EXCEPTIONAL GROUPS ARE GROUPS OF LINEAR TRANSFORMATIONS PRESERVING SYSTEMS OF QUADRICS

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ABSTRACT. Let Φ be a root system of type E_l, and let $G = G(\Phi, R)$ be the Chevalley group of type Φ over a commutative ring R. Consider the adjoint representation $G(\Phi, R) \to GL(N, R)$. We prove that $G(\Phi, R)$ can be characterised as the identity component of the stabiliser of a system of certain quadratic equations. These equations were previously described by the second-named author as equations on the orbit of the highest weight vector.

Classical algebraic groups are often viewed as the groups of automorphisms of some geometric objects; for example, an orthogonal group is the isometry group of a [non-degenerate] bilinear form. Similar descriptions for exceptional groups are less known and generally are harder to obtain, see [GG15] for a detailed review of the situation. In the present paper we identify the [slightly extended] exceptional groups in their adjoint representations with identity components of groups of linear transformations preserving certain ideals generated by explicit quadratic forms. These quadratic forms are exactly the equations on the orbit of the highest weight vector, which were described by the second-named author in [Luz14]. Note that our description works for any commutative ring.

We do not try to recall here all the necessary notions and refer the reader to loc. cit. Let us just fix the notation here. From now on R is a commutative ring, $\Phi = \mathsf{E}_l$, $l = 6, 7, 8$. Let $\{\alpha_1, \ldots, \alpha_l\} = \Pi \subset \Phi$ be a fundamental system in Φ (our numbering of fundamental roots follows Bourbaki [Bou68]). We work with the adjoint representation of $G(\Phi, R)$, which gives us the irreducible action of $G(\Phi, R)$ on a free R-module V of rank $N = 78, 133, 248$ for $\Phi = \mathsf{E}_6, \mathsf{E}_7, \mathsf{E}_8$ respectively. By Λ we denote the set of weights of our representation *with multiplicities*. More precisely, $\Lambda = \Lambda^* \sqcup \Delta$, where $\Lambda^* = \Phi$ is the set of non-zero weights, and $\Delta = \{0_1, \ldots, 0_l\}$ is the set of zero weights. We fix an admissible base e^λ, $\lambda \in \Lambda$ in V. Hence we have the vectors e^α for $\alpha \in \Phi$ and $\widehat{e}^i = e^{0_i}$ for $i = 1, \ldots, l$.

Consider the algebra $\mathrm{Sym}(V^*)$ of polynomial functions on V. Our choice of basis in V identifies $\mathrm{Sym}(V^*)$ with $R[\{x_\alpha\}_{\alpha\in\Phi}, \{\widehat{x}_i\}_{i=1}^l]$. Now we describe a system of quadrics in this algebra. All quadrics in our system in fact have integer coefficients, so they lie in $\mathbb{Z}[\{x_\alpha\}_{\alpha\in\Phi}, \{\widehat{x}_i\}_{i=1}^l]$.

For the rest of the paper, put $k = 4, 5, 7$ for $\Phi = \mathsf{E}_6, \mathsf{E}_7, \mathsf{E}_8$, respectively. Recall that a set of roots $\{\beta_i\}$, $i = 1, \ldots, k, -k, \ldots, -1$ such that $\angle(\beta_i, \beta_{-i}) = \pi/2$ for every $i = 1, \ldots, k$, and $\angle(\beta_i, \beta_j) = \pi/3$ for $i \neq \pm j$, is called a **maximal square**.

The Weyl group $W(\Phi)$ acts transitively on the pairs of orthogonal roots in Φ. Therefore for any pair of orthogonal roots $\alpha, \beta \in \Phi$ there is a unique maximal square containing it. In fact, this square is $\Omega_{\alpha,\beta} = \{\gamma \in \Phi \mid \alpha + \beta - \gamma \in \Phi\}$. Put $\overline{\Omega}_{\alpha,\beta} = \Omega_{\alpha,\beta} \setminus \{\alpha, \beta\}$.

Suppose that $v \in V$, $\alpha, \beta \in \Phi$, and $\Omega = \{\beta_1, \ldots, \beta_{-1}\}$ is a maximal square such that $\beta_1 = \alpha$, $\beta_{-1} = \beta$, and $\beta_i \perp \beta_{-i}$ for every i. We need the following notation

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for the polynomials from [Luz14]:

$$f^{\pi/2}_{\alpha,\beta} = x_\alpha x_\beta - \sum_{i\geq 2} N_{\alpha,-\beta_i} N_{\beta,-\beta_{-i}} x_{\beta_i} x_{\beta_{-i}};$$

$$f^{2\pi/3}_{\alpha,\beta} = \sum_{i\neq\pm 1} N_{\alpha,-\beta_i} x_{\alpha-\beta_i} x_{\beta_i} - x_\alpha \sum_{s=1}^{l} \langle\beta,\alpha_s\rangle \widehat{x}_s;$$

$$f^{\pi}_{\alpha,\beta} = \sum_{i\neq\pm 1} (x_{\alpha-\beta_i} x_{\beta_i-\alpha} - x_{-\beta_i} x_{\beta_i}) - \sum_{s=1}^{l} \langle\alpha,\alpha_s\rangle \widehat{x}_s \cdot \sum_{s=1}^{l} \langle\beta,\alpha_s\rangle \widehat{x}_s.$$

Let us denote by I the ideal in $\mathbb{Z}[\{x_\alpha\}_{\alpha\in\Phi}, \{\widehat{x}_s\}_{s=1}^{l}]$ generated by these polynomials, and let $G_I(R) = \{g \in GL(N, R) \mid f(gx) \in I \text{ for all } f \in I\}$ be the corresponding group of linear transformation preserving this ideal (here $N = |\Phi| + l$). For an arbitrary ideal I, G_I is not an affine group scheme over $\mathbb{Z}$.

Lemma 1. *Let $f_1, \dots, f_s \in \mathbb{Z}[x_1, \dots, x_t]$ be polynomials of degree at most r, and let A be the ideal they generate. Then for the functor $R \mapsto \{g \in GL(t, R) \mid f(gx) \in A \text{ for all } f \in A\}$ to be an affine group scheme, it suffices that the rank of the intersection $A \cap \mathbb{Z}[x_1, \dots, x_t]_r$ does not change under reduction modulo any prime $p \in \mathbb{Z}$.*

Proof. This is [Wat87, Corollary 1.4.6]. □

Lemma 2. *Let $\alpha, \beta, \gamma \in \Phi$ be roots such that α is orthogonal to both β and γ. Then*

(1) $f^{2\pi/3}_{\alpha,-\beta} = -f^{2\pi/3}_{\alpha,\beta}$;
(2) if $\beta + \gamma \in \Phi$, then $f^{2\pi/3}_{\alpha,\beta+\gamma} = f^{2\pi/3}_{\alpha,\beta} + f^{2\pi/3}_{\alpha,\gamma}$;
(3) if $\beta - \gamma \in \Phi$, then $f^{2\pi/3}_{\alpha,\beta-\gamma} = f^{2\pi/3}_{\alpha,\beta} - f^{2\pi/3}_{\alpha,\gamma}$.

Proof. Note that the left-hand sides make sense: if $\alpha \perp \beta$ and $\alpha \perp \gamma$, then $\alpha \perp -\beta$ and $\alpha \perp (\beta \pm \gamma)$. We prove the second assertion (the rest are similar). It is obvious that the term $x_\alpha \sum_{s=1}^{l} \langle\beta, \alpha_s\rangle \widehat{x}_s$ is linear in β. Other terms in $f^{2\pi/3}_{\alpha,\beta}$ can be written as

$$\sum_{\delta\in\overline{\Omega}_{\alpha,\beta}} N_{\alpha,-\delta} x_{\alpha-\delta} x_\delta.$$

Consider a monomial $N_{\alpha,-\delta} x_{\alpha-\delta} x_\delta$, corresponding to some $\delta \in \overline{\Omega}_{\alpha,\beta}$. We claim that this monomial appears either in $f^{2\pi/3}_{\alpha,\beta+\gamma}$ or in $f^{2\pi/3}_{\alpha,\gamma}$ (with the opposite sign). Indeed, $\delta \in \overline{\Omega}_{\alpha,\beta}$ means that $\delta + \overline{\delta} = \alpha + \beta$ for some $\overline{\delta} \in \Phi$. Then $(\gamma, \delta) + (\gamma, \overline{\delta}) = (\gamma, \alpha) + (\gamma, \beta) = -1/2$. Note that we cannot have neither $\gamma = -\delta$ nor $\gamma = -\overline{\delta}$, since $\alpha \perp \gamma$ and $\angle(\alpha, \delta) = \pi/3$. Thus we have two cases:

- $(\gamma, \delta) = 0$ and $(\gamma, \overline{\delta}) = -1/2$. This means that $(\beta + \gamma, \delta) = 1/2$. Moreover, $(\beta + \gamma, \alpha - \delta) = -1/2$, therefore $\alpha + \beta + \gamma - \delta \in \Phi$ and $\alpha + (\beta + \gamma) = \delta + (\alpha + \beta + \gamma - \delta)$. Thus $\delta \in \bar{\Omega}_{\alpha,\beta+\gamma}$, so there is a term $N_{\alpha,-\delta} x_{\alpha-\delta} x_\delta$ in $f^{2\pi/3}_{\alpha,\beta+\gamma}$.
- $(\gamma, \delta) = -1/2$ and $(\gamma, \overline{\delta}) = 0$. This means that $\delta + \gamma \in \Phi$, $(\gamma, \alpha - \delta) = 1/2$ and $\alpha + \gamma = (\alpha - \delta) + (\delta + \gamma)$. Thus $\alpha - \delta \in \bar{\Omega}_{\alpha,\gamma}$, so there is a term $N_{\alpha,\delta-\alpha} x_\delta x_{\alpha-\delta}$ in $f^{2\pi/3}_{\alpha,\gamma}$. It remains to show that $N_{\alpha,-\delta} = -N_{\alpha,\delta-\alpha}$ which follows immediately from identity (C4) in [1].

We proved that every monomial in $f^{2\pi/3}_{\alpha,\beta}$ sums up to zero with some monomial in one of the other two polynomials. Exchanging the roles of β and γ, we can finish the proof. □

Proposition 3. *G_I is a group scheme over $\mathbb{Z}$.*

Proof. We want to apply Lemma 1. Note that the polynomials defining I are not linearly independent, so our first step is to choose a linearly independent subset of them. Note that any linear combination of $f^{\pi/2}$'s contains only monomials $x_\alpha x_\beta$ for $\alpha \perp \beta$; any linear combination of $f^{2\pi/3}$'s contains only monomials $x_\alpha x_\beta$ for $\angle(\alpha,\beta) = 2\pi/3$ and monomials $x_\gamma \widehat{x}_i$; any linear combination of f^{π}'s contains only monomials $x_\alpha x_\beta$ for $\angle(\alpha,\beta) = \pi$ and monomials $\widehat{x}_i\widehat{x}_j$. This means that it suffices to choose some linearly independent subsets for each of three types of polynomials. Now we list the polynomials we choose, and after that we shall prove that they are linearly independent after reduction modulo any prime.

- A monomial $x_\alpha x_\beta$ for $\alpha \perp \beta$ is contained in several $\pi/2$-polynomials, but all of them come from a single maximal square $\Omega = \Omega_{\alpha,\beta}$ (see [Luz14, Section 2]). Therefore we can take one $f^{\pi/2}_{\alpha,\beta}$ for every maximal square Ω and get a linearly independet set such that any other $\pi/2$-equation is a linear combination of the chosen ones.
- Note that any fixed polynomial $f^{2\pi/3}_{\alpha,\beta}$ contains only monomials $x_\gamma x_\delta$ such that $\gamma + \delta = \alpha$, and monomials $x_\alpha \widehat{x}_i$. This means that looking at a monomial that came from one of this equation, we can uniquely determine α. Therefore it remains to choose a linearly independent set of polynomials $f^{2\pi/3}_{\alpha,\beta}$ for a fixed α. Let I_α generate an ideal generated by these polynomials. Consider all roots orthogonal to α; they form a root subsystem Ψ of type A_5, D_6, E_7 when $\Phi = \mathsf{E}_6, \mathsf{E}_7, \mathsf{E}_8$, respectively. For a given α we choose some fundamental root subsystem Π_Ψ in Ψ and take the polynomials $f^{2\pi/3}_{\alpha,\beta_i}$ for $\beta_i \in \Pi_\Psi$. Lemma 2 shows that all the other $f^{2\pi/3}_{\alpha,\beta}$'s are integer linear combinations of those. Note that the rank of Ψ always equals $l-1$, so in total we got $(l-1)|\Phi|$ polynomials of type $2\pi/3$.
- We claim that the π-polynomials $f^{\pi}_{\alpha,\beta}$ generate a linear space of dimension $(l+2)(l-1)/2$. Arguing as in the proof of Lemma 2, we get that $f^{\pi}_{\alpha,\beta} = f^{\pi}_{\beta,\alpha}$. More over, if $\beta_1 \pm \beta_2 \in \Phi$, then $f^{\pi}_{\alpha,\beta_1\pm\beta_2} = f^{\pi}_{\alpha,\beta_1} \pm f^{\pi}_{\alpha,\beta_2}$. Note that $(l+2)(l-1)/2$ is exactly the dimension of the subspace generated by 'zero parts' of the π-polynomials. Choose a set of $(l+2)(l-1)/2$ linearly independent polynomials out of $f^{\pi}_{\alpha,\beta}$.

Now we wish to prove that the chosen polynomials are linearly independent, and they remain linearly independent after reduction modulo any prime. In order to do that we show that for any one of them we can substitute some explicit values of the variables $\{x_\alpha, \widehat{x}_s\}$ such that a given polynomial takes the value 1 while all the remaining polynomials vanish.

- For a polynomial $f^{\pi/2}_{\alpha,\beta}$ we can set $x_\alpha = x_\beta = 1$ and set all other variables to 0. The only non-zero monomial now is $x_\alpha x_\beta$, which is contained only in our polynomial $f^{\pi/2}_{\alpha,\beta}$.
- Fix a root α and choose a base $\beta_1, \ldots, \beta_{l-1}$ in $\Phi^{\perp}_{\alpha} = \{\gamma \in \Phi \mid \gamma \perp \alpha\}$. Note that $\{f_{\beta_i}\}$ is a generating set for $2\pi/3$-polynomials on $\Phi^{\perp}$. Consider all monomials of polynomials in I_α involving zero weights. By our choice of β_i's, these monomials are linear combinations of $x_\alpha \widehat{x}_{\beta_i}$. Taking $x_\alpha = 1$, $x_{\beta_i} = 1$, and setting all the other variables to zero (including the zero ones), we see that exactly one of the chosen polynomials is non-zero, and it equals 1.
- Arguing as in the previous case, we can take $\widehat{x_\alpha} = \widehat{x_\beta} = 1$ and set all the remaining variables to 0 for every pair of the chosen generators for the 'zero parts'.

□

Theorem 4. $G(\Phi, R) \le G_I(R)$.

Proof. Immediately follows from [Luz14, Proposition 1]. □

Recall the definition of the Lie algebra of an algebraic group G: $\mathrm{Lie}(G(K))$ is the kernel of the homomorphism $G(K[\delta]) \to G(K)$ induced by the ring homomorphism $K[\delta] \to K$, $\delta \mapsto 0$, where $K[\delta] = K[x]/(x^2)$ is the ring of dual numbers over K.

Lemma 5. *Let* K *be a field. Suppose the polynomials* $f_1, \dots, f_s \in K[x_1, \dots, x_t]$ *generate an ideal* I. *Then a matrix* $e + z\delta$ *for* $z \in M(t, K)$ *lies in* $\mathrm{Lie}(\mathrm{Fix}_K(f_1, \dots, f_s))$ *if and only if we have*

$$\sum_{1\le i,j\le t} z_{ij} x_i \frac{\partial f_h}{\partial x_j} \in I.$$

for every $h = 1, \dots, s$.

Proof. Follows from the definition of Lie algebra (see also [Wat87, Section 1.5]). □

Proposition 6. *Let* K *be any field. The dimension of* $\mathrm{Lie}(G_I(K))$ *does not exceed* $|\Phi| + l + 1$.

Proof. In our case the partial derivatives are

$$\frac{\partial f^{\pi/2}_{\alpha,\beta}}{\partial x_\gamma} = \pm x_{\alpha+\beta-\gamma}, \text{ if } \alpha+\beta-\gamma \in \Phi;$$

$$\frac{\partial f^{2\pi/3}_{\alpha,\beta}}{\partial x_\alpha} = -\sum_{s=1}^{l} \langle \beta, \alpha_s \rangle \widehat{x}_s;$$

$$\frac{\partial f^{2\pi/3}_{\alpha,\beta}}{\partial x_\gamma} = \pm x_{\alpha-\gamma}, \text{ if } \gamma \in \overline{\Omega}_{\alpha,\beta} \text{ or } \alpha-\gamma \in \overline{\Omega}_{\alpha,\beta};$$

$$\frac{\partial f^{2\pi/3}_{\alpha,\beta}}{\partial \widehat{x}_s} = -\langle \beta, \alpha_s \rangle x_\alpha;$$

$$\frac{\partial f^{\pi}_{\alpha,\beta}}{\partial x_\gamma} = \pm x_{-\gamma}, \text{ if } \pm\gamma \in \overline{\Omega}_{\alpha,\beta} \text{ or } \pm(\alpha-\gamma) \in \overline{\Omega}_{\alpha,\beta};$$

$$\frac{\partial f^{\pi}_{\alpha,\beta}}{\partial \widehat{x}_s} = -\langle \alpha, \alpha_s \rangle \sum_{t=1}^{l} \langle \beta, \alpha_t \rangle \widehat{x}_t - \langle \beta, \alpha_s \rangle \sum_{t=1}^{l} \langle \alpha, \alpha_t \rangle \widehat{x}_t.$$

All the other derivatives are zero.

- Suppose that $\mu = -\lambda$. We claim that $z_{\lambda,-\lambda} = 0$. Take any maximal square Ω containing $-\lambda$ and choose $\alpha \in \Omega$ such that $\alpha \ne \pm\lambda$. There is a unique $\beta \in \Omega$ such that $\alpha \perp \beta$. Then the equation corresponding to $f = f^{\pi}_{\alpha,\beta}$ contains a term $z_{\lambda,-\lambda} x_\lambda \frac{\partial f}{\partial x_{-\lambda}} = \pm z_{\lambda,-\lambda} x_\lambda^2$, and it is the only term containing x_λ^2 in this equation. Moreover, the defining polynomials of I do not contain this kind of term. Therefore $z_{\lambda,-\lambda} = 0$.
- Suppose that $\angle(\lambda, \mu) = 2\pi/3$. We claim that $z_{\lambda,\mu} = 0$. Take any maximal square Ω containing λ and $\alpha = \lambda + \mu$. Let β be the unique root in Ω such that $\alpha \perp \beta$. Then the equation corresponding to $f = f^{2\pi/3}_{\alpha,\beta}$ contains a term $z_{\lambda,\mu} x_\lambda \frac{\partial f}{\partial x_\mu} = \pm z_{\lambda,\mu} x_\lambda^2$. Reasoning as in the first case, we obtain $z_{\lambda,\mu} = 0$.
- Suppose that $\angle(\lambda, \mu) = \pi/2$. We claim that $z_{\lambda,\mu} = 0$. Consider the equation corresponding to $f = f^{\pi/2}_{\lambda,\mu}$. It contains a term $z_{\lambda,\mu} x_\lambda \frac{\partial f}{\partial x_\mu} = \pm z_{\lambda,\mu} x_\lambda^2$. Reasoning as in the first case, we obtain $z_{\lambda,\mu} = 0$.
- Suppose that $\angle(\lambda, \mu) = \pi/3$ and $\rho, \sigma \in \Phi$ are roots such that $\rho - \sigma = \lambda - \mu$. We claim that $z_{\lambda\mu} = \pm z_{\rho\sigma}$. Note that $\langle \lambda, \rho \rangle - \langle \lambda, \sigma \rangle = \langle \lambda, \rho - \sigma \rangle = \langle \lambda, \lambda - \mu \rangle = 1/2$. Our claim is trivial when $\lambda = \rho$. Suppose that $\angle(\lambda, \rho) = \pi/3$, so that

$\lambda \perp \sigma$. Than $\lambda, \mu, \rho, \sigma \in \Omega_{\lambda,\sigma}$. Consider the equation corresponding to $f = f^{\pi/2}_{\lambda,\sigma}$. It contains a term $z_{\lambda,\mu} x_\lambda \frac{\partial f}{\partial x_\mu} = \pm z_{\lambda,\mu} x_\lambda x_\rho$. There is one other term in this equation containing $x_\lambda x_\rho$, namely, $z_{\rho,\sigma} x_\rho \frac{\partial f}{\partial x_\sigma} = \pm z_{\rho,\sigma} x_\rho x_\lambda$. Since $\angle(\lambda, \rho) = \pi/3$, the defining polynomials of I do not contain $x_\rho x_\lambda$, therefore these two terms should cancel, which proves our claim.

Suppose now that $\lambda \perp \rho$. Then $\mu \perp \sigma$ and $\angle(\lambda, \sigma) = \angle(\mu, \rho) = 2\pi/3$. We claim that there exist $\nu, \kappa \in \Phi$ such that $\nu - \kappa = \lambda - \mu = \rho - \sigma$ and $\angle(\lambda, \nu) = \angle(\nu, \rho) = \pi/3$. Indeed, note that $(-\lambda, \mu, \rho)$ is a basis of a root subsystem $\Psi \subseteq \Phi$ of type A_3. By [Luz14, Lemma 4], there exists a root $\epsilon \in \Phi$ such that $\epsilon \perp \lambda$, $\epsilon \perp \rho$ and $\angle(\mu, \epsilon) = 2\pi/3$. But then we can take $\kappa = -\epsilon$ and $\nu = \kappa + \lambda - \mu$. Earlier we proved that in this setting we have $z_{\lambda,\mu} = z_{\nu,\kappa} = z_{\rho,\sigma}$, as desired.

Finally, the only remaining case is $\lambda = -\sigma$, $\mu = -\rho$. But in this case we can find two roots ν, κ not in $\{\pm\lambda, \pm\mu\}$ such that $\nu - \kappa = \lambda - \mu = \rho - \sigma$. We already proved that we have $z_{\lambda,\mu} = z_{\nu,\kappa} = z_{\rho,\sigma}$, as desired.

- Suppose that $\angle(\lambda, \mu) = \pi/3$ and $\rho, \sigma \in \Phi$ are roots such that $\rho - \sigma = \lambda - \mu$. We claim that $z_{\lambda\lambda} = \pm z_{\mu\mu} \pm z_{\rho\rho} \pm z_{\sigma\sigma}$. Our claim is trivial for $\lambda = \rho$. Suppose that $\angle(\lambda, \rho) = \pi/3$ and consider the equation corresponding to $f = f^{\pi/2}_{\lambda,\sigma}$. It contains terms $z_{\lambda,\lambda} x_\lambda \frac{\partial f}{\partial x_\lambda} = \pm z_{\lambda,\lambda} x_\lambda x_\sigma$, $z_{\mu,\mu} x_\mu \frac{\partial f}{\partial x_\mu} = \pm z_{\mu,\mu} x_\mu x_\rho$, $z_{\rho,\rho} x_\sigma \frac{\partial f}{\partial x_\sigma} = \pm z_{\rho,\rho} x_\rho x_\mu$, $z_{\sigma,\sigma} x_\sigma \frac{\partial f}{\partial x_\sigma} = \pm z_{\sigma,\sigma} x_\sigma x_\lambda$. There is only one polynomial among those defining I that contains a term $x_\lambda x_\sigma$ (namely, f), and the same polynomial contains $x_\mu x_\rho$. Therefore our four terms should add up to some multiple of this polynomial, which implies that $z_{\lambda,\lambda} \pm z_{\mu,\mu} \pm z_{\rho,\rho} \pm z_{\sigma,\sigma} = 0$.
- Similar analysis shows that if $\alpha, \beta, \alpha - \beta \in \Phi$, then $z_{\alpha,\beta} = \pm z_{\alpha-\beta, 0_\beta}$. Moreover, $z_{0_\alpha, 0_\alpha} = z_{\alpha,\alpha}$. This shows that we do not need new variables for dealing with zero weights.

Let's sum up. The first three points show that the variables $z_{\lambda,\mu}$ for $(\lambda, \mu) \leq 0$ are not involved in the Lie algebra at all. Next, the variables $z_{\lambda,\mu}$ for $\lambda - \mu \in \Phi$ generate a subspace of dimension $|\Phi|$. Finally, the variables $z_{\lambda,\lambda}$ generate a subspace of dimension $l + 1$. Indeed, we can express any $z_{\lambda,\lambda}$ as a linear combinations of $z_{\mu,\mu}$'s, where $\mu = \mu_1, \dots, \mu_t$ are such roots that every simple root is among the pairwise differences between μ_i's. It is easy to see that we can take $\mu_1, \dots, \mu_t$ with this property such that $t = l + 1$. Therefore, the total dimension of the Lie algebra is no more than $|\Phi| + l + 1$. □

We need to use the following lemma by Waterhouse [Wat87, Theorem 1.6.1].

Lemma 7. *Let G and H be affine group schemes of finite type over $\mathbb{Z}$ with G flat, and let $\varphi\colon G \to H$ be a homomoprhism. Suppose that the following conditions hold for any algebraically closed field K:*

(1) $\dim(G(\Phi, -)_K) \geq \dim_K(\mathrm{Lie}(G^0_I(K))$;
(2) π is injective on $\mathrm{Spec}(K)$-points and on $\mathrm{Spec}(K[x]/(x^2))$-points;
(3) the normaliser of $\pi(G^0(K))$ in $G^0_I(K)$ is contained in $\pi(G(K))$.

Now let G^0_I denote the identity connected component of G_I, and $\overline{G}(\Phi, -) = G(\Phi, -) \times \mathrm{Cent}(GL(N, -))$, where Cent denotes the center. In other words, $\overline{G}(\Phi, R)$ is the product of our Chevalley group $G(\Phi, R)$ and the group of non-zero diagonal matrices in $GL(N, R)$.

Theorem 8. $\overline{G}(\Phi, R) = G^0_I(R)$.

Proof. The adjoint representation together with the inclusion of scalar matrices gives us a homomorphism $\pi\colon \overline{G}(\Phi, -) \to GL(N, -)$. Theorem 4 (and a trivial remark that scalar matrices preserve the ideal I) shows that its image lies in G_I. $\overline{G}(\Phi,-)$ is a connected group scheme, therefore its image lies already in G_I^0. Now we apply lemma 7 to π. Our group schemes are affine of finite type over $\mathbb{Z}$, since they are closed subschemes of $GL(N,-)$. Moreover, $G = G(\Phi,-)$ is flat, since it is smooth. It remains to check the three conditions of the lemma. The first follows from the fact that $\dim(G(\Phi,-)_K) = |\Phi| + l + 1$, and by Proposition 6 we have $\dim_K(\mathrm{Lie}(G_I^0(K)) = \dim_K(\mathrm{Lie}(G_I(K)) \leq |\Phi| + l + 1$. The second condition is true since π is a faithful representation. Finally, it follows from the tables in [Sei87] that there are [almost] no connected algebraic subgroups N such that $G(\Phi,K) \leq N \leq GL(N,K)$. On the other hand, G_I^0 is such a subgroup. Therefore, either we have $G_I^0(K) = G(\Phi,K)$, or $G_I^0(K) \geq SO(\Phi,K)$. It is trivial to show that $SO(\Phi,K)$ does not preserve the ideal I, so the second possibility is excluded. It remains to note that $G(\Phi,K)$ coincides with its own normaliser in $GL(N,K)$, which is well known. □

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