Duflo isomorphism and the Kashiwara–Vergne conjecture in depth 2

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1 Duflo isomorphism

Let k be a field of characteristic zero, $\mathfrak{g}$ a finite dimensional Lie algebra. We define the tensor algebra

$$T\mathfrak{g} = k \oplus \mathfrak{g} \oplus (\mathfrak{g} \otimes \mathfrak{g}) \oplus \dots$$

and the symmetric algebra

$$S\mathfrak{g} = T\mathfrak{g}/\langle x \otimes y - y \otimes x \mid x, y \in \mathfrak{g}\rangle \cong K[x_1, \dots, x_n],$$

where $x_1, \dots, x_n$ is a basis of $\mathfrak{g}$. On the other hand, we have a universal enveloping algebra

$$U\mathfrak{g} := T\mathfrak{g}/\langle x \otimes y - y \otimes x - [x, y] \mid x, y \in \mathfrak{g}\rangle$$

Recall that $\mathfrak{g}$ acts on $\mathfrak{g}$ via the adjoint representation: $\mathrm{ad}_x \colon \mathfrak{g} \to \mathfrak{g}$, $y \mapsto [x, y]$. This action extends to $S\mathfrak{g}$ (and to $U\mathfrak{g}$) by derivation $\mathrm{ad}_x(y^n) = n[x, y]y^{n-1}$. Let us define the invariant elements

$$(S\mathfrak{g})^{\mathfrak{g}} = \{\rho \in S\mathfrak{g} \mid \mathrm{ad}_x(\rho) = 0 \text{ for all } x \in \mathfrak{g}\},$$
$$(U\mathfrak{g})^{\mathfrak{g}} = \{\rho \in U\mathfrak{g} \mid \mathrm{ad}_x(\rho) = 0 \text{ for all } x \in \mathfrak{g}\} = Z(U\mathfrak{g}).$$

We can define a symmetrization map

$$x_1, \dots, x_n \mapsto \frac{1}{n!} \sum_{\sigma \in S_n} x_{\sigma(1)} \dots x_{\sigma(n)},$$

but it is not an isomorphism of algebras.

Теорема 1.1 (Duflo, 1977)**.** There is an isomorphism of algebras $(U\mathfrak{g})^{\mathfrak{g}} \cong (S_{\mathfrak{g}})^{\mathfrak{g}}$ given by $Duf\colon \mathrm{Sym} \circ \partial_{J^{1/2}}$, where

$$J^{1/2} = \left(\det(\frac{e^{\mathrm{ad}_x} - 1}{\mathrm{ad}_x})\right)^{1/2} = \exp\left(\frac{1}{2}\operatorname{Tr}\mathrm{ad}_x + \frac{1}{2}\sum_{k=2}^{\infty} \frac{B_k}{k \times k!}\operatorname{Tr}(\mathrm{ad}_x^k)\right).$$

For example, take $\mathfrak{g} = su(2) = \langle x, y, z\rangle$ with relations $[x, y] = z$, $[y, z] = x$, $[z, x] = y$. Then $(S_{\mathfrak{g}}^{\mathfrak{g}} = \mathbb{R}[x^2 + y + 2 + z^2$, and the Duflo isomorphism is given by

$$Duf\colon x^2 + y^2 + z^2 \mapsto x^2 + y^2 + z^2 + 1/24,$$

and

$$\partial_{J^{1/2}} = 1 + \frac{1}{24}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) + \text{elements of higher order.}$$

The proof was given by Duflo in 1977. In 1978 Kashiwara–Vergne stated a conjecture that implies Duflo isomoprhism. Kontsevich proved Duflo theorem using formally Feynmann graph calculus. Them in 2006 Alekseev and Meinrenken proved KV, and in 2008 Alekseev and Torossian gave another proof.

2 Kashiwara–Vergne conjecture

KV conjecture shows how to transfer convolution product using exp map. It yields two equations.

Recall the CBH formula (Campbell–Baker–Hausdorff):

$$\log(e^x e^y) = ch(x,y) = x + y + \frac{1}{2}[x,y] + \frac{1}{12}\dots.$$

KV conjecture: there exist two Lie series $A(x,y)$, $B(x,y)$ such that

1. $x + y - ch(x,y) = (1 - e^{\mathrm{ad}_x})A + (e^{\mathrm{ad}_y} - 1)B$;
2. $$\mathrm{Tr}_{\mathfrak{g}}(\mathrm{ad}_x \circ \partial_x(A) + \mathrm{ad}_y \circ \partial_y(b)) = \frac{1}{2}\mathrm{Tr}_{\mathfrak{g}}\left(\frac{\mathrm{ad}_x}{e^{\mathrm{ad}_x} - 1} + \frac{\mathrm{ad}_y}{e^{\mathrm{ad}_y} - 1} - \frac{\mathrm{ad}_{ch(x,y)}}{\exp^{\mathrm{ad}_{ch(x,y)}} - 1}\right)$$

where $\partial_x A\colon \mathfrak{g} \to \mathfrak{g}$, $v \mapsto \frac{d}{dt}(A(x+tv,y))|_{t=0}$.

How does this imply Duflo isomorphism? Define a 1-parameter family of products

$$m_t\colon S_{\mathfrak{g}}[t] \to U(\mathfrak{g}[t], t[-,-]),$$

where $t[]$ is a rescaled product. The idea is that m_0 is just the regular commutative product on $S_{\mathfrak{g}}$, and for $t \neq 0$ m_t is a non-commutative product on $U_{\mathfrak{g}}$.

We want to have $m_0 = m_t$ on $\mathfrak{g}$-invariant elements. When trying to write this equality, we get equations on the Lie series A and B that are implied by the KV equations!

Some extensions of Duflo isomorphism to symmetric spaces are proven by Rouviere using "KV method".

There is a group $\widehat{KV}_2$, which is the symmetry group of the KV problem. Let $\widehat{krv}_2$ be its Lie algebra.

Гипотеза 2.1. $\widehat{krv}_2$ is isomorphic to $kt \oplus grt_1$, where grt_1 is the Grothendieck–Teichmüller Lie algebra. Moreover, it is isomorphic to the double shuffle multi zeta values Lie algebra (see Schneps, Zagier).

Now we need to define krv_2 and grt_1.

Consider a completion of the free Lie algebra on two elements: $\widehat{lie}_2 = \prod_{k=1}^{\infty} lie_2^k(x,y)$. Here $x, y \in lie_2^1$, $[x,y] \in lie_2^2$, $[x,[x,y]] \in lie_2^3$, and so on. It is graded by "length" of the Lie words. The universal enveloping algebra of this algebra is isomorphic to free associative algebra:

$$U(\widehat{lie}_2) \cong \widehat{Ass}_2 = \prod_{k=0}^{\infty} Ass_2^k(x,y),$$

where for $k = 1$ we have the elements x, y, for $k = 2$ we have x^2, xy, yx, y^2, and so on. For $a \in \widehat{Ass}_2$ we have

$$a = a_0 + (\partial_x a)x + (\partial_y a)y + \dots,$$

where $a_0 \in k$ is the constant term. For example, for $a = xy - y^2x$ we have $\partial_x(a) = -y^2$, $\partial_y(a) = x$.

Now we can define $tr_2 = Ass_2^+/\langle ab - ba \mid a, b \in Ass_2^+\rangle$, where Ass_2^+ consists of elements with zero constant term. This thing consists of cyclic words in x and y: for $k = 1$ we have $tr(x)$, $tr(y)$, and for $k = 2$ we have $tr(x^2)$, $tr(y^2)$, $tr(xy) = tr(yx)$.

Let der_2 be derivations on $\widehat{lie}_2$, so $u \in der_2$ if $u([a,b]) = [u(a),b] + [a,u(b)]$. Define tangential derivations $tder_2$: $u \in tder_2$ if $u(x) = [x, a(x,y)]$, $u(y) = [y, b(x,y)]$ for some words $a, b \in \widehat{lie}_2$. We can identify u with (a,b). Actually, $tder_2$ is a Lie algebra (Lie bracket = commutator of derivations).

Now we define the divergence

$$\begin{aligned} div\colon tder_2 &\to tr_2, \\ u=(a,b) &\mapsto tr(x\partial_x a + y\partial_y b). \end{aligned}$$

Here's an example. Let's take $u = ([x,y],x)$. This means that $u(x) = [x,[x,y]]$, $u(y) = [y,x]$, and

$$div(u) = \mathrm{tr}(x\partial_x([x,y]) + y\partial_y(x)) = \mathrm{tr}(x(-y)) = -\mathrm{tr}(xy).$$

This div map is a cocycle (in the sense of Chevalley–Eilenberg cohomology): $div([u,v]) = u\cdot div(v) - v\cdot div(u)$ (here $\cdot$ denotes the action of $tder_2$ on tr_2).

3 Grothendieck–Teichmüller algebra

grt_1 is generated by $(0,\psi)\in tder_2$ such that:

1. $\psi(x,y) = -\psi(y,x)$;
2. hexagonal equation: $\psi(x,y)+\psi(y,z)+\psi(z,x) = 0$ if $x+y+z=0$;
3. pentagonal equation: some equation with values on t_4, which comes from the Khono–Drinfeld Lie algebra.

The Khono–Drinfeld Lie algebra t_n is generated by $n(n-1)/2$ elements $t^{i,j} = t^{j,i}$ with relations

1. $[t^{i,j}, t^{k,l}] = 0$ for $\{i,j\}\neq\{k,l\}$;
2. $[t^{i,j} + t_{i,k}, t^{j,k}] = 0$ for $i\neq j$ and $j\neq k$.

So the third equation on grt_1 is the pentagonal equation

$$\psi(t^{1,2},t^{2,34}) + \psi(t^{12,3},t^{3,4}) = \psi(t^{2,3},t^{3,4}) + \psi(t^{1,23},t^{23,4}) + \psi(t^{1,2},t^{2,3}]),$$

where $t^{1,23} = t^{1,2}+t^{1,3}$.

We know a couple of things about grt_1: it is graded by weight of $\psi\in lie_2$ (put $\deg(x) = \deg(y) = 1$):

$$grt_1 = \bigoplus grt_1^m.$$

It admits a decreasing filtration by depth $=$ smallest y-degree in any monomial of ψ. For example, $(0,[x,[x,y]]-[y,[y,x]])\in grt_1$ has depth 1.

We know there exist elements of depth 1 in every odd degree; they are called the Soulé elements.

Define krv to be the set of $u=(a,b)\in tder_2$ such that $[x,a(x,y)]+[y,b(x,y)] = 0$, $div(u) = \mathrm{tr}(\rho(x)+\rho(y)-\rho(x+y))$ for some $\rho\in tr_1$.

Теорема 3.1 (Alekseev–Torossian)**.** There is an injective Lie algebra homomorphism

$$\begin{aligned} grt_1 &\hookrightarrow krv, \\ (0,\psi) &\mapsto (\psi(-x-y,x)+\psi(-x-y,y)). \end{aligned}$$

Take $t=(y,x)$; then $t(x) = [x,y]$, $t(y)=[y,x]$, and $t\in krv$: we have $[x,y]+[y,x]=0$, so $div(t) = \mathrm{tr}(x\partial_x(y)+y\partial_y(x)) = 0$.

How to get isomorphism in depth 2?

1. Modify krv into krv' such that $krv\cong krv'$, and krv' admits the same depth filtration
2. Take the associated graded algebras; we want to show that they are isomorphic for depth 2.

We find that the space $\ker\{\psi \mid \mathrm{tr}(y\partial_y\psi) = 0\}$ contains $gr(krv')^{(2)}$. This kernel can be described as $\{(0,\psi) \mid div((0,\psi)) = 0\}$. Therefore, we need to compute dim of this space. This is the same as the space of polynomials in v, w such that $p(v,w) = -p(w,v)$ and $p(v,w) = -p(v+w,-w)$. As it happens, these two transformations generate the dihedral group D_6.

We computed this dimension, and got an upper bound. For the lower bound, we use the injection $grt_1 \hookrightarrow krv$. We know the elements of depth 1; taking the commutator of two Soulé elements we get $[\sigma_i, \sigma_j]$, which is an element of depth 2. Luckily, the dimension gives a lower bound that coincides with the upper bound.