & & \tau_{>b}X = \tau_{>b}\tau_{\geq a}X
\end{tikzcd} \quad .$$

Notice that $\tau_{\geq a}\tau_{\leq b}X \in \mathcal{D}^{\leq b}$: indeed $\tau_{>b}\tau_{\geq a}\tau_{\leq b}X = \tau_{>b}\tau_{\leq b}X = 0$. Hence necessarily

$$\tau_{\geq a}\tau_{\leq b} \overset{\gamma}{\simeq} \tau_{\leq b}\tau_{\geq a} \quad .$$

□

Remark 14.1.7. Notice that the isomorphism γ above is canonical.

Definition 14.1.8. *One defines :*

$$\tau_{[a,b]}X := \tau_{\geq a}\tau_{\leq b}X \simeq \tau_{\leq b}\tau_{\geq a}X \quad ,$$
$$\mathcal{D}^{[a,b]} := \mathcal{D}^{\geq a} \cap \mathcal{D}^{\leq b} \quad ,$$
$$H^0X := \tau_{[0,0]}X \in D^{[0,0]} =: \mathcal{C} \quad ,$$
$$H^nX := H^0(X[n]) = (\tau_{\geq n}(\tau_{\leq n}X))[n] \quad .$$

15. The core is Abelian

Theorem 15.0.9. *Let $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ be a t-structure on a triangulated category $\mathcal{D}$ and $\mathcal{C} := \mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0}$ its core. Then :*

(1) *$\mathcal{C}$ is Abelian.*
(2) *For $A \longrightarrow B \longrightarrow C \in \mathcal{C}$ the following properties are equivalent :*
   (i) *The sequence $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ is exact in $\mathcal{C}$.*
   (ii) *There exists $d : C \longrightarrow A[1]$ such that $A \longrightarrow B \longrightarrow C \xrightarrow{d} A[1]$ is a distinguished triangle in $\mathcal{D}$ (in fact d is unique).*
(3) *The functor $H^0 : \mathcal{D} \longrightarrow \mathcal{C}$ is cohomological : for any distinguished triangle $X \longrightarrow Y \longrightarrow Z \xrightarrow{+1}$ in $\mathcal{D}$ there is a long exact sequence :*

$$\cdots \longrightarrow H^{-1}(Z) \longrightarrow H^0(X) \longrightarrow H^0(Y) \longrightarrow H^0(Z) \longrightarrow H^1(X) \longrightarrow \cdots \quad .$$

Proof. Proof of(1). First notice that $\mathcal{C}$ is obviously additive.

Let us show that $\mathcal{C}$ has cokernels. Let $f : A \longrightarrow B \in \mathcal{C}$. Choose a distinguished triangle in $\mathcal{D}$:

$$A \longrightarrow B \longrightarrow C \longrightarrow A[1] \quad . \tag{39}$$

As $A \in \mathcal{D}^{[0,0]}$ one has $\tau_{>1}(A[1]) = (\tau_{>0}(A))[1] = 0$ hence $A[1] \in \mathcal{D}^{[-1,-1]}$. For any $T \in \mathcal{C}$ applying $\mathrm{Hom}(T, \cdot)$ to the distinguished triangle (39) gives rise to the long exact sequence

$$\cdots \longrightarrow \mathrm{Hom}(A, T) \longrightarrow \mathrm{Hom}(B, T) \longrightarrow \mathrm{Hom}(C, T) \longrightarrow \mathrm{Hom}(A[1], T) \longrightarrow \cdots \quad . \tag{40}$$

As $T \in \mathcal{D}^{\geq 0}$ one has $\mathrm{Hom}(C, T) \simeq \mathrm{Hom}(\tau_{\geq 0}C, T)$. Second, $\mathrm{Hom}(A[1], T) = 0$ as $A[1] \in \mathcal{D}^{\leq -1}$ and $T \in \mathcal{D}^{\geq 0}$. Applying $\mathrm{Hom}(\cdot, X)$ to (39) for $X \in \mathcal{D}^{>0}$ one obtains that $C \in \mathcal{D}^{\leq 0}$ hence $\tau_{\geq 0}C \in \mathcal{C}$.

Finally equation (40) tells us that $B \longrightarrow \tau_{\geq 0}C$ is a cokernel for $f : A \longrightarrow B$.

Similarly one checks that $H^{-1}C = \tau_{\leq 0}C[-1] \longrightarrow A$ is a kernel for $f : A \longrightarrow B$.

It remains to show that $\mathrm{Coim} f \simeq \mathrm{Im}\ f$. Consider the following diagram obtained from the octahedron axiom :

$$\begin{tikzcd}
 & & & \ker f[1] = \tau_{\leq -1}C \arrow[ddd] \\
 & & I \arrow[ur, dashed] \arrow[dd] & \\
 & & & \\
\ker f = \tau_{\leq 0}C[-1] \arrow[r] & A \arrow[uur, dashed] \arrow[r, "f"'] & B \arrow[r] \arrow[dr] & C \arrow[d] \\
 & & & \mathrm{coker} f = \tau_{\geq 0}C
\end{tikzcd} ,$$

where $I = \mathrm{Im}\ f = \ker(B \longrightarrow \mathrm{coker} f)$. The distinguished triangle

$$\ker f \longrightarrow A \longrightarrow I \xrightarrow{+1}$$

shows that $\mathrm{Im}\ f = I = \mathrm{coker}(\ker f \longrightarrow A) = \mathrm{Coim} f$.

This finishes the proof that $\mathcal{C}$ is Abelian.

Proof of (2).

Suppose (ii). Let $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{+1}$ be a distinguished triangle with $A, B, C \in \mathcal{C}$. By (1) one knows that $\mathrm{coker} f = H^0C = C$ and in a dual way $\ker g = A$. Thus

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is a short exact sequence in $\mathcal{C}$.

Conversely let $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ be a short exact sequence in $\mathcal{C}$. Choose a distinguished triangle

$$A \xrightarrow{f} B \longrightarrow X \xrightarrow{+1} \quad .$$

As $A, B \in \mathcal{C}$ one has $X \in \mathcal{D}^{[-1,0]}$. As f is monic $H^{-1}X = 0$. As $\mathrm{coker} f = H^0 X$ one obtains $X \simeq \mathrm{coker} f = C$ and the result.

Proof of (3). Let $X \longrightarrow Y \longrightarrow Z \xrightarrow{+1}$ be a distinguished triangle.

Step 1 : we first show that if $X, Y, Z \in \mathcal{D}^{\geq 0}$ then $0 \longrightarrow H^0 X \longrightarrow H^0 Y \longrightarrow H^0 Z$ is exact in $\mathcal{C}$.

Notice that $H^0 X = \tau_{\leq 0} X$ and similarly for Y and Z. Hence it is enough to show that for any $T \in \mathcal{C}$ the sequence

$$0 \longrightarrow \mathrm{Hom}(T, \tau_{\leq 0} X) \longrightarrow \mathrm{Hom}(T, \tau_{\leq 0} Y) \longrightarrow \mathrm{Hom}(T, \tau_{\leq 0} Z)$$

is exact. Applying $\mathrm{Hom}(T, \cdot)$ to our triangle we obtain the exact sequence

$$\mathrm{Hom}(T, Z[-1]) \longrightarrow \mathrm{Hom}(T, X) \longrightarrow \mathrm{Hom}(T, Y) \longrightarrow \mathrm{Hom}(T, Z) \quad .$$

As $T \in \mathcal{D}^{\leq 0}$ and $Z[-1] \in \mathcal{D}^{\geq 1}$ the term $\mathrm{Hom}(T, Z[-1])$ vanishes. As $T \in \mathcal{C}$ one has a canonical isomorphism $\mathrm{Hom}(T, \tau_{\leq 0} X) \simeq \mathrm{Hom}(T, X)$ and similarly for Y and Z. The result follows.

Step 2 : let us show that if $Z \in \mathcal{D}^{\geq 0}$ then $0 \longrightarrow H^0 X \longrightarrow H^0 Y \longrightarrow H^0 Z$ is exact. As before we want to show that the rows in the following diagram are exact :

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}(T, H^0 X) \arrow[r] \arrow[d, equal] & \mathrm{Hom}(T, H^0 Y) \arrow[r] \arrow[d, equal] & \mathrm{Hom}(T, H^0 Z) \arrow[d, equal] \\
0 \arrow[r] & \mathrm{Hom}(T, \tau_{\geq 0} X) \arrow[r] & \mathrm{Hom}(T, \tau_{\geq 0} Y) \arrow[r] & \mathrm{Hom}(T, Z)
\end{tikzcd} \quad .$$

Applying the first step, it is enough to show that $\tau_{\geq 0} X \longrightarrow \tau_{\geq 0} Y \longrightarrow Z$ is part of a distinguished triangle. The morphism $X \longrightarrow Y$ gives rise to a commutative diagram :

$$\begin{tikzcd}
\tau_{<0} X \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & \tau_{\geq 0} X \arrow[d] \\
\tau_{<0} Y \arrow[r] & Y \arrow[r] & \tau_{\geq 0} Y
\end{tikzcd} \quad .$$

We first claim that $\tau_{<0} X \simeq \tau_{<0} Y$. By the universal property of the truncation functor $\tau_{<0}$ it is equivalent to showing that for any $T \in \mathcal{D}^{<0}$ the morphism $\mathrm{Hom}(T, X) \longrightarrow \mathrm{Hom}(T, Y)$ is an isomorphism. This follows from applying $\mathrm{Hom}(T, \cdot)$ to our distinguished triangle $X \longrightarrow Y \longrightarrow Z \xrightarrow{+1}$ since $Z \in \mathcal{D}^{\geq 0}$ (hence $Z[-1] \in \mathcal{D}^{\geq 1}$) by assumption.

Applying the octahedron axiom we obtain :

$$\begin{tikzcd}
\tau_{<0} X = \tau_{<0} Y \arrow[r] \arrow[dr] & X \arrow[r] \arrow[d] & \tau_{\geq 0} X \arrow[dd] \\
 & Y \arrow[dr] \arrow[ddr] & \\
 & & \tau_{\geq 0} Y \arrow[d, dashed] \\
 & & Z
\end{tikzcd}$$

and the result.

Step 3 : in a dual way we obtain that if $X \in \mathcal{D}^{\leq 0}$ then $H^0 X \longrightarrow H^0 Y \longrightarrow H^0 Z \longrightarrow 0$ is exact.

Step 4 : the general case. Consider the following diagram obtained from the octahedron axiom :

$$\begin{tikzcd}
\tau_{\leq 0}X \arrow[r] \arrow[dr] & X \arrow[r] \arrow[d] & \tau_{>0}X \arrow[dd, dashed] \\
& Y \arrow[dr] \arrow[ddr] & \\
& & U \arrow[d, dashed] \\
& & Z
\end{tikzcd}$$

Apply H^0 :

$$\begin{tikzcd}
H^0\tau_{\leq 0}X \arrow[r] \arrow[dr] & H^0X \arrow[r] \arrow[d] & 0 \arrow[dd, dashed] \\
& H^0Y \arrow[dr] \arrow[ddr] & \\
& & H^0U \arrow[d, dashed] \\
& & H^0Z
\end{tikzcd}$$

We want to show that $H^0X \longrightarrow H^0Y \longrightarrow H^0Z$ is exact. This follows from the fact that :

- $0 \longrightarrow H^0\tau_{\leq 0}X \longrightarrow H^0X \longrightarrow 0 = H^0\tau_{>0}X$ is exact by step 2.
- $H^0\tau_{\leq 0}X \longrightarrow H^0Y \longrightarrow H^0U \longrightarrow 0$ is exact by step 3.
- $0 = H^0\tau_{>0}X \longrightarrow H^0U \longrightarrow H^0Z$ is exact by applying step 2 to the distinguished triangle $U \longrightarrow Z \longrightarrow \tau_{>0}X[1] \xrightarrow{+1}$.

$\square$

16. Non-degenerate t-structures and t-exact functors

Definition 16.0.10. *The t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ is* non-degenerate *if one of the following equivalent conditions is satisfied :*

(i) $\cap_{n\in\mathbb{Z}}\mathcal{D}^{\leq n} = \{0\}$ *and* $\cap_{n\in\mathbb{Z}}\mathcal{D}^{\geq n} = \{0\}$.
(ii) *If $X \in \mathcal{D}$ satisfies $H^p(X) = 0$ for all $p \in \mathbb{Z}$ then $X = 0$.*

Let us show that the two conditions are indeed equivalent. First assume (i). Consider the distinguished triangle :

$$\tau_{\leq 0}X \longrightarrow X \longrightarrow \tau_{<0}X \xrightarrow{+1} \quad .$$

As

$$H^p(\tau_{\leq 0}X) = \begin{cases} H^p(X) & \text{if } p \leq 0 \\ 0 & \text{otherwise} \end{cases}$$

vanishes in any case, one obtains $\tau_{\leq 0}X \in \mathcal{D}^{\leq p}$ for all $p \in \mathbb{Z}$. Similarly $\tau_{>0}(X) \in \mathcal{D}^{\geq p}$ for all $p \in \mathbb{Z}$. Hence $\tau_{\leq 0}X = \tau_{>0}X = 0$ by assumption (i), hence $X = 0$.

Conversely assume (ii). Let $X \in \cap_{n\in\mathbb{Z}}\mathcal{D}^{\leq n}$. Thus $\tau_{\geq p}X = 0$ for all $p \in \mathbb{Z}$. Hence $H^p(X) = (\tau_{\leq p}(\tau_{\geq p}X))[p] = 0$ and $X = 0$ by (ii). Similarly $\cap_{n\in\mathbb{Z}}\mathcal{D}^{\geq n} = \{0\}$.

Example 16.0.11. The standard t-structure on the derived category $\mathcal{D}(\mathcal{A})$ of an Abelian category is non-degenerate. On the other hand for any triangulated category $\mathcal{D}$ the t-structure $(\{0\}, \mathcal{D})$ is degenerate.

Proposition 16.0.12. *If the t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ is non-degenerate then :*

(i) *the family of functors* (H^i) *is conservative : a morphism* $f : X \longrightarrow Y \in \mathcal{D}$ *is an isomorphism if and only if for all* $i \in \mathbb{Z}$ *the morphisms* $H^i f : H^i X \longrightarrow H^i Y$ *are isomorphisms.*

(ii) *for all* $n \in \mathbb{Z}$,

$$\mathcal{D}^{\leq n} = \{X \in \mathcal{D} \;/\; H^p X = 0 \; for \, p > n\},$$
$$\mathcal{D}^{\geq n} = \{X \in \mathcal{D} \;/\; H^p X = 0 \; for \, p < n\} \quad .$$

Proof. For (i) : clearly if f is an isomorphism then $H^i f$ is an isomorphism ofr all $i \in \mathbb{Z}$. Conversely suppose $H^i F$ is an isomorphism for all $i \in \mathbb{Z}$. Let

$$X \longrightarrow Y \longrightarrow Z \xrightarrow{+1}$$

be a distinguished triangle. The associated long exact sequence

$$H^p X \xrightarrow{\sim} H^p Y \longrightarrow H^p Z \longrightarrow H^{p+1} X \xrightarrow{\sim} H^{p+1} Y$$

forces $H^p Z = 0$ for all $p \in \mathbb{Z}$ hence $Z = 0$ as the t-structure is non-degenerate, hence f is an isomorphism.

For (ii) : if $X \in \mathcal{D}^{\leq n}$ then $H^p X = 0$ for $p > n$. Conversely suppose $H^p X = 0$ for $p > n$. As $H^p(X) \simeq H^p(\tau_{\geq n+1} X)$ for $p > n$ one has $H^p(\tau_{\geq n+1})X) = 0$ for all $p \in \mathbb{Z}$. Thus $\tau_{\geq n+1} X = 0$ i.e. $X \in \mathcal{D}^{\leq n}$. □

Definition 16.0.13. *Let* $F : \mathcal{D}_1 \longrightarrow \mathcal{D}_2 E$ *be a triangulated functor. Let* $(\mathcal{D}_i^{\leq 0}, \mathcal{D}_i^{\geq 0})$, $i = 1, 2$ *be t-structures on* $\mathcal{D}_i$, *with core* $\mathcal{C}_i$.

One says that :

- F *is left t-exact if* $F(\mathcal{D}_1^{\geq 0}) \subset \mathcal{D}_2^{\geq 0}$.
- F *is right t-exact if* $F(\mathcal{D}_1^{\leq 0}) \subset \mathcal{D}_2^{\leq 0}$.
- F *is t-exact if it is both right and left t-exact.*

We denote by ${}^p F : \mathcal{C}_1 \longrightarrow \mathcal{C}_2$ *the functor* $H_0 \circ F \circ i_1$.

Proposition 16.0.14. *Let* $F : \mathcal{D}_1 \longrightarrow \mathcal{D}_2$ *be left t-exact. Then :*

(1) *For all* $X \in \mathcal{D}_1$ *one has a canonical isomorphism*

$$\tau_{\leq 0}(F(\tau_{\leq 0} X))) \simeq \tau_{\leq 0} F(X) \quad .$$

In particular if $X \in \mathcal{D}_1^{\geq 0}$ *then* $H^0(F(X)) \simeq {}^p F(H^0(X))$.

(2) ${}^p F : \mathcal{C}_1 \longrightarrow C_2$ *is left-exact.*

Symmetric statement for F *right t-exact.*

Proof. For (1). Consider the exact triangle $\tau_{\leq 0} X \longrightarrow X \longrightarrow \tau_{>0} X \xrightarrow{+1}$. Applying F one obtains the exact triangle

$$F(\tau_{\leq 0} X) \longrightarrow F(X) \longrightarrow F(\tau_{>0} X) \xrightarrow{+1} \tag{41}$$

As F is left t-exact, $F(\tau_{>0} X) \in \mathcal{D}_2^{\geq 1}$. Hence it is enough to show that for any $A \longrightarrow B \longrightarrow C \xrightarrow{+1}$ with $C \in \mathcal{D}^{\geq 1}$ then $\tau_{\leq 0} A \simeq \tau_{\leq 0} B$; equivalently that for any $W \in \mathcal{D}^{\leq 0}$ the morphism $\mathrm{Hom}(T, A) \longrightarrow \mathrm{Hom}(T, B)$ is an isomorphism. This follows from the long exact sequence obtained by applying $\mathrm{Hom}(T, \cdot)$ to our triangle.

For (2). Let $0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$ be an exact sequence in $\mathcal{C}_1$, hence $X \longrightarrow Y \longrightarrow Z \xrightarrow{+1}$ is an exact triangle in $\mathcal{D}_1$. Applying F one obtains the exact triangle $F(X) \longrightarrow F(Y) \longrightarrow F(Z) \xrightarrow{+1}$. Applying H^0 and as $F(Z) \in \mathcal{D}_2^{\geq 0}$ one obtains :

$$0 \longrightarrow H^0 F(X) \longrightarrow H^0 F(Y) \longrightarrow H^0 F(Z) \quad ,$$

hence ${}^p F$ is left t-exact. □

17. Glueing of t-structures

17.1. **The glued t-structure.** Consider the following topological situation :

$$U \overset{o}{\hookrightarrow} X \overset{/}{\hookleftarrow} Y \quad .$$

Denoting by $\mathcal{D}_Z$ the derived category of sheaves with constructible cohomology $\mathcal{D}_c(\mathbb{Z}_Z)$ on a space Z we have the following diagram of adjunctions between triangulated categories :

$$\begin{tikzcd}
\mathcal{D}_U \arrow[d, bend right=60, "j_!"'] \arrow[d, bend left=60, "j_*"] \\
\mathcal{D}_X \arrow[u, "j^!=j^*"] \arrow[d, bend right=60, "i^*"'] \arrow[d, bend left=60, "i^!"] \\
\mathcal{D}_Y \arrow[u, "i_*=i_!"]
\end{tikzcd}$$

satisfying the following identities

$$\mathrm{id} \xrightarrow{\sim} j^! j_! \quad ,$$
$$j^* j_* \xrightarrow{\sim} \mathrm{id} \quad ,$$
$$i^* i_* \xrightarrow{\sim} \mathrm{id} \quad ,$$
$$\mathrm{id} \xrightarrow{\sim} i^! i_! \quad ,$$
$$j^* i_* = 0 \quad \text{hence} \quad i^! j_* = i^* j_! = 0 \quad ,$$

where we wrote j_* rather than Rj_*, etc ...

From now on we will assume we are given abstracly three categories $\mathcal{D}_Y$, $\mathcal{D}_U$ and $\mathcal{D}_X$ and functors i^*, j^*, $j_!$, j_*, i_*, $i^!$ satisfying the previous relations. Suppose that $(\mathcal{D}_Y^{\leq 0}, \mathcal{D}_Y^{\geq 0})$ and $(\mathcal{D}_U^{\leq 0}, \mathcal{D}_U^{\geq 0})$ are t-structures on $\mathcal{D}_Y$ and $\mathcal{D}_U$ respectively. We would like to "glue them" together into a t-structure on $\mathcal{D}_X$. Our candidate is the following :

Definition 17.1.1. *We define :*

$$\mathcal{D}_X^{\leq 0} := \{F \in \mathcal{D}_X \ / \ i^*F \in \mathcal{D}_Y^{\leq 0} \text{ and } j^*F \in \mathcal{D}_U^{\leq 0}\} \quad .$$
$$\mathcal{D}_X^{\geq 0} := \{F \in \mathcal{D}_X \ / \ i^!F \in \mathcal{D}_Y^{\leq 0} \text{ and } j^!F \in \mathcal{D}_U^{\leq 0}\} \quad .$$

Theorem 17.1.2. $(\mathcal{D}_X^{\leq 0}, \mathcal{D}_X^{\geq 0})$ *defines a t-structure on $\mathcal{D}_X$.*

Proof. We have to show that the three axioms (i), (ii) and (iii) of t-structures are satisfied.

For (i) : clearly $\mathcal{D}_X^{<0} \subset \mathcal{D}_X^{\leq 0}$ and $D_X^{>0} \subset \mathcal{D}_X^{\geq 0}$ as the same is true for $\mathcal{D}_Y$ and $\mathcal{D}_U$.

For (ii) : Let $F, G \in \mathcal{D}_X$. We have an exact triangle :

$$i_! i^! G \longrightarrow G \longrightarrow j_* j^* G \xrightarrow{+1} \quad .$$

Applying $\mathrm{Hom}(F, \cdot)$ to this triangle we obtain the long exact sequence :

$$\cdots \longrightarrow \mathrm{Hom}(i^*F, i^!G) \longrightarrow \mathrm{Hom}(F, G) \longrightarrow \mathrm{Hom}(j^*F, j^*G) \longrightarrow \cdots \quad .$$

If $F \in \mathcal{D}_X^{\leq 0}$ and $G \in \mathcal{D}_X^{>0}$ then $i^*F \in D_Y^{\leq 0}$ and $i^!G \in \mathcal{D}_Y^{>0}$ hence $\mathrm{Hom}(i^*F, i^!G) = 0$. Similarly $j^*F \in \mathcal{D}_U^{\leq 0}$ and $j^!G \in \mathcal{D}_U^{>0}$ hence $\mathrm{Hom}(j^*F, j^!G) = 0$. The long exact sequence implies $\mathrm{Hom}(F, G) = 0$ in this case, thus $\mathrm{Hom}(\mathcal{D}_X^{\leq 0}, \mathcal{D}_X^{>0}) = 0$.

For (iii) : Let $F \in \mathcal{D}_X$. We want to produce an exact triangle : $A \longrightarrow F \longrightarrow B \xrightarrow{+1}$ with $A \in \mathcal{D}_X^{\leq 0}$ and $B \in \mathcal{D}_X^{>0}$.

Let us first analyse the problem, thus suppose such a triangle exists. As j^* is exact and $j^*A \in \mathcal{D}_U^{\leq 0}$ and $j^*B = j^!B \in \mathcal{D}_U^{>0}$, necessarily $j^*A = \tau_{\leq 0} j^*F$ and $j^*B = \tau_{>0} j^*F$.

As $j^* = j^!$ is right-adjoint to j_* we get from $j^*A = \tau_{\leq 0} j^*F$ a morphism

$$j_! \tau_{\leq 0} j^* F \longrightarrow A$$

which becomes an isomorphism after applying j^*. Applying the octahedron axiom we obtain the diagram :

$$\begin{tikzcd}
j_!j^!A = j_!\tau_{\leq 0}j^*F \arrow[r] \arrow[dr] & A \arrow[r] \arrow[d] & i_*i^*A \arrow[d, dashed] \\
 & F \arrow[r] \arrow[dr] & G \arrow[d, dashed] \\
 & & B
\end{tikzcd}$$

hence an exact triangle

$$i_*i^*A \longrightarrow G \longrightarrow B \xrightarrow{+1} \quad .$$

Applying $i^!$ we obtain

$$i^*A \longrightarrow i^!G \longrightarrow i^!B \xrightarrow{+1} \quad .$$

As $i^*A \in \mathcal{D}_Y^{\leq 0}$ we obtain $i^*A = \tau_{\leq 0}i^!G$ necessarily.

Using our analysis we now construct A and B. Start from the morphism

$$\tau_{\leq 0}j^*F \longrightarrow j^*F \quad .$$

By adjunction we obtain a morphism

$$j_!\tau_{\leq 0}j^*F \longrightarrow F \quad .$$

Complete it into an exact triangle

$$j_!\tau_{\leq 0}j^*F \longrightarrow F \longrightarrow G \xrightarrow{+1} \quad .$$

Consider the morphism $\tau_{\leq 0}i^!G \longrightarrow G$. By adjunction it gives $i_*\tau_{\leq 0}i^!G \longrightarrow G$ which we complete into an exact triangle

$$i_*\tau_{\leq 0}i^!G \longrightarrow G \longrightarrow B \xrightarrow{+1} \quad .$$

Consider the diagram given by the octahedron axiom :

$$\begin{tikzcd}
 & & i_*\tau_{\leq 0}i^!G \arrow[dr] & & \\
 & & & G \arrow[dr] & \\
 & A \arrow[r] \arrow[uur, dashed] & F \arrow[ur] \arrow[rr] & & B \\
j_!\tau_{\leq 0}j^*F \arrow[ur, dashed] \arrow[urr] & & & &
\end{tikzcd}$$

which produces A and B.

We have to check that A and B satisfy the required properties, namely $A \in \mathcal{D}_X^{\leq 0}$ and $B \in \mathcal{D}_X^{>0}$. But

$$j^*A = j^*(j_!\tau_{\leq 0}j^*F) = \tau_{\leq 0}j^*F \in \mathcal{D}_U^{\leq 0}$$

where the first equality follows from applying j^* to the triangle

$$j_!\tau_{\leq 0}j^*F \longrightarrow A \longrightarrow i_*\tau_{\leq 0}i^!G \xrightarrow{+1}$$

and $j^*i_* = 0$. On the other hand :

$$i^*A = i^*i_*\tau_{\leq 0}i^!G = \tau_{\leq 0}i^!G \in \mathcal{D}_Y^{\leq 0} \quad .$$

Hence $A \in \mathcal{D}_X^{\leq 0}$.

SImilarly :

$$j^!B = \tau_{>0}j^*F \in \mathcal{D}_U^{>0} \quad .$$
$$i^!B = \mathrm{cone}(\tau_{\leq 0}i^!G \longrightarrow i^!G) = \tau_{>0}i^!G \in \mathcal{D}_Y^{>0} \quad .$$

Hence $B \in \mathcal{D}_X^{\geq 0}$.

$\square$

Proposition 17.1.3. (i) *The functors* $j^* : \mathcal{D}_X \longrightarrow \mathcal{D}_U$ *and* $i_* : \mathcal{D}_Y \longrightarrow \mathcal{D}_X$ *are* t*-exact.*

(ii) *The functors* $j_! : \mathcal{D}_U \longrightarrow \mathcal{D}_X$ *and* $i^* : \mathcal{D}_X \longrightarrow D_Y$ *are right* t*-exact (but not* t*-exact in general).*

(iii) *The functors* $j_* : \mathcal{D}_U \longrightarrow \mathcal{D}_X$ *and* $i^! : \mathcal{D}_X \longrightarrow \mathcal{D}_Y$ *are left* t*-exact.*

Proof. Consider first $i_* : \mathcal{D}_Y \longrightarrow \mathcal{D}_X$. As $i^*i_* \xrightarrow{\sim} \mathrm{id}$ and $j^*i_* = 0$ one has $i_*\mathcal{D}_Y^{\leq 0} \subset \mathcal{D}_X^{\leq 0}$ by definition of $\mathcal{D}_X^{\leq 0}$. As $\mathrm{id} \xrightarrow{\sim} i^!i_*$ and $j^!i_* = 0$ one has $i_*\mathcal{D}_Y^{\geq 0} \subset \mathcal{D}_X^{\geq 0}$. Hence i_* is t-exact (hence ${}^pi_* : \mathcal{C}_Y \longrightarrow \mathcal{C}_X$ is exact and i_* commutes with H^0).

Consider next $j^* : \mathcal{D}_X \longrightarrow \mathcal{D}_U$. This time $j^*\mathcal{D}_X^{\leq 0} \subset \mathcal{D}_U^{\leq 0}$ and $j^*\mathcal{D}_X^{\geq 0} = j^!\mathcal{D}_X^{\geq 0} \subset \mathcal{D}_U^{\geq 0}$ by definition of the t-structure on $\mathcal{D}_X$ and the result.

For (ii) : As $\mathrm{id} \xrightarrow{\sim} j^*j_!$ one has $j^*j_!\mathcal{D}_U^{\leq 0} \subset \mathcal{D}_U^{\leq 0}$. As $i^*j_! = 0$ this implies $j_!\mathcal{D}_U^{\leq 0} \subset \mathcal{D}_X^{\leq 0}$. On the other hand $i^*\mathcal{D}_X^{\leq 0} \subset \mathcal{D}_Y^{\leq 0}$ by definition.

Statement (iii) is dual to (ii).

□

Corollary 17.1.4. *We have the following diagram of adjunctions*

$$\begin{tikzcd}
 & \mathcal{C}_U \arrow[d, phantom] & \\
\arrow[d, "H^0 j_!"'] & \arrow[u, "j^! = j^*"] & \arrow[d, "H^0 j_*"] \\
 & \mathcal{C}_X & \\
\arrow[d, "H^0 i^*"'] & \arrow[u, "i_* = i_!"] & \arrow[d, "H^0 i^!"] \\
 & \mathcal{C}_Y &
\end{tikzcd}$$

satisfying the following identities

$$\begin{aligned}
\mathrm{id} &\xrightarrow{\sim} j^!(H^0 j_!) \ , \\
j^*(H^0 j_*) &\xrightarrow{\sim} \mathrm{id} \ , \\
(H^0 i^*)i_* &\xrightarrow{\sim} \mathrm{id} \ , \\
\mathrm{id} &\xrightarrow{\sim} (H^0 i^!)i_! \ , \\
j^*i_* = 0 \quad \textit{hence} \quad (H^0 i^!)(H^0 j_*) &= (H^0 i^*)(H^0 j_!) = 0 \ .
\end{aligned}$$

Let us now study what the distinguished triangles in $\mathcal{D}$

$$j_!j^!\mathcal{F} \longrightarrow \mathcal{F} \longrightarrow i_*i^*\mathcal{F} \xrightarrow{+1} \tag{42}$$

and

$$i_!i^!\mathcal{F} \longrightarrow \mathcal{F} \longrightarrow j_*j^*\mathcal{F} \xrightarrow{+1} \tag{43}$$

become in $\mathcal{C}$.

Lemma 17.1.5. *Let* $\mathcal{F} \in \mathcal{C}$*. One has the following exact sequences in* $\mathcal{C}$ *:*

$$0 \longrightarrow {}^pi_*H^{-1}i^*\mathcal{F} \longrightarrow {}^pj_!{}^pj^*\mathcal{F} \longrightarrow \mathcal{F} \longrightarrow {}^pi_*{}^pi^*\mathcal{F} \longrightarrow 0 \tag{44}$$

and

$$0 \longrightarrow {}^pi_!{}^pi^!\mathcal{F} \longrightarrow \mathcal{F} \longrightarrow {}^pj_*{}^pj^*\mathcal{F} \longrightarrow {}^pi_*H^{-1}i^!\mathcal{F} \longrightarrow 0 \ . \tag{45}$$

Proof. From the triangle (42) one deduces the long exact sequence :

$$\cdots \longrightarrow H^{-1}\mathcal{F} \longrightarrow H^{-1}i_*i^*\mathcal{F} \longrightarrow H^0 j_!j^!\mathcal{F} \longrightarrow H^0\mathcal{F} \longrightarrow H^0 i_*i^*\mathcal{F} \longrightarrow H^1 j_!j^!\mathcal{F} \longrightarrow \cdots \ .$$

As $\mathcal{F} \in \mathcal{C}$ one has $H^{-1}\mathcal{F} = 0$ and $H^0\mathcal{F} = \mathcal{F}$. As i_* commutes with H^0 (hence with H^i, $i \in \mathbb{Z}$) the exact sequence (44) will follow from $H^1 j_!j^!\mathcal{F} = 0$. As $\mathcal{F} \in \mathcal{D}_X^{\leq 0}$ one has $j^!\mathcal{F} \in \mathcal{D}_U^{\leq 0}$. As $j_!$ is right t-exact it follows that $j_!j^!\mathcal{F} \in \mathcal{D}_X^{\leq 0}$ hence $H^1 j_!j^!\mathcal{F} = 0$.

The proof of (45) starting from (43) is similar. □

Corollary 17.1.6. (a) *The essential image of* ${}^p i_* : \mathcal{C}_Y \longrightarrow \mathcal{C}_X$ *is*

$$\{\mathcal{F} \in \mathcal{C}_X \ / \ F \simeq H^0 i_* i^* \mathcal{F}\} = \{\mathcal{F} \ / \ j^*\mathcal{F} = 0\} \ .$$

(b) *The essential image of* ${}^p j_* = H^0 j_* : \mathcal{C}_Y \longrightarrow \mathcal{C}_X$ *identifies with*

$$\{F \ / \ F \simeq H^0 j_* j^* F\} = \{F \ / \ H^0 i_! \mathcal{F} = H^1 i_! \mathcal{F} = 0\} \ .$$

Proof. If $\mathcal{F} \simeq i_*\mathcal{G}$ then $H^0 i^* \mathcal{F} \simeq \mathcal{G}$ by adjunction hence $\mathcal{F} \simeq i_* H^0 i^* \mathcal{F} = H^0 i_* i^* \mathcal{F}$ as H^0 commutes with i_*. We deduce from the sequence (45) that $\mathcal{F} \stackrel{\sim}{\longrightarrow} H^0 i_* i^* \mathcal{F}$ if and only if ${}^p j_! {}^p j^* \mathcal{F} = 0$. As $\mathrm{id} \stackrel{\sim}{\longrightarrow} j^! {}^p j_!$ the functor ${}^p j^!$ is fully faithful and $\mathcal{F} \stackrel{\sim}{\longrightarrow} H^0 i_* i^* \mathcal{F}$ if and only if ${}^p j^* \mathcal{F} = 0$.

Similarly : if $\mathcal{F} \simeq {}^p j_* \mathcal{F}$ then as $j^* {}^p j_* \stackrel{\sim}{\longrightarrow} \mathrm{id}$ one has $j^* \mathcal{F} \simeq \mathcal{G}$ hence $\mathcal{F} \simeq H^0 j_* j^* \mathcal{F}$. Contemplating the sequence (44) shows this is the case if and only if ${}^p i_* {}^p j^! \mathcal{F} = {}^p i_* H^1 i^! \mathcal{F} = 0$, hence if and only if $H^0 i^! \mathcal{F} = H^1 i^! \mathcal{F} = 0$ as ${}^p i_*$ is fully faithful. □

Lemma 17.1.7. *Let* $\mathcal{F} \in \mathcal{C}_X$. *The exact sequences*

$$0 \longrightarrow i_* H^0 i^! \mathcal{F} \longrightarrow \mathcal{F}$$

and

$$\mathcal{F} \longrightarrow i_* H^0 i^* \mathcal{F} \longrightarrow 0$$

obtained from the sequences (44) and (45) are respectively the biggest subobject and the biggest quotient of $\mathcal{F}$ *contained in the essential image of* $\mathcal{C}_Y$.

Proof. Let $\mathcal{G} \in \mathcal{C}_Y$. Then $\mathrm{Hom}(i_*\mathcal{G}, \mathcal{F}) = \mathrm{Hom}(\mathcal{G}, H^0 i^! \mathcal{F}) = \mathrm{Hom}(i_*\mathcal{G}, i_* H^0 i^! \mathcal{F})$ where the first equality follows from adjunction and the second one from the fact that i_* is fully faithful.

The second statement is dual to the first one. □

18. Intermediate extensions

Definition 18.0.8. *Let* $\mathcal{G} \in \mathcal{C}_U$. *One says that* $\mathcal{F} \in \mathcal{C}_X$ *extends* $\mathcal{G}$ *if* $j^*\mathcal{F} \stackrel{\sim}{\longrightarrow} \mathcal{G}$.

Example 18.0.9. Both $H^0 j_! \mathcal{G}$ and $H^0 j_* \mathcal{G}$ extend $\mathcal{G}$.

Proposition 18.0.10. *There exists a pair* $(\mathcal{F} \in \mathcal{C}_X, \alpha : j^*\mathcal{G} \stackrel{\sim}{\longrightarrow} \mathcal{G})$, *unique up to unique isomorphism, extending* $\mathcal{G}$ *and satisfying*

(i) $i^*\mathcal{F} \in \mathcal{D}_Y^{<0}$.
(ii) $i^!\mathcal{F} \in \mathcal{D}_Y^{>0}$.

It will be called the intermediate extension *of* $\mathcal{G}$ *and denoted* $j_{!*}G$.

Proof. First notice that :

$$\begin{aligned} i^*\mathcal{F} \in \mathcal{D}_Y^{<0} &\iff H^0 i^* \mathcal{F} = 0 \\ &\iff \text{any quotient of } \mathcal{F} \text{ in } \mathcal{C}_Y \text{ is } 0 \text{ by lemma (17.1.7).} \end{aligned}$$

Similarly $i^!\mathcal{F} \in \mathcal{D}_Y^{<0}$ if and only if $H^0 i^! \mathcal{F} = 0$, if and only if any subobject of $\mathcal{F}$ in $\mathcal{C}_Y$ is 0.

Let us prove first the unicity of the intermediate extension. Suppose $\alpha : j^*\mathcal{F} \stackrel{\sim}{\longrightarrow} \mathcal{G}$ and $\alpha' : j^*\mathcal{F}' \stackrel{\sim}{\longrightarrow} \mathcal{G}$ are two such intermediate extension. Applying $\mathrm{Hom}(\mathcal{F}, \cdot)$ to the exact triangle

$$i_! i^! \mathcal{F}' \longrightarrow \mathcal{F}' \longrightarrow j_* j^* \mathcal{F}' \stackrel{+1}{\longrightarrow}$$

one obtains the long exact sequence :

$$\cdots \longrightarrow \mathrm{Hom}(i^*\mathcal{F}, i^!\mathcal{F}') \longrightarrow \mathrm{Hom}(\mathcal{F}, \mathcal{F}') \longrightarrow \mathrm{Hom}(j^*\mathcal{F}, j^*\mathcal{F}') \longrightarrow \mathrm{Hom}(i^*\mathcal{F}, i^!\mathcal{F}'[1]) \longrightarrow \cdots \ .$$

As $i^*\mathcal{F} \in \mathcal{D}_Y^{<0}$ and $i^!\mathcal{F}' \in \mathcal{D}_Y^{>0}$ (thus $i^!\mathcal{F}'[1] \in \mathcal{D}_Y^{\geq 0}$) we obtain

$$\mathrm{Hom}(\mathcal{F}, \mathcal{F}') \stackrel{\sim}{\longrightarrow} \mathrm{Hom}(\mathcal{G}, \mathcal{G}) \ .$$

Hence there exists a unique isomorphism between $\mathcal{F}$ and $\mathcal{F}'$ compatible with α and α'.

Let us show the existence of the intermediate extension. Considering the adjunctions $(j_!, j^! = j^*, j_*)$ we get a canonical morphism $j_!\mathcal{G} \longrightarrow j_*\mathcal{G}$ in $\mathcal{D}$ which induces the identity of $\mathcal{G}$ once composed with j^*. Applying H^0 we obtain a canonical morphism

$$H^0 j_!\mathcal{G} \longrightarrow H^0 j_*\mathcal{G}$$

in $\mathcal{C}$ inducing the identity of $\mathcal{G}$ once composed with j^*. Define the intermediate extension M as the image of this morphism :

$$\begin{tikzcd} & M \arrow[dr, hook] & \\ H^0 j_!\mathcal{G} \arrow[ur, two heads] \arrow[rr] & & H^0 j_*\mathcal{G} \end{tikzcd} .$$

Obviously $j^*M = G$. It remains to show that M satisfies the conditions (i) and (ii) for intermediate extensions.

First $(H^0 i^*)(H^0 j_!\mathcal{G}) = 0$ hence $H^0 j_!\mathcal{G}$ has no quotient in $\mathcal{C}_Y$. A fortiori M has no such quotient.

Dually $H^0 j_*\mathcal{G}$ has no subobject in $\mathcal{C}_Y$ hence also M. □

Example 18.0.11. If $\mathcal{D} = \mathcal{D}_c(\mathbb{Z}_X)$ with its standard t-structure then $j_{!*}\mathcal{G} = j_!\mathcal{G}$.

Lemma 18.0.12.

$$\begin{aligned} j_{!*}\mathcal{G} &= H^0 j_! G / i_* H^0 i^! j_!\mathcal{G} \quad \text{where } i_* H^0 i^! j_!\mathcal{G} \text{ is the biggest subobject of } H^0 j_! G \text{ in } \mathcal{C}_Y \ , \\ &= \ker(H^0 j_*\mathcal{G} \longrightarrow i_* H^0 i^* j_*\mathcal{G}) \quad \text{where } i_* H^0 i^* j_*\mathcal{G} \text{ is the biggest quotient of } H^0 j_* G \text{ in } \mathcal{C}_Y \ . \end{aligned}$$

Proof. Put $\mathcal{F} = j_*\mathcal{G}$ in the triangle $j_! j^! \mathcal{F} \longrightarrow \mathcal{F} \longrightarrow i_* i^* \mathcal{F} \xrightarrow{+1}$. On the other hand put $\mathcal{F} = j_!\mathcal{G}$ in the triangle $i_! i^! \mathcal{F} \longrightarrow \mathcal{F} \longrightarrow j_* j^* \mathcal{F} \xrightarrow{+1}$. One gets the diagram

$$\begin{tikzcd} & j_!\mathcal{G} \arrow[r] \arrow[d, equal] & j_*\mathcal{G} \arrow[r] \arrow[d, equal] & i_* i^* j_*\mathcal{G} \arrow[r] & {} \\ i_! i^! j_!\mathcal{G} \arrow[r] & j_!\mathcal{G} \arrow[r] & j_*\mathcal{G} \arrow[r] & i_! i^! j_!\mathcal{G}[1] \arrow[r] & {} \end{tikzcd}$$

Hence $i_* i^* j_*\mathcal{G} \xrightarrow{\sim} i_! i^! j_!\mathcal{G}[1]$. As i_* is fully faithful on $\mathcal{D}$ we deduce : $i^* j_*\mathcal{G} \xrightarrow{\sim} i_! i^! \mathcal{G}[1]$. Applying H^0 to the second line of the previous diagram we obtain the exact sequence :

$$H^0(j_*\mathcal{G}[-1]) = 0 \longrightarrow i_* H^0 i^! j_!\mathcal{G} \longrightarrow H^0 j_*\mathcal{G} \longrightarrow i_* H^0 i^* j^*\mathcal{G} \longrightarrow H^1(j_!\mathcal{G}) \ .$$

As $j_*\mathcal{G} \in \mathcal{D}_X^{\geq 0}$ one has $H^0(j_*\mathcal{G}[-1]) = 0$. Similarly $j_!\mathcal{G} \in \mathcal{D}_X^{\leq 0}$ hence $H^1(j_!\mathcal{G}) = 0$ which gives the result. □

Proposition 18.0.13. *Simple objects in $\mathcal{C}_X$ are the ${}^p i_*\mathcal{F}$, where $\mathcal{F} \in \mathcal{C}_Y$ is simple, and the $j_{!*}\mathcal{G}$, $\mathcal{G}$ simple in $\mathcal{C}_U$.*

Proof. Follows immediately from the exact sequence of Abelian categories :

$$0 \longrightarrow \mathcal{C}_Y \longrightarrow \mathcal{C}_X \longrightarrow \mathcal{C}_U \longrightarrow 0 \ .$$

□

19. Application to perverse sheaves

Recall our setting :

$$X = X_m \supset X_{m-1} \supset \cdots \supset X_0 \supset X_{-1}$$

is an evenly stratified space with $\dim_{\mathbb{C}} X_i = i$ and we consider the middle perversity for simplicity.

We define the perverse t-structure on $\mathcal{D}_c^b(X)$ inductively on $m := \dim_{\mathbb{C}} X$ as follows :

- if $m = 0$ we consider the standard t-structure. Hence a perverse sheaf is just a sheaf.

- the inductive step works as follows. Consider the diagram

$$\begin{tikzcd} X_{m-1} \arrow[r, hook, "/"] \arrow[d, equal] & X \arrow[d, equal] & X \setminus X_{m-1} \arrow[l, hook', "0"'] \arrow[d, equal] \\ Y \arrow[r, hook, "/"] & X & U \arrow[l, hook', "0"'] \end{tikzcd} .$$

  By induction hypothesis we have the perverse t-structure on $\mathcal{D}_Y := \mathcal{D}_c^b(Y)$. On $\mathcal{D}_U := \mathcal{D}_c^b(U)$ we put the standard t-structure *shifted* by $m_U := \dim_{\mathbb{C}} U$:

$$\begin{cases} {}^p\mathcal{D}_U^{\leq 0} := {}^{st}\mathcal{D}_U{}^{\leq -m_U}, \\ {}^p\mathcal{D}_U^{\geq 0} := {}^{st}\mathcal{D}_U{}^{\geq -m_U}. \end{cases}$$

  We then define the perverse t-structure on $\mathcal{D}_X := \mathcal{D}_c^b(X)$ by glueing the two t-structures on $\mathcal{D}_Y$ and $\mathcal{D}_U$.

Let us show that we recover our previous definition of perverse sheaves.

Proposition 19.0.14. *Let $\mathcal{F} \in \mathcal{D}_c^b(X)$. Then :*

(1)

$$\begin{aligned} \mathcal{F} \in {}^p\mathcal{D}_X{}^{\leq 0} &\iff k_s^*\mathcal{F} \in {}^{st}\mathcal{D}_X{}^{\leq -m_S} \text{ for any stratum } k_S : S \longrightarrow X \\ &\iff \dim_{\mathbb{C}} \operatorname{supp} H^i\mathcal{F} \leq -i \text{ for any } i \in \mathbb{Z}. \end{aligned}$$

(2)

$$\begin{aligned} \mathcal{F} \in {}^p\mathcal{D}_X{}^{\geq 0} &\iff k_S^!\mathcal{F} \in {}^{st}\mathcal{D}_X{}^{\geq -m_S} \text{ for any stratum } k_S : S \longrightarrow X \\ &\iff k_S^* D_X\mathcal{F} \in {}^{st}\mathcal{D}_X{}^{\leq -m_S} \text{ for any stratum } k_S : S \longrightarrow X \\ &\iff \dim_{\mathbb{C}} \operatorname{supp} H^i D_X\mathcal{F} \leq -i \text{ for any } i \in \mathbb{Z}. \end{aligned}$$

(3) $\mathcal{F} \in \mathcal{C}_X$ *if and only $\mathcal{F}$ is a perverse sheaf in the sense of the definition 13.2.1.*

Proof. The proof is by induction on $m := \dim_{\mathbb{C}} X$.

If $m = 0$ this is obvious.

The inductive step works as follows. With the above notations $\mathcal{F} \in {}^p\mathcal{D}_X{}^{\leq 0}$ if and only if

$$\begin{cases} j^*\mathcal{F} \in {}^p\mathcal{D}_U{}^{\leq 0}, \\ \quad i^*\mathcal{F} \in {}^p\mathcal{D}_Y^{\leq 0}. \end{cases}$$

By definition of the t-structure on $\mathcal{D}_U$ and by induction hypothesis this is equivalent to saying that $k_S^*\mathcal{F} \in {}^{st}\mathcal{D}_S{}^{\leq -m_S}$ for any stratum S. This proves (1).

The proof of (2) is similar and (3) follows. □

19.1. **Intermediate extension for perverse sheaves.** Once more consider the situation $U \overset{0}{\hookrightarrow} X \overset{/}{\hookleftarrow} Y$ and let $\mathcal{E}$ be a local system on U (hence $\mathcal{E}[m]$ is a perverse sheaf on U). Recall that the intermediate extension $j_{!*}\mathcal{E}[m]$ is the unique extension $\mathcal{F} \in \mathrm{Perv}(X)$ of $\mathcal{E}[m]$ such that $i^*\mathcal{F} \in {}^p\mathcal{D}_Y{}^{<0}$ and $i^!\mathcal{F} \in {}^p\mathcal{D}_Y{}^{>0}$. Let us check it coincides with Deligne's extension.

Notice that :

$$\begin{aligned} i^*\mathcal{F} \in {}^p\mathcal{D}_Y{}^{<0} &\iff i^*\mathcal{F}[-1] \in {}^p\mathcal{D}_Y{}^{\leq 0} \\ &\iff \dim_{\mathbb{C}} \operatorname{supp} H^j i^*\mathcal{F}[-1] \leq -j \text{ for any } j \in \mathbb{Z} \\ &\iff \dim_{\mathbb{C}} \operatorname{supp} H^j i^*\mathcal{F} < -j \text{ for any } j \in \mathbb{Z} \\ &\iff \dim_{\mathbb{C}} \operatorname{supp} H^j\mathcal{F} \leq -j \text{ for any } j \in \mathbb{Z} \text{ with strict inequality for } j \neq -m. \end{aligned}$$

Similarly for $i^!\mathcal{F}$ replacing $\mathcal{F}$ by $D_X\mathcal{F}$.

Hence the result.

References

[1] A'Campo N., Pseudomanifold structures on complex analytic spaces, chapter *IV* in *Intersection Cohomology* by Borel et al.

[2] Banagl M., Topological invariants of stratified spaces, Springer Monographs in Mathematics, 2007

[3] Borel A., Sheaf theoretic intersection cohomology, in *Intersection cohomology*, Borel and alii, Progress in Math. 50, Birkhaüser

[4] Friedman G., Stratified fibrations an the intersection homology of the regular neighbourhoods of bottom strata, *Topol. Appl.* **134** (2003) 69-109

[5] Friedman G., An introduction to intersection homology with general perversity functions, available at http://faculty.tcu.edu/gfriedman/

[6] Gajer P., The intersection Dold-Thom theorem, *Topology* **35** (1996) 939-967

[7] Gelfand S.I., Manin Yu.I., Homological algebra

[8] Godement R., Théorie des faisceaux, Hermann, 1973

[9] Goreski M., MacPherson R., Intersection homology theory, *Topology* **19** (1980) 135-162

[10] Goreski M., MacPherson R., Intersection homology II, *Invernt. Math.* **71** (1983) 77-129

[11] Goreski M., MacPherson R., Simplicial intersection homology, *Invent. Math* **84** (1986) 432-433

[12] Goresky M., MacPherson R., Stratified Morse theory, *Ergebnisse* **14**, Springer (1988)

[13] Haefliger A., Introduction to piecewise linear intersection homology, chapter *I* in *Intersection Cohomology* by Borel et al.

[14] Iversen B., Cohomology of sheaves, Springer 1986

[15] Kashiwara M., Schapira P., Sheaves on Manifolds, Grundlehren der mathematischen Wissenschaften **292**, Springer, 1990

[16] King H.C., Topological invariance of intersection homology without sheaves, *Topology Appl.* **20** (1985) 149-160

[17] Kleiman S., The development of intersection homology theory, *Pure and applied Math. Quarterly*, special issue in honor of R. MacPherson (2007)

[18] MacPherson R., Intersection homology and perverse sheaves, preprint IHES 1990

[19] Rourke C.P., Sanderson B.J., Piecewise linear topology, *Ergebnisse der Mathematik* **69**, Springer (1972)

[20] Verdier J.L., Stratifications de Whitney et théorème de Bertini-Sard, *Invent. Math.* **36** (1976) 295-312