Elementary subgroup of an isotropic reductive group is perfect

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1 Introduction

Let R be a commutative ring with 1, and let G be an isotropic reductive algebraic group over R. In [7] Victor Petrov and the second author introduced a notion of an elementary subgroup $E(R)$ of the group of points $G(R)$. In this note we prove that, as one might expect from the split case (e.g., [10]) as well as from the field case (e.g., [12]), under natural assumptions the elementary subgroup of a reductive group is perfect.

More precisely, assume that G is isotropic in the following strong sense: it possesses a parabolic subgroup that intersects properly any semisimple normal subgroup of G. Such a parabolic subgroup P is called *strictly proper*. Denote by $E_P(R)$ the subgroup of $G(R)$ generated by the R-points of the unipotent radicals of P and of an opposite parabolic subgroup P^-. The main theorem of [7] states that $E_P(R)$ does not depend on the choice of P, as soon as for any maximal ideal M of R all irreducible components of the relative root system of G_{R_M} (see [6, Exp. XXVI, §7] for the definition) are of rank ≥ 2. Under this assumption, we call $E_P(R)$ the *elementary subgroup* of $G(R)$ and denote it simply by $E(R)$. In particular, $E(R)$ is normal in $G(R)$. This definition of $E(R)$ generalizes the well-known definition of an elementary subgroup of a Chevalley group (or, more generally, of a split reductive group), as well as several other definitions of an elementary subgroup of isotropic classical groups and simple groups over fields [2, 12, 13, 14, 3].

By the structure constants of a root system we mean the structure constants of the corresponding semisimple complex Lie group, or, in other words, constants appearing in the Chevalley commutator formulas for the corresponding Chevalley group. They are among ± 1, ± 2, ± 3.

Theorem 1. *Let G be an isotropic reductive algebraic group over a commutative ring R. Assume that for any maximal ideal M of R all irreducible components of the relative root system of G_{R_M} are of rank ≥ 2, and, if one of the irreducible components of the (usual) root system of G_{R_M} is of type $B_2 = C_2$ or G_2, that the residue field R_M/MR_M is not isomorphic to $\mathbb{F}_2$. Then $E(R) = [E(R), E(R)]$.*

Observe that the first condition of the theorem ensures that the the elementary subgroup $E(R)$ of $G(R)$ is correctly defined, while the second one essentially eliminates the well-known cases where the elementary subgroup of a *split* reductive group is not perfect. Thus, the result is the strongest possible. One should note that, if we assume only that the rank of relative root systems of G_{R_M} is ≥ 1, the question whether individual subgroups $E_P(R)$ are perfect is of separate interest.

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The proof of Theorem 1 is based on the notion of relative root subschemes (with respect to a parabolic subgroup) of an isotropic group introduced in [7], the generalized Chevalley commutator formula [7, Lemma 9], and localization in the Quillen—Suslin style [11]. To shorten the proof, we also make use of the classification of Tits indices of isotropic reductive groups over local rings obtained in [8].

2 An abstract definition of relative roots

In this section we recall the notion of an (abstract) system of relative roots introduced in [7] and prove a technical lemma.

Let Φ be a reduced root system in a Euclidean space with a scalar product $(-,-)$. Let $\Pi=\{\alpha_1,\ldots,\alpha_l\}$ be a fixed system of simple roots of Φ; if Φ is irreducible, we assume that the numbering follows Bourbaki [5]. Let D be the Dynkin diagram of Φ. We identify nodes of D with the corresponding simple roots in Π. For a subgroup $\Gamma\subseteq\mathrm{Aut}\,(D)$ and a Γ-invariant subset $J\subseteq\Pi$, consider the projection

$$\pi=\pi_{J,\Gamma}\colon\ \mathbb{Z}\,\Phi\longrightarrow\mathbb{Z}\,\Phi/\left\langle\Pi\setminus J;\ \alpha-\sigma(\alpha),\ \alpha\in J,\ \sigma\in\Gamma\right\rangle.$$

The set $\Phi_{J,\Gamma}=\pi(\Phi)\setminus\{0\}$ is called the system of *relative roots* corresponding to the pair (J,Γ). The *rank* of $\Phi_{J,\Gamma}$ is the rank of $\pi(\mathbb{Z}\,\Phi)$ as a free abelian group.

It is clear that any relative root $A\in\Phi_{J,\Gamma}$ can be represented as a unique linear combination of relative roots from $\pi(\Pi)$. We say that $A\in\Phi_{J,\Gamma}$ is a *positive* (resp. *negative*) relative root, if it is a non-negative (respectively, a non-positive) linear combination of the elements of $\pi(\Pi)$. The sets of positive and negative relative roots will be denoted by $\Phi^+_{J,\Gamma}$ and $\Phi^-_{J,\Gamma}$ respectively. By the *level* $\mathrm{lev}(A)$ of a relative root A we mean the sum of coefficients in its decomposition.

Observe that Γ acts on the set of irreducible components of the root system Φ. If this action is transitive, the system of relative roots $\Phi_{J,\Gamma}$ is *irreducible*. Clearly, any system of relative roots $\Phi_{J,\Gamma}$ is a disjoint union of irreducible ones; we call them the *irreducible components* of $\Phi_{J,\Gamma}$.

We will need the following lemma.

Lemma 1. *Let Φ be a root system with a fixed set of simple roots Π, Γ be a subgroup of $\mathrm{Aut}\,(D)$, and J be a Γ-invariant subset of Π. If a relative root $A\in\Phi_{J,\Gamma}$ lies in an irreducible component of rank ≥ 2, then there exist such non-collinear $B,C\in\Phi_{J,\Gamma}$ that $A=B+C$ and all relative roots $iB+jC\in\Phi_{J,\Gamma}$, $i,j>0$, $(i,j)\neq(1,1)$, have the same sign as A and satisfy $|\,\mathrm{lev}(iB+jC)|>|\,\mathrm{lev}(A)|$.*

Proof. We can assume that the root system Φ is irreducible, and that A is a positive relative root, that is, $\pi^{-1}(A)\subseteq\Phi^+$.

Assume first that $A=k\pi(\alpha_r)$, where $\alpha_r\in\Pi$ is a simple root and $k>0$ is a positive integer. Let $\alpha_s\in J$ be a simple root such that the Γ-orbits of α_s and α_r are distinct, and α_s is at the least possible distance from α_r on the Dynkin diagram. It is easy to see that for any $\alpha\in\pi^{-1}(A)$ there exists $\beta\in\pi^{-1}(\alpha_s)$ such that $(\alpha,\beta)<0$, and, consequently, $\alpha+\beta\in\Phi$. Indeed, we have $m_s(\alpha)=0$ by definition, thus we can take for β the sum of all simple roots in the Dynkin diagram chain between α_s and the nearest simple root appearing in the decomposition of α. Now set $B=\pi(\alpha+\beta)$ and $C=\pi(-\beta)$. It is clear that any root in $\pi^{-1}(iB+jC)$, $i,j>0$, contains the summand $i\alpha_r$ in its decomposition, and thus is a positive root. Then $iB+jC$ is a positive relative root for any $i,j>0$. Moreover, one sees that $\mathrm{lev}(iB+jC)=\mathrm{lev}(A)$ if and only if $i=j=1$. Since $\pi(\alpha)=k\pi(\alpha_r)$, and $\pi(-\beta)=-\pi(\alpha_s)$, the roots B and C are non-collinear.

Consider the case where $A\neq k\pi(\alpha_r)$ for any $\alpha_r\in J$. For any $\alpha\in\pi^{-1}(A)$ there exists a sequence of simple roots $\beta_1,\ldots,\beta_n\in\Pi$ such that $\alpha=\beta_1+\ldots+\beta_n$ and $\beta_1+\ldots+\beta_i\in\Phi$ for any $1 \leq i \leq n$. Let i be the least possible index such that $\beta_{i+1}, \dots, \beta_n \in \Delta$. Then $\beta_i \in J$ and $\pi(\beta_1 + \ldots + \beta_{i-1} + \beta_i) = A$. Set $B = \pi(\beta_1 + \ldots + \beta_{i-1})$ and $C = \pi(\beta_i)$. Since B and C are positive relative roots, we have $\mathrm{lev}(iB + jC) > \mathrm{lev}(A)$ for any $i, j > 0$ distinct from $i = j = 1$. The relative roots B and C are non-collinear since otherwise we would have had $A = k\pi(\beta_i)$ for some $k > 0$. $\square$

3 Isotropic reductive groups and relative root subschemes

In this section we recall some basic notions pertaining to reductive groups over rings; see [6, 7, 9] for more detailed exposition.

Let R be a commutative ring with 1, and let G be a reductive group scheme, or reductive group for short, over R (see [6] for the definition). We denote by G^{ad} and G^{sc} the corresponding adjoint and simply connected semisimple groups, respectively.

Any reductive algebraic group G over R is split locally in the fpqc topology on $\mathrm{Spec}\, R$. If G is of constant type over R (that is, the root system of G is the same at any point of $\mathrm{Spec}\, R$), then G is a twisted form of a split reductive algebraic group G_0 over R, given by a cocycle $\xi \in H^1_{fpqc}(R, \mathrm{Aut}\,(G_0))$. Recall that the connected component of $\mathrm{Aut}\,(G_0)$ is precisely G_0^{ad}. The group G is *of inner type*, if ξ is in the image of the natural map $H^1_{fpqc}(R, G_0^{ad}) \to H^1_{fpqc}(R, \mathrm{Aut}\,(G_0))$. One can always find a finite Galois extension S of R such that G_S is of inner type. The Galois group $\mathrm{Gal}(S/R)$ acts on the Dynkin digram of each $G_{\overline{k(s)}}$, where $\overline{k(s)}$ is the algebraic closure of the residue field of a point $s \in \mathrm{Spec}\, R$, via a $*$-*action* (see [8, 9]).

Recall that G is called isotropic, if it contains a proper parabolic subgroup P over R. Recall that we call a parabolic subgroup P of G *strictly proper*, if it does not contain any semisimple normal subgroup of G. We set

$$E_P(R) = \langle U_P(R), U_{P^-}(R) \rangle\,,$$

where P^- is any parabolic subgroup of G opposite to P, and U_P and U_{P^-} are the unipotent radicals of P and P^- respectively. The main theorem of [7] states that $E_P(R)$ does not depend on the choice of a strictly proper parabolic subgroup P, as soon as for any maximal ideal M in R all irreducible components of the relative root system of G_{R_M} are of rank ≥ 2. Under this assumption, we call $E_P(R)$ the elementary subgroup of $G(R)$ and denote it simply by $E(R)$.

Let $P = P^+$ be a parabolic subgroup of G, and P^- be an opposite parabolic subgroup. Let $L = P^+ \cap P^-$ be their common Levi subgroup. It was shown in [7] that we can represent $\mathrm{Spec}(R)$ as a finite disjoint union

$$\mathrm{Spec}(R) = \coprod_{i=1}^{m} \mathrm{Spec}(R_i),$$

so that the following conditions hold for $i = 1, \dots, m$:

- for any $s \in \mathrm{Spec}\, R_i$ the root system of $G_{\overline{k(s)}}$ is the same;
- for any $s \in \mathrm{Spec}\, R_i$ the type of the parabolic subgroup $P_{\overline{k(s)}}$ of $G_{\overline{k(s)}}$ is the same;
- if S_i/R_i is a Galois extension of rings such that G_{S_i} is of inner type, then for any $s \in \mathrm{Spec}\, R_i$ the Galois group $\mathrm{Gal}(S_i/R_i)$ acts on the Dynkin diagram D_i of $G_{\overline{k(s)}}$ via the same subgroup of $\mathrm{Aut}\,(D_i)$.

From here until the end of this section, assume that $R = R_i$ for some i (or just extend the base). Denote by Φ the root system of G, by Π a set of simple roots of Φ, by D the corresponding Dynkin diagram. Then the $*$-action on D is determined by a subgroup Γ of $\mathrm{Aut}\, D$. Let J be the subset of Π such that $\Pi \setminus J$ is the type of $P_{\overline{k(s)}}$ (that is, the set of simple roots of the Levi sugroup $L_{\overline{k(s)}}$). Then J is Γ-invariant. The system of relative roots $\Phi_{J,\Gamma}$ is called *the system of relative roots corresponding to* P and denoted also by Φ_P. If R is a local ring and P is a minimal parabolic subgroup of G, then Φ can be identified with the relative root system of G in the sense of [6, Exp. XXVI §7] (see also [4] for the field case), as was shown in [7, 9].

To any relative root $A \in \Phi_P$ one associates a finitely generated projective R-module V_A and a closed embedding

$$X_A : W(V_A) \to G,$$

where $W(V_A)$ is the affine group scheme over R defined by V_A, which is called a *relative root subscheme* of G. These subschemes possess several nice properties similar to that of elementary root subgroups of a split group, see [7, Th. 2]. In particular, they are subject to certain commutator relations which generalize the Chevalley commutator formula.

More precisely, assume that $A, B \in \Phi_P$ satisfy $mA \neq -kB$ for any $m, k \geq 1$. Then there exists a polynomial map

$$N_{ABij} \colon V_A \times V_B \to V_{iA+jB},$$

homogeneous of degree i in the first variable and of degree j in the second variable, such that for any R-algebra R' and for any for any $u \in V_A \otimes_R R'$, $v \in V_B \otimes_R R'$ one has

$$[X_A(u), X_B(v)] = \prod_{i,j>0} X_{iA+jB}(N_{ABij}(u, v)) \tag{1}$$

(see [7, Lemma 9]).

In a strict analogy with the split case, for any R-algebra R' we have

$$E(R') = \langle X_A(V_A \otimes_R R'),\ A \in \Phi_P \rangle$$

(see [7, Lemma 6]).

We will aslo use the following statement which is a slight extension of [7, Lemma 10].

Lemma 2. *Consider $A, B \in \Phi_P$ satisfying $A + B \in \Phi_P$ and $mA \neq -kB$ for any $m, k \geq 1$. Denote by Φ_0 an irreducible component of Φ such that $A, B \in \pi(\Phi_0)$.*

(1) *In each of the following cases*

(a) *structure constants of Φ_0 are invertible in R (for example, if Φ_0 is simply laced);*

(b) *$A \neq B$ and $A - B \notin \Phi_P$;*

(c) *Φ_0 is of type B_l, C_l, or F_4, and $\pi^{-1}(A + B)$ consists of short roots;*

(d) *Φ_0 is of type B_l, C_l, or F_4, and there exist long roots $\alpha \in \pi^{-1}(A)$, $\beta \in \pi^{-1}(B)$ such that $\alpha + \beta$ is a root;*

the map $N_{AB11} : V_A \times V_B \to V_{A+B}$ is surjective.

(2) *If $A - B \in \Phi_P$ and $\Phi_0 \neq G_2$, then*

$$\mathrm{im}\ N_{AB11} + \mathrm{im}\ N_{A-B,2B,1,1} + \sum_{v \in V_B} \mathrm{im}\,(N_{A-B,B,1,2}(-, v)) = V_{A+B},$$

where $\mathrm{im}\ N_{A-B,2B,1,1} = 0$ *if* $2B \notin \Phi_P$.

Proof. (1) By [7, Lemma 4] any $\gamma \in \pi^{-1}(A + B)$ decomposes as $\gamma = \alpha + \beta$, $\alpha \in \pi^{-1}(A)$, $\beta \in \pi^{-1}(B)$. Let S_τ be any member of an affine fpqc-covering $\coprod \operatorname{Spec} S_\tau \to \operatorname{Spec} R$ that splits G. Set

$$\Psi = \{iA + jB \mid i, j > 0,\ (i, j) \neq (1, 1),\ iA + jB \in \Phi_P\}.$$

Then in the notation of [7, Th. 2], over S_τ the commutator $[X_A(e_\alpha), X_B(e_\beta)]$, computed modulo the subgroup $\langle X_C(V_C),\ C \in \Psi \rangle$, is of the form $x_\gamma(\pm c) = X_{A+B}(\pm c e_\gamma)$, where $c = \pm 1, \pm 2, \pm 3$ is the corresponding structure constant. If (a) holds, then c is invertible. If (b), (c) or (d) holds, then c necessarily equals ± 1. Indeed, in the only dubious case (d) one should note that, due to the transitive action of the Weyl group of the Levi subgroup on the roots of the same shape (see [1]), *any* long root $\gamma \in \pi^{-1}(A+B)$ decomposes as a sum of long roots. Hence c is always invertible. This implies that $\mathrm{im}\,(N_{AB11})_\tau = V_{A+B} \otimes S_\tau$. Since $\mathrm{im}\ N_{AB11}$ is a submodule of V_{A+B} defined over the base ring, we have $\mathrm{im}\ N_{AB11} = V_{A+B}$.

(2) See [7, Lemma 10]. ☐

Lemma 3. *Suppose that Φ_0 is an irreducible component of Φ such that $\Phi_0 \cong C_l$, $l > 2$, P is a parabolic subgroup of type $\Pi \setminus J$, where $J = \{\alpha_i, \alpha_l\}$, $2i = l$ (α_i is short, α_l is long), so that $\Phi_{0,P} \cong C_2$. Denote $\pi(\alpha_i) = A_1$ and $\pi(\alpha_l) = A_2$. Then*

$$\mathrm{im}\,(0, N_{A_1,A_1+A_2,1,1}) + \sum_{v \in V_A} \mathrm{im}\ f_v = V_{A_1+A_2} \oplus V_{2A_1+A_2},$$

where $f_v = (N_{A_1,A_2,1,1}(v,-), N_{A_1,A_2,2,1}(v,-))\colon V_{A_2} \to V_{A_1+A_2} \oplus V_{2A_1+A_2}$.

Proof. Let S_τ be any member of an affine fpqc-covering $\coprod \mathrm{Spec}\, S_\tau \to \mathrm{Spec}\, R$ that splits G. Suppose that $\gamma \in \pi^{-1}(2A_1 + A_2)$ is a short root. We can find short roots $\alpha \in \pi^{-1}(A_1)$, $\beta \in \pi^{-1}(A_1 + A_2)$ such that $\gamma = \alpha + \beta$. Therefore,

$$[X_{A_1}(e_\alpha), X_{A_1+A_2}(e_\beta)] = x_\gamma(\pm 1) = X_{2A_1+A_2}(\pm e_\gamma).$$

Hence, $e_\gamma \in \mathrm{im}\,(N_{A_1,A_1+A_2,1,1})_\tau$. Now let $\gamma \in \pi^{-1}(2A_1 + A_2)$ be a long root. Take $\alpha \in \pi^{-1}(A_1)$, $\beta \in \pi^{-1}(A_2)$ such that $\beta \neq \alpha_l$ and $\gamma = 2\alpha + \beta$ (note that α is short, β is long). Therefore

$$[X_{A_1}(e_\alpha), X_{A_2}(e_\beta)] = x_{\alpha+\beta}(\pm 1) x_{2\alpha+\beta}(\pm 1) = X_{A_1+A_2}(\pm e_{\alpha+\beta}) X_{2A_1+A_2}(\pm e_\gamma).$$

Finally, any $\gamma \in \pi^{-1}(A_1 + A_2)$ is a short root, so there exist short roots $\alpha \in \pi^{-1}(A_1)$, $\beta \in \pi^{-1}(A_2)$ such that $\alpha + \beta = \gamma$, hence $[X_{A_1}(e_\alpha), X_{A_2}(e_\beta)] = X_{A_1+A_2}(\pm e_\gamma)$. Combining these results and noting that the modules in question are defined over the base ring, we are done. ☐

4 Proof of Theorem 1

Let R be a commutative ring with 1, G be a reductive group over R, P be a strictly proper parabolic subgroup of G. For any ideal $I \subseteq R$ we write

$$E_P(I) = \langle U_P(I), U_{P^-}(I) \rangle \leq G(R).$$

We denote by $R[Y, Z]$ a ring of polynomials in two variables Z and Y over R.

Lemma 4. *Let G be a reductive group scheme over a commutative ring R, and let P and P' be two strictly proper parabolic subgroups of G such that $P \leq P'$ or $P' \leq P$. Then for any integer $m > 0$ there exists an integer $k > 0$ such that*

$$E_P(Z^k R[Z]) \leq E_{P'}(Z^m R[Z]).$$

Proof. Without loss of generality, we can assume that over R we have two sets of relative root subschemes $X_A(V_A)$, $A \in \Phi_P$, and $X_B(V_B)$, $B \in \Phi_{P'}$, corresponding to P and P' respectively.

Then, if $P \leq P'$, by [7, Lemma 12] there exists an integer $k > 0$ such that for any $A \in \Phi_P$ and any $v \in V_A$ one can find relative roots $B_i \in \Phi_{P'}$, elements $v_i \in V_{B_i}$, and integers $n_i > 0$ ($1 \leq i \leq m$), such that

$$X_A(Y^k v) = \prod_{i=1}^{m} X_{B_i}(Y^{n_i} v_i),$$

and hence $X_A(Y^k v) \in E_{P'}(YR[Y])$. Substituting R by $R[Z]$ and Y by Z^m, we obtain

$$E_P(Z^k R[Z]) \le E_{P'}(Z^m R[Z]).$$

If, conversely, $P' \le P$, we have $U_P \le U_{P'}$. Let $\Psi^\pm \subseteq \Phi^+$ be two closed sets of roots corresponding to $U_{P^\pm}$. Then we have $\pi(\Psi^\pm) \subseteq \Phi^\pm_{P'}$, where $\pi : \Phi \to \Phi_{P'}$ is the canonical projection. By [7, Lemma 6] the map

$$X_\Psi\colon \mathrm{W}\Big(\bigoplus_{A\in\pi(\Psi^\pm)} V_A\Big) \to \mathrm{U}_{P^\pm}, \quad (v_A)_A \mapsto \prod_A X_A(v_A),$$

where the product is taken in any fixed order respecting the level of relative roots in $\Phi_{P'}$, is an isomorphism of schemes over R. Therefore, $U_{P^\pm}(Z^m R[Z]) \le U_{P'^\pm}(Z^m R[Z])$. $\square$

Lemma 5. *In the setting of Theorem 1, assume moreover that R is a local ring. Then for any integer $m > 0$ there exists an integer $k > 0$ such that for any R-algebra R' one has*

$$E_P(Z^k R'[Z]) \subseteq [E_P(Z^m R'[Z]), E_P(Z^m R'[Z])].$$

Proof. Let $\mathrm{der}(G)$ be the algebraic derived subgroup of the reductive group scheme G (see [6, Exp. XXII, 6.2]). Then, clearly, $E_P(R) \subseteq \mathrm{der}(G)(R)$. Since $\mathrm{der}(G)$ is a semisimple group, we can assume that G is semisimple. Moreover, since the canonical projection $G^{sc} \to G$, where G^{sc} is the simply connected semisimple group corresponding to G, is surjective on $U^\pm(R)$, we can assume that G is simply connected. Any simply connected semisimple group is a direct product of simply connected semisimple groups that cannot be decomposed into a product of smaller semisimple groups. These groups G_i, $i = 1, \dots, n$, are Weil restrictions of certain simple reductive groups G'_i over a finite étale extension S of R: $G_i = \mathrm{R}_{S/R}(G'_i)$ [6, Exp. XIV Prop. 5.10]. Note that the group of R-points $G_i(R)$ is canonically isomorphic to the group of S-points $G'_i(S)$. This isomorphism also respects the embedding $P_i(R) \to G_i(R)$, for any parabolic subgroup P_i of G_i. Then, clearly, we can assume from the very beginning that G is a simple reductive group, and the root system Φ of G is irreducible.

Note that by Lemma 4 we can substitute P by any other strictly proper parabolic subgroup P' of G such that $P \le P'$ or $P' \le P$. Further, over R we have a set of relative root subschemes $X_A(V_A)$, $A \in \Phi_P$, corresponding to P.

We are going to show by induction on $|\,\mathrm{lev}\, A|$ that for any $A \in \Phi_P$ there exists an integer $k = k(A) > 0$ such that for any R-algebra R' and any $v \in V_A \otimes_R R'$ one has

$$X_A(Z^k v) \in [E_P(ZR'[Z]), E_P(ZR'[Z])]. \tag{2}$$

The claim of the lemma then follows by substituting Z by Z^m and R' by $R[Z]$, and taking the final k to be the maximum of all $k(A)$, $A \in \Phi_P$.

Recall that by Lemma 1 there exist non-collinear relative roots $B, C \in \Phi_P$ such that $A = B + C$ and and all roots $iB + jC \in \Phi_{J,\Gamma}$, $i, j > 0$, $(i, j) \neq (1, 1)$, have the same sign as A and satisfy $|\,\mathrm{lev}(iB + jC)| > |\,\mathrm{lev}(A)|$. Assume that the map $N_{BC11} : V_B \times V_C \to V_A$ is surjective. Then by the generalized Chevalley commutator formula (1) we have that for any R-algebra R', and any $v \in V_A \otimes_R R'[Z]$,

$$X_A(Z^k v) = [X_B(Zu_{10}), X_C(Z^{k-1}u_{01})] \cdot \prod_{\substack{i,j>0;\\(i,j)\neq(1,1)}} X_{iB+jC}(Z^{i+j(k-1)}u_{ij})$$

for some $u_{ij} \in V_{iB+jC} \otimes_R R'[Z]$, $i, j > 0$, $(i, j) \neq (1, 1)$. Then, by the inductive hypothesis, (2) holds for k large enough.

Now for any relative root $A \in \Phi_P$ (for a suitable choice of the parabolic subgroup P) we either show that for *any* decomposition $A = B + C$, where B and C are non-collinear, the map N_{BC11} is surjective; or provide an explicit decomposition of $X_A(Z^k v)$, $v \in V_A \otimes_R R'$, into a product of commutators in $E_P(ZR'[Z])$, so that (2) is satisfied for k large enough.

Assume first that all structure constants of the root system Φ of G are invertible in R; this includes the case where Φ is simply laced. Then by Lemma 2 the map N_{BC11} is surjective for any decomposition $A = B + C$, where B and C are non-collinear.

Consider the case where $\Phi = \Phi_P = C_2$, so G is split. Let $\Pi = \{A_1, A_2\}$, $\Phi^+ = \{A_1, A_2, A_1 + A_2, 2A_1 + A_2\}$. Let M be the maximal ideal of R. By the hypothesis of Theorem 1, $R/M \ncong F_2$, hence we can take $\varepsilon \in R \setminus M$ such that $\varepsilon^2 - \varepsilon \in R \setminus M = R^*$. If the root $A \in \Phi_P$ is long, we can assume that $A = 2A_1 + A_2$. Let

$$g_1(s,t) = [X_{A_1}(s), X_{A_2}(t)] = X_{A_1+A_2}(st)X_{2A_1+A_2}(s^2t)$$

and

$$g_2(s,t,u) = [X_{A_2}(u), [X_{A_1+A_2}(s), X_{-A_2}(t)]] = X_{A_1+A_2}(-stu)X_{2A_1+A_2}(-s^2t^2u).$$

Therefore,

$$g_1(Z^2, -Z^{k-4}\varepsilon(\varepsilon^2-\varepsilon)^{-1}v) \cdot g_2(Z, Z\varepsilon, -Z^{k-4}(\varepsilon^2-\varepsilon)^{-1}v) = X_{2A_1+A_2}(Z^k v).$$

If the root $A \in \Phi_P$ is short, we can assume that $A = A_1 + A_2$, hence

$$g_1(Z, Z^{k-1}v) \cdot X_{2A_1+A_2}(-Z^{k+1}v) = X_{A_1+A_2}(Z^k v).$$

This means that (2) holds for these roots for any $k \geq 5$.

Consider the case where $\Phi = \Phi_P = G_2$, so G is split. Let $\Pi = \{A_1, A_2\}$, $\Phi^+ = \{A_1, A_2, A_1 + A_2, 2A_1 + A_2, 3A_1 + A_2, 3A_1 + 2A_2\}$. By the hypothesis of Theorem 1, $R/M \ncong F_2$, hence we can take $\varepsilon \in R \setminus M$ such that $\varepsilon^2 - \varepsilon \in R \setminus M = R^*$. If the root A is long, we can assume that $A = 3A_1 + 2A_2$. Then

$$[X_{A_2}(Zv), X_{3A_1+A_2}(Z^{k-1})] = X_{3A_1+2A_2}(Z^k v).$$

Therefore, (2) holds for long roots for any $k \geq 2$. If the root A is short, we can assume that $A = 2A_1 + A_2$. Then

$$[X_{A_1}(s), X_{A_2}(t)] = X_{A_1+A_2}(st) \cdot X_{2A_1+A_2}(s^2t) \cdot X_{3A_1+A_2}(s^3t) \cdot X_{3A_1+2A_2}(s^3t^2).$$

Hence,

$$\begin{aligned}
&[X_{A_1}(Z\varepsilon), X_{A_2}(-(\varepsilon^2-\varepsilon)^{-1}Z^{k-2}v)]^{-1} \cdot [X_{A_1}(Z), X_{A_2}(-\varepsilon(\varepsilon^2-\varepsilon)^{-1}Z^{k-2}v)] = \\
&\qquad X_{2A_1+A_2}(Z^k v)X_{3A_1+A_2}((\varepsilon+1)Z^{k+1}v)X_{3A_1+2A_2}(\varepsilon(\varepsilon^2-\varepsilon)^{-1}Z^{2k+1}v), \quad (3)
\end{aligned}$$

and the roots $3A_1 + A_2$ and $3A_1 + 2A_2$ are long. This means that (2) holds for short roots for any $k \geq 3$.

We are left with the case where Φ is of type B_l, C_l, or F_4. Recall that by the hypothesis of Theorem 1 G contains a split torus of rank ≥ 2. Hence in the F_4 case the classification of Tits indices over local rings [8] says that G is a split group. Hence we can assume that P is a Borel subgroup of G, and $\Phi_P = \Phi$ is a root system of type F_4. Then if the root $A \in \Phi_P$ is short, by Lemma 2 the map N_{BC11} is surjective for any non-collinear $B, C \in \Phi_P$ such that $A = B + C$. If this root is long, then it belongs to the long root subsystem of F_4, which has type D_4. Then (for example, by Lemma 1) we can find two long roots $B, C \in \Phi_P$, such that $B + C = A$, and necessarily, $iB + jC$ is not a root for any $i, j > 0$ distict from $i = j = 1$. Then

$$X_A(Z^k v) = [X_B(Zu_{10}), X_C(Z^{k-1}u_{01})]$$

for some $u_{10} \in V_B \otimes_R R'[Z]$ and $u_{01} \in V_C \otimes_R R'[Z]$.

Consider the case where Φ is of type B_l, $l \geq 3$. By the classification of Tits indices over local rings [8], we can assume that P is a parabolic subgroup of type $\Pi \setminus J$, where Π is a set of simple roots of Φ and $J = \{\alpha_1, \alpha_2\}$. Then Φ_P can be identified with a root system of type B_2. One readily sees, using the fact that $l \geq 3$, that for any relative root $A \in \Phi_P$ and any pair $B, C \in \Phi_P$ satisfying $A = B + C$, we can find a pair of long roots $\beta \in \pi^{-1}(B)$, $\gamma \in \pi^{-1}(C)$ such that $\beta + \gamma$ is a root (one can assume that A is one of two simple roots of Φ_P, due to the lifting of the relative Weyl group [6, Exp. XXVI Th. 7.13 (ii)]). Then by Lemma 2 the map N_{BC11} is surjective.

It remains to consider the case where Φ is of type C_l, $l \geq 3$. First assume that P is a parabolic subgroup of type $\Pi \setminus J$, where $J = \{\alpha_i, \alpha_j\}$ for two short simple roots α_i, α_j of Φ. Then Φ_P can be identified with a root system of type BC_2. One readily sees that for all extra-short and short relative roots $A \in \Phi_P$ the set $\pi^{-1}(A)$ consists of short roots, and hence by Lemma 2 the map N_{BC11} is surjective for any decomposition $A = B + C$. Let A be a long root. Let A_1 and A_2 be a short and an extra-short simple roots of Φ_P. We can assume without loss of generality that $A = 2A_1 + 2A_2$. Take $k \geq 4$. Then by Lemma 2 (2) and by the generalized Chevalley commutator formula, for any R-algebra R', and any $v \in V_A \otimes_R R'[Z]$, we have

$$X_A(Z^k v) = [X_{A_1}(Zu_1), X_{2A_2}(Z^{k-2}u_2)] \cdot X_{A_1+2A_2}(Z^{k-1}u_3)$$

for some $u_1 \in V_{A_1} \otimes_R R'[Z]$, $u_2 \in V_{2A_2} \otimes_R R'[Z]$, and $u_3 \in V_{A_1+2A_2} \otimes_R R'[Z]$. Further, by Lemma 2 (1) and by the generalized Chevalley commutator formula, since $\pi^{-1}(A_1 + 2A_2)$ consists of short roots, we have

$$X_{A_1+2A_2}(Z^{k-1}u_3) = [X_{A_1+A_2}(Zu_4), X_{A_2}(Z^{k-3}u_5)]$$

for some $u_4 \in V_{A_1+A_2} \otimes_R R'[Z]$ and $u_5 \in V_{A_2} \otimes_R R'[Z]$. Hence (2) holds for A for any $k \geq 4$.

By the classification of Tits indices over local rings [8], the only remaining case is when P is a parabolic subgroup of type $\Pi \setminus J$ for $J = \{\alpha_i, \alpha_l\}$, where $l = 2i$. Now α_i is short, α_l is long, and Φ_P can be identified with a root system of type $B_2 = C_2$. As in Lemma 3, we put $A_1 = \pi(\alpha_i)$, $A_2 = \pi(\alpha_l)$. Then if the root $A \in \Phi_P$ is short, by the lifting of the relative Weyl group [6, Exp. XXVI Th. 7.13 (ii)]), we can assume that $A = A_1 + A_2$. By Lemma 3 for any R-algebra R', and any $v \in V_A \otimes_R R'[Z]$, we have

$$X_A(Z^k v) = \prod_i [X_{A_1}(Zv_i), X_{A_2}(Z^{k-1}u_1)]$$

for some $u_1 \in V_{A_2} \otimes_R R'[Z]$, $v_i \in V_{A_1} \otimes_R R'[Z]$. If the root $A \in \Phi_P$ is long, we can assume that $C = 2A_1 + A_2$. By Lemma 3 for any R-algebra R', and any $v \in V_{2A_1+A_2} \otimes_R R'[Z]$, we have

$$X_A(Z^k v) = [X_{A_1}(Zu_1), X_{A_1+A_2}(Z^{k-1}u_2)] \cdot \prod_i [X_{A_1}(Zv_i), X_{A_2}(Z^{k-2}u_3)]$$

for some $u_1 \in V_{A_1} \otimes_R R'[Z]$, $u_2 \in V_{A_1+A_2} \otimes_R R'[Z]$, $u_3 \in V_{A_2} \otimes_R R'[Z]$, $v_i \in V_{A_1} \otimes_R R'[Z]$. Hence (2) holds for A for any $k \geq 3$. □

Proof of Theorem 1. Recall that we can represent $\mathrm{Spec}(R)$ as a finite disjoint union $\mathrm{Spec}(R) = \coprod_{i=1}^{n} \mathrm{Spec}(R_i)$, so that $E(R) = \prod_{i=1}^{n} E(R_i)$ and for any $1 \leq i \leq n$ we have

$$E(R_i) = \left\langle X_A(V_A),\ A \in \Phi_{P_{R_i}} \right\rangle,$$

for a set of root subschemes X_A, $A \in \Phi_{P_{R_i}}$, over R_i. Hence we can assume that

$$E(R) = \langle X_A(V_A),\ A \in \Phi_P \rangle$$

from the very beginning.

We show that $[E(R), E(R)]$ contains any $X_A(v)$, $A \in \Phi_P$, $v \in V_A$, by induction on $|\operatorname{lev} A|$. Take

$$I = \{s \in R \mid X_A(tsv) \in [E(R), E(R)] \ \forall\ t \in R\}.$$

By [7, Th. 2] for any $u, u' \in V_A$ we have

$$X_A(u+u') = X_A(u)X_A(u')\prod_{i>0} X_{iA}(u_i)$$

for some $u_i \in V_{iA}$. Hence by the inductive hypothesis I is an ideal of R. If $I \neq R$, let M be a maximal ideal of R containing I. Let F_M denote both the localization homomorphism $R \to R_M$ and the induced homomorphism $G(R[Y,Z]) \to G(R_M[Y,Z])$. By Lemma 5 there is an $m > 0$ such that we can represent the element $F_M(X_A(Z^m Yv))$ of $G(R_M[Y,Z])$ as a product

$$F_M(X_A(Z^m Yv)) = \prod_{i=1}^{n}[h_i(Y,Z), g_i(Y,Z)],$$

for some $h_i(Y,Z), g_i(Y,Z) \in E(ZR_M[Y,Z])$, $1 \le i \le n$. By [7, Lemma 15] there exist

$$h_i'(Y,Z),\ g_i'(Y,Z) \in E_P(R[Y,Z], ZR[Y,Z]) \le E_P(R[Y,Z]) = E(R[Y,Z])$$

and $s \in R \setminus M$ such that $F_M(h_i'(Y,Z)) = h_i(Y,sZ)$ and $F_M(g_i'(Y,Z)) = g_i(Y,sZ)$ for all i. Hence

$$F_M\big(X_A((sZ)^m Yv)\big) = F_M\Big(\prod_{i=1}^{n}[h_i'(Y,Z), g_i'(Y,Z)]\Big).$$

Then by [7, Lemma 14] there exists $t \in R \setminus M$ such that

$$X_A((tsZ)^m Yv) = \prod_{i=1}^{n}[h_i'(Y,tZ), g_i'(Y,tZ)].$$

Substituting Z by 1 and Y by an arbitrary element of R, we see that $(ts)^m \in I$. But $(ts)^m \in R \setminus M$, a contradiction. $\square$

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