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CHEVALLEY GROUPS OF TYPE E_6 IN THE 27-DIMENSIONAL REPRESENTATION

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The present paper is devoted to a detailed computer study of the action of the Chevalley group $G(\mathrm{E}_6, R)$ on the minimal module $V(\varpi_1)$. Our main objectives are an explicit choice and a tabulation of the signs of structure constants for this action, compatible with the choice of a positive Chevalley base, the construction of multilinear invariants and equations on the matrix entries of matrices from $G(\mathrm{E}_6, R)$ in this representation, and an explicit tabulation of root elements. Bibliography: 170 *titles.*

The present paper, which is a sequel of [163, 164], is devoted to a detailed computer study of the action of the Chevalley group $G(\mathrm{E}_6, R)$ on the minimal module $V(\varpi_1)$. This paper is of a technical nature and is based mainly on

• the realization of the 27-dimensional module as an internal Chevalley module in the unipotent radical of the standard parabolic subgroup of typoe P_7 in the Chevalley group $G(\mathrm{E}_7, R)$,

• extensive computer calculations carried out with the help of the general purpose computer algebra system `Mathematica`.

We take into account results related to the exceptional 27-dimensional Jordan algebra. However, we prefer not to invoke them, but rather to obtain independent verifications by direct computation.

Here our main objectives are an explicit choice and a tabulation of the signs of structure constants for this action, compatible with the choice of a positive Chevalley base in [163], the construction of multlinear invariants and equations on matrix entries for the elements of $G(\mathrm{E}_6, R)$ in this representation, and an explicit tabulation of root elements in this representation. For convenience of use, the resulting tables are reproduced in three indexations:

• the natural indexation,

• the indexation related to the A_5-branching,

• the indexation related to the D_5-branching.

The present article is essentially a common initial fragment of several joint papers, under way or upcoming, devoted to

• the K-theory of exceptional groups;

• the study of overgroups of $G(\mathrm{E}_6, R)$ in the general linear group $\mathrm{GL}(27, R)$;

• the study of some classes of subroups in $G(\mathrm{E}_6, R)$;

• the study of the unipotent radical of parabolic subgroups;

• the geometry of root subgroups;

• generation problems.

1. Structure constants

The notation pertaining to roots, weights, Lie algebras, algebraic groups, and representations is utterly standard and follows [2, 3, 4, 21, 23]; see also [158, 130], where one can find many further references. We do not recall the definition of Chevalley groups and their subgroups, which can be found, for example, in [1, 2, 5, 21, 25–30, 123, 145, 146, 158, 160, 162, 164]. In the present section, we merely fix the notation that we use in what follows.

First let Φ be a reduced irreducible root system of rank l, $\Pi = \{\alpha_1, \dots, \alpha_l\}$ be a fundamental system in Φ, and Φ^+ and Φ^- be the corresponding sets of positive and negative roots. The elements of Π are called fundamental roots, and we always use the same numbering as in [3]. Since the main body of the present paper is devoted mostly to the type $\Phi = \mathrm{E}_6$ and partly to the type $\Phi = \mathrm{E}_7$, we are interested exclusively in the case where all

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the roots of Φ have the same length. Such root systems are called *simply-laced*, as opposed to *multiply-laced* systems. As usual, $W = W(\Phi)$ denotes the Weyl group of Φ; w_α is the root reflection with respect to $\alpha \in \Phi$, and the $s_i = w_{\alpha_i}$, $1 \le i \le l$, are fundamental reflections, i.e., Coxeter generators.

The construction of Chevalley groups is usually based on the choice of a Chevalley base in a complex simple Lie algebra L of type Φ. Recall that the choice of a Cartan subalgebra H in L determines the root decomposition $L = H \bigoplus \sum L_\alpha$, where the L_α are one-dimensional subspaces invariant with respect to H. For every root $\alpha \in \Phi^+$, we choose a nonzero root vector $e_\alpha \in L_\alpha$ and identify the root α with a linear functional on H such that $[h, e_\alpha] = \alpha(h)e_\alpha$. The restriction of the Killing form of the Lie algebra L to H is nondegenerate and, thus, establishes a canonical isomorphism $H \cong H^*$. In the sequel, we use this isomorphism to regard roots as elements of H. However, it is usually more convenient to work with coroots $h_\alpha = 2\alpha/(\alpha,\alpha) \in H$. Thus, any choice of a nonzero $e_\alpha \in L_\alpha$, $\alpha \in \Phi^+$, uniquely determines $e_{-\alpha} \in L_{-\alpha}$, $\alpha \in \Phi^+$, such that $[e_\alpha, e_{-\alpha}] = h_\alpha$. The set $\{e_\alpha, \alpha \in \Phi; h_\alpha, \alpha \in \Pi\}$ is a base of the Lie algebra L, which is called a Weyl base. Moreover, $[h_\alpha, e_\beta] = A_{\alpha\beta}e_\beta$, where the $A_{\alpha\beta} = 2(\alpha,\beta)/(\alpha,\alpha) \in \mathbb{Z}$ are Cartan numbers. The structure constants $N_{\alpha\beta}$ are now defined by the relation $[e_\alpha, e_\beta] = N_{\alpha\beta}e_{\alpha+\beta}$. A Weyl base can be normalized in such a way that all structure constants $N_{\alpha\beta}$ are integers; in this case it is called a *Chevalley base*, and the set $\{e_\alpha, \alpha \in \Phi\}$ is called a *Chevalley system*.

For simply-laced systems, one always has $N_{\alpha\beta} = 0, \pm 1$, so that we need only fix the *signs* of the structure constants. We fix a **positive** Chevalley base, which is characterized by the property that $N_{\alpha\beta} > 0$ for all extra-special pairs (see [155], [46], [86], and [163]). For simply-laced systems, this condition means precisely that $N_{\alpha_i\beta} = +1$ whenever $\alpha_i + \beta \in \Phi$ has the following property: if $\alpha_j + \gamma = \alpha_i + \beta$ for a fundamental root α_j and a positive root γ, then $j > i$.

The signs of the structure constants in a positive Chevalley base are tabulated in [163], and for reader's convenience in comparing them with our present calculations, we reproduce Table 3 therefrom. This table contains structure constants for the positive half of a Chevalley system e_α, $\alpha \in \Phi^+$, in a Lie algebra of E_6 type. The other three quarters of the complete structure constant table can easily be reconstructed with the help of relations. For a cross-check, the calculations in [163] have been performed by two entirely different methods.

- First, by Tits' inductive algorithm [155]; see also [46].
- Second, by the Frenkel–Kac cocycle [75]; see also [76, 104, 142].

Calculations by Tits' algorithms have been performed from time immemorial [45, 86], but usually for a different order of roots. On the other hand, the obvious choice of the bilinear form and signs in the Frenkel–Kac algorithm does not usually lead to a positive Chevalley base. In [163], we keep the Cartan form but modify signs; in other words, there $\varepsilon(\alpha) = -1$ for some positive roots. On the other hand, [50, 54] describe another construction of structure constants in a positive Chevalley base, where the signs of all positive roots are equal to $+1$, but the bilinear form is not given by Cartan numbers any more.

Now, let $G = G(\Phi, R)$ be a simply connected Chevalley group of type Φ over R. The choice of a Chevalley base determines, in particular, a split maximal torus $T = T(\Phi, R)$ in G and a parametrization of unipotent root subgroups X_α, $\alpha \in \Phi$, elementary with respect to the torus T. We fix such a parametrization, and let $x_\alpha(\xi)$ be an elementary root unipotent corresponding to $\alpha \in \Phi$, $\xi \in R$. One has

$$X_\alpha = \{x_\alpha(\xi) \mid \xi \in R\}.$$

For two elements x and y of a group G, we denote by $[x, y]$ their left-normed commutator $xyx^{-1}y^{-1}$. The Chevalley commutator formula asserts that

$$[x_\alpha(\xi), x_\beta(\eta)] = \prod x_{i\alpha+j\beta}(N_{\alpha\beta ij}\xi^i\eta^j)$$

for every $\alpha, \beta \in \Phi$ such that $\alpha + \beta \neq 0$, and $\xi, \eta \in R$. The product on the right-hand side is taken over all roots of the form $i\alpha + j\beta \in \Phi$, $i, j \in \mathbb{N}$, in an arbitrary but fixed order. The structure constants $N_{\alpha\beta ij}$ of the Chevalley group do not depend on ξ and η. Clearly, $N_{\alpha\beta 11} = N_{\alpha\beta}$ are precisely the structure constants of the Lie algebra L in the corresponding Chevalley base. In a simply-laced root system, the only positive linear combination of roots α and β, which may possibly be a root, is their sum $\alpha + \beta$. Here the Chevalley commutator formula can only take the form $[x_\alpha(\xi), x_\beta(\eta)] = e$ if $\alpha + \beta$ is not a root or the form

$$[x_\alpha(\xi), x_\beta(\eta)] = x_{\alpha+\beta}(N_{\alpha\beta}\xi\eta)$$

if $\alpha + \beta$ is a root. Thus, in this case the tables of structure constants for Lie algebras, constructed in [163], are simultaneously tables of structure constants for Chevalley groups of types E_6, E_7, and E_8. For other root systems, see formulas in [46], as well as tables and further references in [164].

2. Weyl modules

Usually we consider a Chevalley group $G = G(\Phi, R)$ together with its action on a *Weyl module* $V = V(\omega)$ for some dominant weight ω. In the present paper, we limit ourselves to the case where the highest weight ω of the module V is fundamental, $\omega = \varpi_r$. However, in many problems of structure theory it is much more convenient to carry out calculations in some representations that are not fundamental, or even in reducible representations, if this leads to more amenable equations.

Let, as above, Φ be a reduced irreducible root system of rank l. Denote by $Q(\Phi)$ its root lattice and by $P(\Phi)$ its weight lattice. Recall that $P(\Phi)$ consists of integer linear combinations of fundamental weights $\varpi_1, \dots, \varpi_l$. By definition, the fundamental weights form a base dual with respect to the base $\alpha_1{}^\vee, \dots, \alpha_l{}^\vee$, where $\alpha^\vee = 2\alpha/(\alpha, \alpha)$. As usual, $P_{++}(\Phi)$ denotes the cone of *dominant* integral weights, which are nonnegative integer linear combinations of fundamental weights $\varpi_1, \dots, \varpi_l$.

We fix a dominant weight $\omega \in P_{++}(\Phi)$, and let $V = V(\omega)$ be the Weyl module of the group G with highest weight ω. The corresponding representation $G \longrightarrow \mathrm{GL}(V)$ will be denoted by $\pi = \pi(\omega)$. Further, $\Lambda = \Lambda(\omega)$ is the *multiset* of weights of the module $V = V(\omega)$ *with multiplicities*. To denote the *set* of weights *without* multiplicity, we usually write $\overline{\Lambda}(\omega)$. In the present paper, we are mostly interested in microweight modules; see [4, 123, 126, 130, 158, 162] and references therein. For a microweight representation, all weights are extremal and thus have multiplicity 1. This means that, in this case, $\Lambda = \overline{\Lambda}(\omega)$ coincides with the Weyl orbit of the highest weight $\Lambda = W\omega$.

In what follows, we fix an *admissible* base v^λ, $\lambda \in \Lambda$, of the module V. Recall that a base is called admissible if the following two conditions are fulfilled.

- Every vector v^λ is a vector of weight λ, provided that λ is regarded as a weight *without* multiplicity.
- The action of root unipotents $x_\alpha(\xi)$, $\alpha \in \Phi$, $\xi \in R$, in the base v^λ, $\lambda \in \Lambda(\omega)$, is described by matrices whose entries are polynomials of ξ with integer coefficients.

Matsumoto's lemma (see [123] and [146]) asserts that for a microweight representation, an admissible base can be normalized in such a way that

$$x_\alpha(\xi)v^\lambda = v^\lambda + c_{\lambda\alpha}\xi v^{\lambda+\alpha},$$

where the action structure constants $c_{\lambda\alpha}$ are all equal to ± 1. In fact, in what follows we always choose a *crystal* base, where all structure constants $c_{\lambda\alpha}$ are equal to $+1$ for fundamental and negative fundamental roots: $c_{\lambda\alpha} = +1$ if $\alpha \in \pm\Pi$. The existence of such a base immediately follows from general results by George Lusztig and Masaki Kashiwara, elementary proofs are supplied in [162] and [6].

We conceive a vector $a \in V$, $a = \sum v^\lambda a_\lambda$, as a coordinate *column* $a = (a_\lambda)$, $\lambda \in \Lambda$. Within this framework, an element b of the contragradient module V^* should be visualized as a coordinate *row* $b = (b_\lambda)$, $\lambda \in \Lambda$. Of course, with respect to the weights Λ^* of the contragradient module V^* itself the picture is precisely opposite: the elements of V^* are personified by *columns* $b = (b_\lambda)$, $\lambda \in \Lambda^*$, while the elements of V should be writted as *rows* $a = (a_\lambda)$, $\lambda \in \Lambda^*$. Let us emphasize that, in this paper, we index *both* the columns and the rows by weights of the module V. This means that $\lambda, \mu, \nu, \dots$ always belong to Λ. In other words, we always index the coordinates of a vector in V^* by weights of V and regard them as rows, whereas usually they would be indexed by weights of the module V^* itself and depicted as columns.

One of the chief technical aspects is that the elements of these columns and rows are not linearly ordered. They are merely partially ordered with respect to the usual order on Λ, determined by the choice of the fundamental system Π. Recall that $\lambda \geq \mu$ with respect to this order, provided that $\lambda - \mu = \sum m_i\alpha_i$, where $m_i \geq 0$. With the above interpretation of V, it is customary to conceive the elements of the Chevalley group itself as matrices $g = (g_{\lambda\mu})$, $\lambda, \mu \in \Lambda$, with respect to the base v^λ. As usual, the columns of such a matrix are the coordinate columns of the vectors gv^μ, $\mu \in \Lambda$, with respect to the base v^λ, $\lambda \in \Lambda$. We often denote the μth column of a matrix g by $g_{*\mu}$ and the λth row by $g_{\lambda *}$.

3. Internal Chevalley modules

As usual, we denote by $U = U(\Phi, R)$ the subgroup generated by all root subgroups X_α corresponding to the positive roots $\alpha \in \Phi^+$, and by $B = TU$ the standard Borel subgroup. For a subset $J \subseteq \Pi$ of fundamental roots, we denote by Φ_J the subsystem in Φ generated by J, by $P_J \geq B$ the corresponding standard parabolic subgroup in G, by L_J its Levi factor, which is a reductive group of type Φ_J, and finally by

$$U_J = \langle X_\alpha,\ \alpha \in \Phi^+ \setminus \Phi_J \rangle$$

its unipotent radical. In the case where $R = K$ is a field, we have

$$L_J = T\langle X_\alpha,\ \alpha \in \Phi_J\rangle.$$

With the above notation, P_J is the semidirect product $P_J = L_J \leftthreetimes U_J$, where L_J acts on the normal subgroup U_J by conjugation.

Simultaneously with P_J, we consider the opposite parabolic subgroup P_J^- with the same Levi subgroup L_J and the opposite unipotent radical

$$U_J^- = \langle X_{-\alpha},\ \alpha \in \Phi^+ \setminus \Phi_J\rangle.$$

In fact, we are mostly interested in the case of maximal parabolic subgroups. Let us fix r, $1 \le r \le l$, and set $J = J_r = \Pi \setminus \{\alpha_r\}$. The corresponding maximal parabolic subgroup, its Levi subgroup, and its unipotent radical will be denoted by P_r, L_r, and U_r, respectively. Set $\Sigma_r = \Phi^+ \setminus \Phi_J$ and denote by $\Sigma_r(h)$ the set of roots $\alpha \in \Sigma_r$ of α_r-level h. In other words,

$$\Sigma_r(h) = \{\alpha = \sum m_i\alpha_i,\ m_r = h\}.$$

Clearly, $\Sigma_r(h) = \emptyset$ for $h > h_r$. Here h_r denotes the coefficient with which α_r ocurs in the linear expansion of the highest root. The union of all $\Sigma_r(h)$, $h \ge k$, will be denoted by $\widetilde{\Sigma}_r(k)$. One has $\widetilde{\Sigma}_r(1) = \Sigma_r$. Set $U_r(h) = \prod X_\alpha$, where the product is taken over all roots $\alpha \in \widetilde{\Sigma}_r(h)$ in an arbitrary order. The Chevalley commutator formula implies that $U_r(h)$ is a subgroup of U_r; in particular, $U_r(1) = U_r$. Actually, $U_r(h)$ coincides almost always with the hth term of the descending central series of the group U_r. However, we defined $U_r(h)$ as the product of all root subgroups corresponding to the roots of level $\ge h$ just to avoid the discussion of small exceptions.

The structure of consecutive factors $U_r(h)/U_r(h+1)$ as L_r-modules is well known; for example, see [36, 132]. Namely, for all r, $1 \le r \le l$, and h, $h \le h_r$, the quotient group

$$U_r(h)/U_r(h+1) \cong \oplus X_\alpha, \qquad \alpha \in \Sigma_r(h),$$

is the Weyl module of L_r with the highest weight ω, where ω is the element of highest height in $\Sigma_r(h)$. Let us clarify that the right-hand side of the above formula is a direct sum of Abelian groups, whereas in the definition of $U_r(h)$ the right-hand side was the product of subgroups in a given group. The above statement is a trivial special case of the main results of [36], with irreducible modules replaced by Weyl modules, not to mention very bad primes.

Actually, we are mostly interested not in the Levi subgroup L_r itself but in its commutator subgroup[1] $G_r = [L_r, L_r]$. If r is fixed, we write simply Σ and $\Sigma(h)$ instead of Σ_r and $\Sigma_r(h)$, etc. Thus, $\Delta = \Phi \cup \Sigma \cup (-\Sigma)$. The largest, and usually the most interesting one, is the first quotient $\Sigma/\Sigma(2)$. In most usual applications, the following two cases are of particular importance:

- the unipotent radical U_r is Abelian, $U_r(2) = 1$,
- the unipotent radical U_r is extraspecial, $U_r(2) = X_\delta$, where δ is the highest root of Φ.

The primary example of the Abelian case is the 27-dimensional module for E_6, which can be personified as U_7 in the Chevalley group of type E_7. The primary example of the extraspecial case is the 56-dimensional module for E_7, which can be personified as $U_8/U_8(2)$ in the Chevalley group of type E_8. In fact, most of the calculations in the present paper have been performed – or double-checked! – precisely in this realization of the Chevalley group of type $G(\mathrm{E}_6, R)$, and we discuss it minutely in Sec. 5.

4. Root systems of type E_l

In the present paper, as in [163], we use the hyperbolic realization of root systems of type E_l in the $(l+1)$-dimensional Minkowski space [16], $240 = 2\binom{8}{2}+2\binom{8}{3}+2\binom{8}{1}+2\binom{8}{1}$. This realization is *dramatically* more convenient for practical calculations than the following usual realizations in a Euclidean space:

- the realization of E_l as elements of minimal norm in the ring of Cayley integer octaves, $240 = 2\binom{8}{1}+2^4{\cdot}7+2^4{\cdot}7$, where $7 = 2^2+2+1$ is the number of lines – and their complements – on a Fano plane. This has been the favorite realization used by Coxeter and Freudenthal.

[1]As usual, speaking of normalizers, commutator subgroups, central series, etc., for subgroups in G, we understand the commutator subgroup of L_r *in the sense of algebraic group theory* rather than the commutator subgroup of its group of rational points!

• Cartan's realization, $240 = 4\binom{8}{2} + 2^7$, presented in most standard textbooks on Lie groups and Lie algebras. The point is that in the hyperbolic realization, ALL COEFFICIENTS OF ROOTS IN THE STANDARD ORTHONORMAL BASE ARE INTEGERS. At the same time, we preserve the usual numbering of fundamental roots; see [3]. Since all calculations of the present paper materially depend on this realization of root systems of types E_6 and E_7, we briefly recall it. See also [91] or [163], where one can find many further details. Of course, this realization is tantamount to the usual realization of E_8 in terms of compact octaves $\mathbb{O}$, the isomorphism being established by $SL(2,\mathbb{O}) \cong Spin(9,1,\mathbb{R})$ (see [37]). Since in the present paper we are interested in the computational aspect rather than in differential geometry or the geometry of numbers, we do not dwell on this theme.

Consider the real vector space $U = \mathbb{R}^{l,1}$ of dimension $l+1$ with a nondegenerate symmetric inner product $(\ ,\ )\colon U \times U \to \mathbb{R}$ of signature $(l,1)$. We fix an orthonormal base $e_0, e_1, \dots, e_l$ such that $(e_0,e_0) = -1$ and $(e_i,e_i) = 1$ for all $1 \le i \le l$. We are primarily interested in the senior case $l = 8$. Denote by $L \le \mathbb{R}^{8,1}$ the lattice consisting of all vectors $v \in \mathbb{R}^{8,1}$ the coordinates $\lambda, \mu_1, \dots, \mu_8$ of which in the base $e_0, e_1, \dots, e_8$ are integers,

$$v = \lambda e_0 + \mu_1 e_1 + \dots + \mu_8 e_8, \quad \lambda, \mu_1, \dots, \mu_8 \in \mathbb{Z}.$$

Denote by Φ the set of all vectors $v \in L$ the coordinates of which satisfy the following system of equations:

$$\begin{cases} 3\lambda - (\mu_1 + \dots + \mu_8) = 0, \\ -\lambda^2 + \mu_1^2 + \dots + \mu_8^2 = 2. \end{cases}$$

The solutions of this system, subject to the inequalities

$$0 \le \lambda \le 3, \quad \mu_1 \ge \dots \ge \mu_8,$$

are listed in the following table.

λ	μ_1	μ_2	μ_3	μ_4	μ_5	μ_6	μ_7	μ_8
0	1	0	0	0	0	0	0	-1
1	1	1	1	0	0	0	0	0
2	1	1	1	1	1	1	0	0
3	2	1	1	1	1	1	1	1

It is easy to verify that all other integer solutions can be obtained from these ones by means of the following two transformations:

• an arbitrary permutation of $\mu_1, \dots, \mu_8$;

• a simultaneous change of signs of all coordinates.

Thus, up to sign, every element of Φ has the following form:

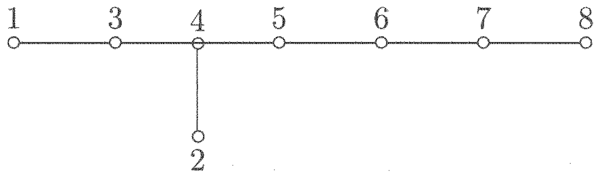
$$\begin{cases} \beta_{ij} = e_i - e_j, \quad i > j, \\ \gamma_{ijh} = e_0 + e_i + e_j + e_h, \\ \eta_{ij} = 2e_0 + e_1 + \dots + \widehat{e_i} + \dots + \widehat{e_j} + \dots + e_8, \\ \zeta_i = 3e_0 + e_1 + \dots + 2e_i + \dots + e_8, \end{cases}$$

where the indices $i, j, h = 1, \dots, 8$ are pairwise distinct, and the hat ^ over an index signifies that this index should be omitted.

It is easy to see that the restriction of the inner product to the hyperplane V orthogonal to the test vector $3e_0 + e_1 + \dots + e_8$ is positive definite and Φ is a root system of type E_8 in V (see [16, 163]). We fix the following fundamental system $\Pi = \{\alpha_1, \dots, \alpha_8\}$ in Φ:

$$\alpha_1 = e_2 - e_1, \quad \alpha_2 = e_0 + e_1 + e_2 + e_3, \quad \alpha_3 = e_3 - e_2, \quad \alpha_4 = e_4 - e_3,$$
$$\alpha_5 = e_5 - e_4, \quad \alpha_6 = e_6 - e_5, \quad \alpha_7 = e_7 - e_6, \quad \alpha_8 = e_8 - e_7.$$

The numbering of fundamental roots follows [3] and is reproduced in the diagram



The roots β_i, γ_{ijh}, η_{ij}, and ζ_i, listed above, form the set Φ^+ of positive roots with respect to the fundamental system Π.

Now, to get the root system of type E_7 it suffices to take roots of E_8 such that $\mu_8 = 0$ or, what is the same, such that α_8 does not occur in their expansion with respect to fundamental roots. Similarly, to get the root system of type E_6 it suffices to take roots of E_8 such that $\mu_7 = \mu_8 = 0$ or, what is the same, such that *both* α_7 and α_8 do not occur in their expansion with respect to fundamental roots.

In the sequel, for roots we employ a system of notation known as *Dynkin notation*, in other words, Dynkin diagrams with coordinate marks. In this notation, a root $\alpha \in \Phi^+$ such that

$$\alpha = p\alpha_1 + q\alpha_2 + r\alpha_3 + s\alpha_4 + t\alpha_5 + u\alpha_6 + v\alpha_7 + w\alpha_8$$

for some $p, q, r, s, t, u, v, w \in \mathbb{N}_0$ is denoted simply by

$$\alpha = {prstuvw \atop q}.$$

The roots of E_7 and E_6 are denoted likewise. The sum of coefficients

$$\mathrm{ht}(\alpha) = p + q + r + s + t + u + v + w$$

is called the height of a root α. For a given choice of a fundamental system, there is a unique root of maximal height, which is called the maximal root or the highest root. With respect to the above choice of fundamental systems, the highest roots of E_6 and E_7 are

$$2e_0 + e_1 + \ldots + e_6 = (2, 1, 1, 1, 1, 1, 1) = {12321 \atop 2},$$

$$2e_0 + e_2 + \ldots + e_7 = (2, 0, 1, 1, 1, 1, 1, 1) = {234321 \atop 2}.$$

The heights of these roots are $11 = 12 - 1$ and $17 = 18 - 1$, respectively. To save room in the tables, we use the *string Dynkin notation*, when a root α is denoted by the string of its coefficients $pqrstuvw$. In this notation, the highest roots of E_6 and E_7 are written as 122321 and 2234321, respectively.

For roots, all of their coefficients in the expansion with respect to a fundamental base have the same sign. However, when one wishes to record an integer linear combination of fundamental roots the coefficients of which have different signs, it is customary to put minuses not in front of digits but *over* them. For example, the Dynkin notation of $\alpha_1 - \alpha_2$ would be ${10000 \atop \bar{1}}$.

In what follows, we always arrange positive roots in *height lexicographic* order. This order is *regular*, in the sense that the roots of smaller height always precede the roots of larger height, and *lexicographic* on the roots of a given height. We write $\alpha \prec \beta$ to denote that α precedes β with respect to this order. By definition, this means that either $\mathrm{ht}(\alpha) < \mathrm{ht}(\beta)$ or $\mathrm{ht}(\alpha) = \mathrm{ht}(\beta)$ and the integer represented by the string Dynkin form of the root α is *larger* than the integer represented by the string Dynkin form of the root β.

In [163], one can find `Mathematica` code that generates roots of E_l in the hyperbolic base and in the fundamental base. We do not reproduce this code here; it simply translates the above definitions into the `Mathematica` language. However, for reader's convenience we reproduce the answer, especially as in [163] positive roots are listed in the Dynkin notation but not in hyperbolic form. The actual calculations were performed in hyperbolic form, so that one of the main objects we need from [163] is the following list `positiveE6`:

```
{{0,-1,1,0,0,0,0},{1,1,1,1,0,0,0},{0,0,-1,1,0,0,0},
{0,0,0,-1,1,0,0},{0,0,0,0,-1,1,0},{0,0,0,0,0,-1,1},
```

```
{0,-1,0,1,0,0,0},{1,1,1,0,1,0,0},{0,0,-1,0,1,0,0},
{0,0,0,-1,0,1,0},{0,0,0,0,-1,0,1},{0,-1,0,0,1,0,0},
{1,1,0,1,1,0,0},{1,1,1,0,0,1,0},{0,0,-1,0,0,1,0},
{0,0,0,-1,0,0,1},{1,0,1,1,1,0,0},{0,-1,0,0,0,1,0},
{1,1,0,1,0,1,0},{1,1,1,0,0,0,1},{0,0,-1,0,0,0,1},
{1,0,1,1,0,1,0},{0,-1,0,0,0,0,1},{1,1,0,0,1,1,0},
{1,1,0,1,0,0,1},{1,0,1,0,1,1,0},{1,0,1,1,0,0,1},
{1,1,0,0,1,0,1},{1,0,0,1,1,1,0},{1,0,1,0,1,0,1},
{1,1,0,0,0,1,1},{1,0,0,1,1,0,1},{1,0,1,0,0,1,1},
{1,0,0,1,0,1,1},{1,0,0,0,1,1,1},{2,1,1,1,1,1,1}}.
```

Once this list has been generated, it is not particularly important where it came from. The initial fragment of this list of length six,

```
{{0,-1,1,0,0,0,0},{1,1,1,1,0,0,0},{0,0,-1,1,0,0,0},
{0,0,0,-1,1,0,0},{0,0,0,0,-1,1,0},{0,0,0,0,0,-1,1}}
```

was taken as the system of fundamental roots `rootbaseE6`. After that, to get the string Dynkin form, for each of the other roots a system of linear equations was solved, which yielded their linear expressions in terms of `rootbaseE6`. We do not reproduce the lists `positiveE7` and `rootbaseE7`, which are generated similarly, but, for convenience, in Tables 1 and 2 we give the lists of positive roots of E_6 and E_7 with respect to the height lexicographic order.

5. The weights of E_6

By definition, the base of fundamental weights `weightbaseE6` is dual to the base of fundamental roots `rootbaseE6`. The only subtlety one should take into account is that, in the hyperbolic realization, the roots of E_6 belong to a 7-dimensional rather than a 6-dimensional space. To get rid of the extradimension, in constructing the weights one has to check their orthogonality – with respect to the *Euclidean* inner product! – to the test vector

```
testvectorE6={3,-1,-1,-1,-1,-1,-1};
```

thus, the fundamental weights $\varpi_1, \dots, \varpi_6$ are solutions of the following systems of linear equations:

```
omegaE6[i_]:=LinearSolve[Append[rootbasebisE6,testvectorE6],
              Table[If[j==i,1,0],{j,1,7}]] /; 1<=i<=6
```

The list `rootbasebisE6` differs from the list `rootbaseE6` in exactly one position. Namely, its second component is `{-1,1,1,1,0,0,0}`. The objective is to ensure that the solution is orthogonal to α_2 in the sense of the *hyperbolic* inner product. Since all other α_i's do not contain e_0, for them it does not matter, whether we take the Euclidean or the hyperbolic inner product. That is why they are not changed. A different but equivalent approach would be the multiplication of the matrix `Append[rootbasebisE6,testvectorbisE6]`, where

```
testvectorbisE6={3,1,1,1,1,1,1};
```

from the right by the Gram matrix `Diagonal Matrix[{-1,1,1,1,1,1,1}]`. Now the evaluation of

```
weightbaseE6=Table[omegaE6[i],{i,6}]
```

returns the coordinates of the fundamental weights in the orthonormal base of the space $\mathbb{R}^{6,1}$:

```
{{1,-1/3,2/3,2/3,2/3,2/3,2/3},{2,1,1,1,1,1,1},
{2,1/3,1/3,4/3,4/3,4/3,4/3},{3,1,1,1,2,2,2},
{2,2/3,2/3,2/3,2/3,5/3,5/3},{1,1/3,1/3,1/3,1/3,1/3,4/3}}.
```

A dominant (integer) weight is precisely a linear combination of fundamental weights with nonnegative integer coefficients. Equivalently one could stipulate that all of its inner products with fundamental roots are nonnegative integers. The following function gives a test for a weight to be dominant:

```
dominantE6Q[u_]:=And@@Table[Block[{xxx=hip[u,rootbaseE6[[i]]},
                                  xxx>=0&&IntegerQ[xxx]],{i,6}]]
```

Here `hip` denotes the hyperbolic inner product in $\mathbb{R}^{l,1}$, which can be introduced as follows:

```
hip[u_,v_]:=-u[[1]]*v[[1]]+Sum[u[[i]]*v[[i]],{i,2,Length[u]}]
                          /; Length[u]==Length[v]
hip[u_]:=hip[u,u]
```

The following miserable code has been used to generate the weights $\overline{\Lambda}(\omega)$ of the representation with the highest weight ω. To avoid infinite recursion, one should check that ω is indeed dominant:

```
weightsE6[u_]:=Block[{i,j,list={u},len=hip[u],we},
                For[j=1,j<=Length[list],j++,
                For[i=1,i<=6,i++,we=list[[j]]-rootbaseE6[[i]];
                           If[hip[we]<=len&&!MemberQ[list,we],
                                 list=Append[list,we]]]];
                   Return[list]] /; dominantE6Q[u]
```

This text is so *terribly much* retarded and inefficient that the generation of even 27 weights of $V(\omega_1)$ takes about 0.02 seconds! Probably, in this case one would have been better off even with the plain recursion. However, since the largest representation that we need to consider has dimension 248, we made absolutely no attempt to optimize the code.

6. The 27-dimensional module for E_6

Now the evaluation of

```
minimalE6=weightsE6[omegaE6[1]]
```

returns the list of 27 weights of the minimal module. These weights are reproduced in Table 3, together with their expressions in Dynkin form with respect to the base of fundamental roots and to the base of fundamental weights. These expressions are obtained as solutions of the following systems of linear equations:

```
rootformE6[u_]:=LinearSolve[Transpose[rootbaseE6],u]
weightformE6[u_]:=LinearSolve[Transpose[weightbaseE6],u]
```

Observe that, as opposed to the positive roots, the weights of $V(\varpi_1)$ are listed in this table not in the increasing order, but in the *decreasing* order, starting with the highest weight ϖ_1, which has height 8, and ending with $-\varpi_6$, which has height -8.

However, it is often more convenient to use for calculations the realization of $V(\varpi_1)$ as an internal Chevalley module in the group $G(\mathrm{E}_7, R)$. Namely, consider the standard parabolic subgroup $P_7 \leq G(\mathrm{E}_7, R)$. The commutator subgroup $G_7 = [L_7, L_7]$ of its Levi subgroup is the simply connected Chevalley group of type E_6, and the conjugation action of G_7 on the Abelian unipotent radical U_7 converts U_7 to a $G(\mathrm{E}_6, R)$ module $V(\varpi_1)$. Now, Theorem 1 in [162] implies that the elements $x_\alpha(1)$, $\alpha \in \Sigma = \Sigma_7$, corresponding to a positive Chevalley base form a crystal base for the module $V(\varpi_1)$. It follows that we can read off the action structure constants merely by choosing in the structure constants table for the Lie algebra of type E_7 the rows corresponding to the roots of E_6 and the columns corresponding to the roots from Σ.

Remark. For historical accuracy one might observe that the actual course of events has been precisely the opposite. First, Eugene Plotkin concocted tables of action structure constants for this and some other representations we used to check the concordance of signs in the initial version of the decomposition of unipotents. This has been done partly by adapting the tables by Paul Gilkey and Gary Seitz [86] and additional unpublished tables, they sent us, partly by hand, and partly by a program written in `Pascal`. After that the first-named author calculated these tables once more by a `Mathematica` program, noticed the positivity property and described how to read off these structure constants directly from the weight diagram. This has culminated in *a priori* proofs for the decomposition of unipotents in microweight representations. After that Roger Carter interpreted this property as an immediate consequence of the positivity of canonical bases, and precise in this way it is expounded in [162].

Out of the positive roots of E_7 we choose those belonging to E_6 and those belonging to Σ:

```
positiveE6insideE7=Select[positiveE7,Last[rootformE7[#]]==0&]
minimalE6insideE7=Select[positiveE7,Last[rootformE7[#]]==1&]
```

Observe, though, that the roots of E_6 inside E_7 *in the same order* could be obtained simply by appending 0 to the roots of E_6 in the usual order:

```
positiveE6insideE7=Map[Append[#,0]&,positiveE6]
```

The weights of $V(\varpi_1)$ as roots of E_7 are listed in order of increasing height, in Table 4. This table is much more convenient for use than the previous one, since now ALL COEFFICIENTS OF THESE WEIGHTS both in their expansion in the fundamental base of E_7 and in the corresponding hyperbolic base are NONNEGATIVE INTEGERS.

As a matter of fact, we performed all calculations precisely in this realization. In particular, to construct the action structure constants it suffices to pick out in the structure constant table for E_7, constructed in [163], the rows with indices corresponding to the positions of members of `positiveE6insideE7`, and the columns with indices corresponding to the positions of members of `minimalE6insideE7`, on the list `positiveE7`. The result is reproduced in Table 6.

Out of pure curiosity we repeated also the calculations of [163], necessary to construct this table. In doing that, we obtained another striking confirmation of an opinion, expressed by Donbald Knuth, that computers become better and better, while everything else becomes worse and worse. In fact, even by means of a plain recurrent algorithm the calculation of the whole table now takes less than 10 seconds − less than what was required in the mid-1990s to calculate just one row of this table!

In view of the following obvious fact (for example, consult [162, Lemma 4] for the proof), the action structure constants for negative roots are also seen from this table.

Lemma. *For any $\alpha \in \Phi^+$ and any $\lambda \in \Lambda$, one has*

$$c_{\lambda+\alpha,-\alpha} = c_{\lambda,\alpha}.$$

In fact, in many practical calculations it is more convenient to use not the table of action structure constants but the *matrix of signs* of the representation $V(\varpi_1)$ the rows and columns of which are indexed by weights of this representation. The entry of this matrix in the position (λ, μ) equals the constant $c_{\lambda,\mu-\lambda}$ with which the root element $x_{\mu-\lambda}(1)$ adds v^λ to v^μ, or 0 if $\mu - \lambda$ is not a root. The matrix of signs for the representation $V(\varpi_1)$ for the numbering of weights as roots of E_7 is reproduced in Table 7.

7. The weight diagram of $V(\varpi_1)$

Important tools in the study and use of representations of exceptional groups are their *weight diagrams*. Weight diagrams were defined by Eugene Dynkin in 1951 and have been used by his students, especially by Ernest Vinberg. However, the diagrams never made their way to the published works of the Moscow school, as Eugene Dynkin himself explained to the authors, mainly because of the purely technical difficulties to insert a picture in a mathematical text in the precomputer era. Weight diagrams visualize the action of group elements on a module and, in some sense, are a substitute of the usual matrix calculations. In the case of microweight representations, they coincide with *crystal graphs* of Kashiwara. A detailed discussion of wight diagrams and many further references can be found in [158, 130, 162].

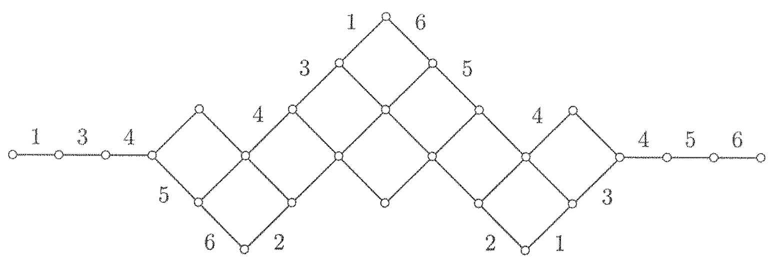
Initially, weight diagrams emerged in representation theory as a purely combinatorial object, describing the Bruhat order on the factors of the Weyl group; we refer to this as the *first look* (see the bibliography in [158, 130]). Soon afterwards, experts became aware that weight diagrams are very convenient to specify the action of a Lie group or a Lie algebra *up to signs*; this is called the *second look* (see [123, 146, 158, 130]). Finally, fairly recently it has become clear that, in fact, weight diagrams completely describe action *including the signs*; this is the idea of the *third look*.

The *weight diagram* of a representation π is an oriented marked graph or, in the terminology of Kashiwara, a *colour* graph, constructed as follows.

• Its vertices correspond to the weights of the representation π, usually with multiplicities.

• Two vertices λ and μ are joined by an arrow with *mark i* (= of *color i*) directed from μ to λ, provided that $\lambda - \mu = \alpha_i$ is the ith fundamental root.

The arrowheads are usually not set, but the whole diagram is read in the positive direction, usually from right to left, and bottom-up. If the weights λ and/or μ are multiple, then the precise sense of what exactly is meant by $\lambda - \mu = \alpha_i$ is needed. In general, the solution of the multiplicity problem is *highly nontrivial* and was obtained only in famous works by Lusztig and Kashiwara in the context of quantum groups. Shortly thereafter Littelmann invented a strikingly beautiful elementary but rather ingenious approach to the construction of these graphs – the *path model*. However, for microweight representations all multiplicities are equal to 1, so that there are no further complications.



Weight diagrams are often used as a shorthand graphical imprint of the corresponding *weight graph*. The vertices of a weight graph are again the weights of a representation π, but now the arrows correspond to *all* positive roots and not just to fundamental ones, as in the case of weight diagram. In other words, weights λ and μ are joined by an arrow with mark $\alpha \in \Phi^+$ directed from μ to λ, provided that $\lambda - \mu = \alpha$. Weight graphs possess some very strong regularity properties and crop up in an immense number of puiblications on combinatorics, finite geometries, sphere packings, and all that (see [42] and references therein). The weight graph of type (E_6, ϖ_1) – or its complement! – is usually called the *Schläfli graph*. It first appeared in the study of algebraic surfaces. Namely, consider the algebraic surface obtained from the complex projective plane $\mathbb{P}^2$ by blow-up of l point in general position. The Schläfli graph describes a configuration of 27 lines occurring on this surface in the case $l = 6$.

More precisely, let $\mathbb{P}^3 = \mathbb{P}^3(K)$ be the three-dimensional projective space over $\mathbb{C}$. We use the expression *cubic hypersurface* as an abbreviation for a *smooth surface* $X \subseteq \mathbb{P}^3$ *of degree* 3. Let (x_0, x_1, x_2, x_3) be the homogeneous coordinates in $\mathbb{P}^3$. Then X is the set of solutions of a cubic equation

$$\sum a_{ijkl} x_0^i x_1^j x_2^k x_3^l = 0, \quad i + j + k + l = 3.$$

Almost any choice of the 20 coefficients a_{ijkl} leads to a smooth surface: allowable choices form an open subset of $\mathbb{P}^{19}$.

A famous classical theorem of algebraic geometry asserts that, on every cubic hypersurface X, there are 27 lines, and the configuration of these lines does not depend on the surface. Namely, if Y and Z are two lines on X, then either $Y \cap Z = \emptyset$ or Y and Z intersect transversally at a single point. It is customary in algebraic geometry to render this configuration by a graph whose vertices correspond to the lines and two lines are joined by a bond if and only if the corresponding lines intersect.

The theorem that we have just stated asserts that the resulting graph does not depend on the choice of X. Thus, for explicit calculations we can choose a particularly simple model. The most famous cubic hypersurface is the *Fermat cubic surface* defined by the equation

$$x_0^3 + x_1^3 + x_2^3 + x_3^3 = 0.$$

Let $\eta = e^{2\pi i/3}$ be a primitive root of 1 of degree 3. Then the 27 lines can be described as follows:

$$\begin{bmatrix} 1 \\ ij \end{bmatrix} = \{(x_0, x_1, x_2, x_3) \in \mathbb{P}^3 \mid x_0 + \eta^i x_1 = 0,\ x_2 + \eta^j x_3 = 0\},$$

$$\begin{bmatrix} 2 \\ ij \end{bmatrix} = \{(x_0, x_1, x_2, x_3) \in \mathbb{P}^3 \mid x_0 + \eta^i x_2 = 0,\ x_1 + \eta^j x_3 = 0\},$$

$$\begin{bmatrix} 3 \\ ij \end{bmatrix} = \{(x_0, x_1, x_2, x_3) \in \mathbb{P}^3 \mid x_0 + \eta^i x_3 = 0,\ x_1 + \eta^j x_2 = 0\},$$

where $0 \le i, j \le 2$ in each case. The fact that the lines appear in bunches of three each, immediately resembles Freudenthal's construction of E_6. Of course, this is not a coincidence. Solving a few systems of algebraic equations, it is easy to derive the following rules:

- $\begin{bmatrix} r \\ ij \end{bmatrix}$ intersects $\begin{bmatrix} r \\ kl \end{bmatrix}$ if and only if $i = k$ or $j = l$;
- $\begin{bmatrix} 1 \\ ij \end{bmatrix}$ intersects $\begin{bmatrix} 2 \\ kl \end{bmatrix}$ if and only if $i - j \equiv k - l \pmod 3$;
- $\begin{bmatrix} 1 \\ ij \end{bmatrix}$ intersects $\begin{bmatrix} 3 \\ kl \end{bmatrix}$ if and only if $i + j \equiv k - l \pmod 3$;
- $\begin{bmatrix} 2 \\ ij \end{bmatrix}$ intersects $\begin{bmatrix} 3 \\ kl \end{bmatrix}$ if and only if $i + j \equiv k + l \pmod 3$.

Drawing the corresponding graph, it is easy to perceive that it is precisely the *complement* of the weight graph of the representation (E_6, ϖ_1). Namely, in the weight graph every weight is joined with exactly 16 other weights, whereas in the graph describing the configuration of 27 lines, every line intersects precisely 10 other lines, $27 = 1 + 16 + 10$.

The above construction of 27 lines displayed the symmetry of type $3\,\mathrm{A}_2$. There are other descriptions which distinctly thrust forward different kinds of symmetry. For example, blowing up six points $p_1, \ldots, p_6 \in \mathbb{P}^2$ one emphasizes the symmetry of type A_5. Namely, in this case the 27 lines can be described as follows (see [24, Theorem 4.9]):

- 6 exceptional lines $\pi : X \longrightarrow \mathbb{P}^2$;
- 15 strict transforms of the lines p_ip_j, $i \neq j$;
- 6 strict transforms of the conics passing through $p_1, \dots, \widehat{p_i}, \dots, p_6$, $1 \leq i \leq 6$.

In [18], the same configuration is described in terms of the symmetry of type D_5.

8. Restrictions to A_5 and D_5

Proofs of results on Chevalley groups over rings usually proceed either by induction on the dimension of the ring (such proofs are called arithmetic) or by induction on the rank of the group (such proofs are called geometric) or both. For the group of type E_6, the most usual geometric proofs are based either on the reduction to the group of type D_5, using elements from the group A_5, or on the reduction to the group of type A_5, using elements from the group D_5. These groups inevitably arise in such analysis, since in the process of reduction to smaller ranks one usually deletes one of the extreme nodes of the Dynkin diagram. Now, deleting α_1 or α_6 in the system E_6 yields D_5, while deleting α_2 gives A_5. Thus, these are precisely the Levi subgroups of the maximal parabolic subgroups P_1, P_6, or P_2, respectively. The case of E_7 is somewhat more complicated, because in this case *three* distinct subgroups of types E_6, D_6, and A_6 naturally emerge in reduction to smaller rank.

Therefore, in many calculations one has to understand how the 27-dimensional E_6-module branches upon restriction to subsystems of types A_5 and D_5. How the restrictions of $V(\varpi_1)$ to subgroups of these types decompose into irreducibles, is well known. This is also easy to see directly from the weight diagram. As explained in [130], it suffices to strike out the edges marked by 2 or 1, respectively.

Hence,

$$V(\mathrm{E}_6, \varpi_1) \downarrow \mathrm{A}_5 = V(\mathrm{A}_5, \varpi_1) \oplus V(\mathrm{A}_5, \varpi_4) \oplus V(\mathrm{A}_5, \varpi_1),$$
$$V(\mathrm{E}_6, \varpi_1) \downarrow \mathrm{D}_5 = V(\mathrm{D}_5, 0) \oplus V(\mathrm{D}_5, \varpi_4) \oplus V(\mathrm{D}_5, \varpi_1).$$

Observe that the numbering of fundamental weights on the right-hand side is the usual numbering in systems of types A_5 and D_5, respectively, rather than their numbering as fundamental roots of E_6. Expessing this in a downright fashion, we can say that the restriction of the 27-dimensional module $V(\varpi_1)$ to A_5 decomposes into two natural 6-dimensional modules and a 15-dimensional module, which is the fourth exterior power of the natural one. Similarly, the restriction of the 27-dimensional module to D_5 decomposes into three summands: a 1-dimensional module, a 16-dimensional half-spin module, and a 10-dimensional natural module.

Considering $V(\varpi_1)$ as an internal Chevalley module in the Chevalley group of type E_7, we can detect these decomposition as follows. Since among all the roots only α_2 has a nonzero coefficient at e_0, the A_5-components can be told apart by the value of the coefficient at e_0, which can take only the values 0, 1, and 2 for weights of $V(\varpi_1)$. Thus, to get the weights in an order compatible with the A_5-branching, it suffices to select from the weight lists all weights with a prescribed value of the *first* component and then to join the resulting lists again:

```
minimalE6branchA5=Apply[Join,Table[
                    Select[minimalE6insideE7,First[#]==i&],
                                                   {i,0,2}]]
```

On the other hand, the D_5-components are conveniently discerned directly by their coefficient at α_1, which can also take only the values 0, 1, and 2 for weights of $V(\varpi_1)$. Therefore, in this case one can first apply `rootformE7`, and then treat the resulting lists *exactly* in the same way as above:

```
minimalE6branchD5=Apply[Join,Table[
             Select[minimalE6insideE7,First[rootformE7[#]]==i&],
                                                   {i,0,2}]]
```

9. The matrix of signs

Now we are in a position to calculate explicitly the tables of action structure constants for the represnetation $V(\varpi_1)$ and the shape of root elements in this representation. In [163], we calculated the structure constants of the Lie algebra of type E_l by two different methods: the inductive algorithm `mu[l,i,j]` and the Frenkel–Kac cocycle `kac[l,i,j]`. The resulting table of structure constants for E_6 is reproduced as Table 5. From the table for E_7 we select the part we need:

```
minimalE6insideE7sign=Table[
```

```
nu[7,positiveE6insideE7[[i]],minimalE6insideE7[[j]]],
                                {i,1,36},{j,1,27}]]
```

The function nu[7,x,y] defined in [163] expresses the structure constants for E_7. It differs from the function mu in that its arguments are the roots themselves rather than their positions on the list of positive roots. Actually, we have not referred to an electronic version of [163] but have calculated all necessary signs afresh. With great satisfaction we ascertained an enormous growth of computational power as compared with the 1990s. Even with the help of the recurrent algorithm, the calculation of all necessary action structure constants required less than 10 seconds. The resulting table is reproduced as Table 6.

It is often more convenient to use the table of signs in a slightly different form: not as the table of action structure constants but as the *matrix of signs*:

```
signtable=Table[
        nu[7,minimalE6insideE7[[j]]-minimalE6insideE7[[i]],
                minimalE6insideE7[[i]]],
                        {i,1,27},{j,1,27}]]
```

As opposed to the table of action structure constants the rows of which are indexed by roots and the columns of which are indexed by weights; here both the rows and the columns are indexed by weights of the module $V(\varpi_1)$. The entry of this matrix in the position (λ, μ) equals the coefficient with which the root element e_α, where $\alpha = \lambda - \mu$, adds v^μ to v^λ. The matrix of signs is reproduced in Table 7; all subsequent tables rely on this one.

Below we reproduce the results in three numberings of weights: the *natural* one, the one related to the A_5-branching, and the one related to the D_5-branching. Table 8 specifies the correspondence between these numberings. By tradition, we number weights starting with the highest one, as customary in algebra, rather than starting with the lowest one, which most Computer Algebra packages would do by default. This means that, apart from the numbering of rows and columns, the matrix of signs in Table 9 is obtained from the matrix in Table 7 by passing to the transpose with regard to the *skew* diagonal. Therefore, apart from some typographical delicacies – various \vrule, \tablerule, etc. – Table 9 is the table form of the following matrix:

```
minimalE6insideE7signtable=Reverse[Transpose[Reverse[signtable]]]
```

Tables 12 and 15 are obtained thereof by renumbering weights in accordance with the A_5-numbering or the D_5-numbering, respectively. This can be done in many ways, for example by calculating the corresponding permutations. However, since we work with *tiny* lists, and the whole calculation takes fractions of a second, it does not make any sense to strive for efficiency. The following function returns the *position* of y on the list x:

```
search[x_,y_]:=Nest[First,Position[x,y],2]
```

Recall that the built-in function Position returns the *list* of positions formatted as a list, so that we have to get rid of *two* pairs of braces. It remains only to pass from the natural numbering to the A_5-numbering. This can be done, for example, as follows:

```
minimalE6branchA5signtable=Table[minimalE6insideE7signtable
        [search[minimalE6insideE7,minimalE6branchA5[[i]]],
        search[minimalE6insideE7,minimalE6branchA5[[j]]]]],
                        {i,1,27},{j,1,27}]
```

For the D_5-numbering, we can do precisely the same, but the weights should be taken from the list minimalE6branchD5.

We decompose the resulting matrices into blocks in accordance with the A_5-branching or the D_5-branching, respectively, to emphasize their structure patterns.

For instance, in the North-West and the South-East corners of Table 12 one clearly sees two copies of the natural 6-dimensional representation of the group $SL(6, R)$ corresponding to the subsystem $\langle \alpha_1, \alpha_3, \alpha_4, \alpha_5, \alpha_6 \rangle$. On the other hand, in the North-East and the South-West corners one sees the action of the group $SL(2, R)$ corresponding to the highest root δ, which glues these 6-dimensional representations together.

In turn, in Table 15 one clearly sees that, under the D_5-numbering, for any element of the Lie algebra L of type E_6 the last 10 components of the first column are all equal to 0. Precisely this fact underlies the PROOF FROM THE BOOK for this case (see [7, 8]).

10. Elementary root unipotents

Now we can proceed with the calculation of root unipotents for the above three root numberings. First, we calculate root elements e_α:

```
rooteE6insideE7[h_]:=Table[
            If[minimalE6insideE7[[i]]-minimalE6insideE7[[j]]==
                                    positiveE6insideE7[[h]],
                        minimalE6insideE7sign[[h,j]],0],
                                    {i,1,27},{j,1,27}]
```

For the natural numbering, the resulting elements e_α are listed in Table 10. Here $1 \leq h \leq 36$ is the number of a *positive* root. Now, from Sec. 7 we know that the root element e_{-a} corresponding to a negative root is the transpose e_α^t of e_α. Hence, the remaining 36 root elements are just the transposes of the 36 elements listed in the table.

The matrices `rooteE6branchA5[h_]` and `rooteE6branchD5[h_]` are defined similarly. Obviously, on the right-hand side instead of h we should indicate the position of a weight on the list `minimalE6branchA5[[h]]` or, respectively, on the list `minimalE6branchD5[[h]]`, inside `minimalE6insideE7`. Thereby we can use the function `search` defined above. The results are reproduced in Tables 13 and 16.

Now we can easily describe the remaining root elements as matrices. For example, since the representation $V(\varpi_1)$ is microweight, we have $e_\alpha^2 = 0$. Hence, $x_\alpha(\xi) = e + \xi e_\alpha$. This shows that root elements can be defined as follows:

```
rootxE6insideE7[h_,x_]:=IdentityMatrix[27]+x*rooteE6insideE7[h]
```

Obviously, defining the elements

`rootxE6branchA5[h_,x_]` or `rootxE6branchD5[h_,x_]`

we should use

`rooteE6branchA5[h_]` or `rooteE6branchD5[h_]`,

instead of `rooteE6insideE7[h]`

As usual, we set $w_\alpha(\varepsilon) = x_\alpha(\varepsilon)x_{-\alpha}(-\varepsilon^{-1})x_\alpha(\varepsilon)$, where $\varepsilon \in R^*$. Thus, we can now define $w_\alpha(\varepsilon)$ as a matrix:

```
rootwE6insideE7[h_,x_]:=rootxE6insideE7[h,x].
                  Transpose[rootxE6insideE7[h,-Power[x,-1]].
                        rootxE6insideE7[h,x]
```

Finally, the *semisimple* root element $h_\alpha(\varepsilon) = w_\alpha(\varepsilon)w_\alpha(1)^{-1}$, where $\varepsilon \in R^*$, can be expressed in terms of $w_\alpha(\varepsilon)$,

```
roothE6insideE7[h_,x_]:=rootwE6insideE7[h,x].rootwE6insideE7[h,-1]
```

or, at wish, directly in terms of the elements $x_\alpha(\xi)$.

However, in calculations that invoke only a few of the root elements, we prefer to store and use e_α and $x_\alpha(\xi)$ as *sparse* matrices in format `SparseArray`. In that case, we draw 27×27-matrices only at the very last stage of calculations, and only when we actually need such a matrix. We do not reproduce the codes needed for such calculations. Knowing the tables of signs everybody that has basic familiarity with the `Mathematica` language can easily write such codes for himself.

11. Invariant cubic form

Another paramount tool in the study of Chevalley groups of type E_6 in the 27-dimensional representation is the invariant cubic form Q, its complete polarization F, and its 27 partial derivatives $f_1, \ldots, f_{27}$ describing the highest weight orbit.

This form was first constructed by Leonhard Dickson back in 1905. The most elementary constructions were proposed by Claude Chevalley and Hans Freudenthal in 1951–1952 (see [48, 49, 77–82]). Freudenthal worked over a field of characteristic 0, for characteristics $\neq 2, 3$ this form has been used by Tony Springer, Jacque Tits, George Seligman, Nathan Jacobson, Ferdinand Veldkamp, Arjeh Cohen, Bruce Cooperstein, and others (for example, see [100, 101, 139–141, 143, 144, 153, 154, 165–167], [68–71], [52] and references therein). In [140], Tony Springer gave an axiomatic description of the form Q. Manifestedly, this is precisely the norm form of the exceptional 27-dimensional Jordan algebra. A priori the characteristics 2 and 3 may cause problems here. However, as was discovered in a slightly different language by Michael Aschbacher [31–33], this is not the case. All usual proofs of identification invoke difficult geometric results: the classification of simple algebraic groups or local characterization of buildings, due to Tits.

• **Construction in terms of** $3\mathrm{A}_2$**.** The remarkable Freudenthal's construction displays the symmetry with respect to $3\mathrm{A}_2$. Let $M(3,R)$ be the full matrix ring of degree 3 over a commutative ring R. Further, let

$$V = \{(x,y,z) \mid x,y,z \in M(3,K)\}$$

be the free $M(3,R)$-module of rank 3, viewed as a free R-module of rank 27. Define a cubic form F on V by the formula

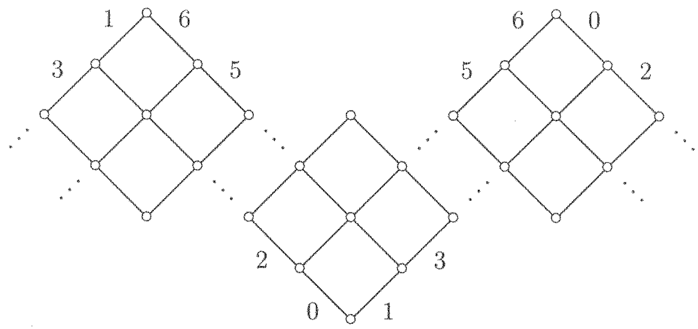
$$F((x,y,z)) = \det(x) + \det(y) + \det(z) - \operatorname{tr}(xyz).$$

Then $\mathrm{G}_{\mathrm{sc}}(\mathrm{E}_6,K)$ may be identified with the group $\operatorname{Isom}(F,K)$ consisting of all transformations $g \in \mathrm{GL}(V) \cong \mathrm{GL}(27,K)$ such that $F(g(x,y,z)) = F(x,y,z)$.

In the next figure, we see 3×3-matrices on the weight diagram of the 27-dimensional representation. This picture corresponds to the restriction of π to the subsystem of type $3\mathrm{A}_2$ generated by α_0 and all fundamental roots, except for α_4:

$$3\mathrm{A}_2 = \langle\alpha_1,\alpha_3\rangle + \langle\alpha_5,\alpha_6\rangle + \langle\alpha_2,\alpha_0\rangle.$$

This branching is implemented by cutting the diagram along the edges marked with 4 and introducing additional edges marked with 0, which join vertices whose difference equals α_0. The easiest way to do that is to consider the Bruhat order on the *affine* Weyl group $\overline{W} = \overline{W}(\Phi)$. Then the induced Bruhat order on $\overline{W}(\mathrm{E}_6)/W(\mathrm{D}_5)$ can be depicted by an infinite strip consisting of weight diagrams of type (E_6,ϖ_1) glued together by the edges marked with 0. After that it remains only to cut this infinite picture by edges marked with 4 and take three consecutive connected components of the resulting picture:



This picture clearly shows three 3×3-matrices and the action of three copies of $\mathrm{SL}(3,K)$ on them. Every 9-dimensional component is isomorphic to the tensor product of the *vector* representation of one copy of the group $\mathrm{SL}(3,K)$ and the *covector* representation of another copy of $\mathrm{SL}(3,K)$. Moreover, the three copies of $\mathrm{SL}(3,K)$ are cyclically permuted, so that each one of them appears once in each role. This is a visual incarnation of triality.

• **Construction in terms of** A_5**.** The initial construction of the form due to Dickson displayed symmetry of type A_5. In 1951, the invariant meaning of this construction was clarified by Chevalley [48]. Similar constructions have later been used by Shult and Aschbacher [31].

Consider the restriction of π to the subsystem of type A_5. As we know, it amounts to the decomposition $V = U \oplus \bigwedge^2(U^*) \oplus U$, where U is the natural 6-dimensional representation of $\mathrm{SL}(6,R) = G(\mathrm{A}_5,R)$. Additional symmetry with respect to $\mathrm{A}_1 = (\mathrm{A}_5)^\perp$ can be described as follows. This copy of A_1 is generated by α_0, and the corresponding subgroup $G(\mathrm{A}_1,K) \cong \mathrm{SL}(2,K)$ leaves $\bigwedge^2(U^*)$ invariant but glues together two copies of U. The resulting 12-dimensional representation is the tensor product of the natural representations of $\mathrm{SL}(6,K)$ and $\mathrm{SL}(2,K)$.

In terms of the above decomposition, the exterior product supplies two trilinear forms on V. Namely, the exterior product of two vectors and one bivector on the dual space is a scalar. Likewise, the exterior product of three bivectors is a scalar. The desired trilinear form on V is obtained as the sum of the above two forms. A detailed description of the form F in this realization and an explicit table of signs may be found in [31].

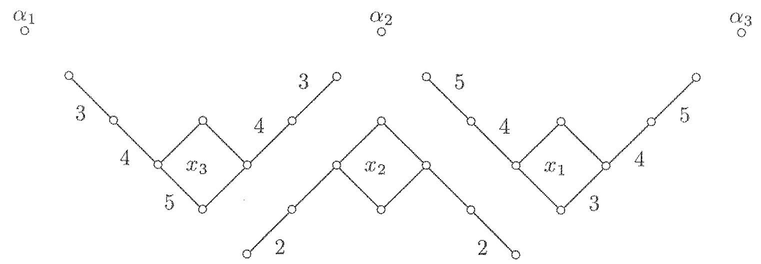
• **Construction in terms of** D_4**.** Another construction, which operates in terms of the Cayley–Dickson algebra and the exceptional 27-dimensional Jordan algebra, is extremely popular in the theory of algebraic groups. It exposes symmetry with respect to D_4 and a relationship with F_4. This construction can be described as follows. Let $\operatorname{char} K \neq 2,3$, and further let C be the *split octonion algebra* over the field K corresponding to a quadratic form N. This is a unique *composition algebra* of dimension 8 over K with 0 divisors. In other words, $N(xy) = N(x)N(y)$ for all $x, y \in C$. In [70, 37, 144], one can find a realization of this algebra in terms of Zorn matrices and many further references. Denote by $x \mapsto x^*$ the canonical involution of C and by $T(x) = x + x^*$ the trace on C. Next, pick up three nonzero scalars $\gamma_1, \gamma_2, \gamma_3 \in K$ and define an involution of the ring $M(3, C)$ as follows:

$$a \mapsto g^{-1}(a^*)^t g, \quad g = \operatorname{diag}(\gamma_1, \gamma_2, \gamma_3).$$

Let J be the set of Hermitian matrices with respect to this involution. In other words, J consists of all matrices of the form

$$a = \begin{pmatrix} \alpha_1 & x_3 & \gamma_1^{-1}\gamma_3 x_2^* \\ \gamma_2^{-1}\gamma_1 x_3^* & \alpha_2 & x_1 \\ x_2 & \gamma_3^{-1}\gamma_2 x_1^* & \alpha_3 \end{pmatrix},$$

where $\alpha_1, \alpha_2, \alpha_3 \in K$ and $x_1, x_2, x_3 \in C$. Then J is closed under the multiplication $ab = \frac{1}{2}(a \cdot b + b \cdot a)$, where $a \cdot b$ is the usual matrix multiplication in $M(3, C)$. With respect to this multiplication, J is a Jordan algebra (see [70, 144]). The following picture shows how the three scalar and the three octonion components fit in the 27-dimensional representation V:



Now the usual determinant expansion

$$\det(a) = \alpha_1\alpha_2\alpha_3 + T(x_1x_2x_3) - \alpha_1\gamma_3^{-1}\gamma_2 N(x_1) - \alpha_2\gamma_1^{-1}\gamma_3 N(x_2) - \alpha_3\gamma_2^{-1}\gamma_1 N(x_3)$$

defines a cubic form on J. However, it is usually more convenient to consider not the form det itself but its polarization to a trilinear form F on J such that $F(x, x, x) = \det(x)$.

12. Invariant cubic form, continued

Yet another construction of the form, where the symmetry with respect to the whole Weyl group $W(\mathrm{E}_6)$ is obvious, and a proof of identification, which does not use anything beyond elementary linear algebra, are sketched in [158]. This construction is discussed in somewhat more detail in [162], see also [6].

This construction is based on the known fact that the monomials of the invariant cubic form constitute one orbit with respect to the action of the Weyl group $W(\mathrm{E}_6)$. We call a triple of weights (λ, μ, ν) of the representation $V(\varpi_1)$ a *triad*, provided that λ, μ, and ν are pairwise orthogonal roots of E_7 or, what is the same, their pairwise differences $\lambda - \mu$, $\lambda - \nu$, and $\mu - \nu$ are not roots. In Aschbacher's terminology [31], a triple of weights (λ, μ, ν) is a triad if and only if the corresponding weight vectors v^λ, v^μ, and v^ν generate a *special plane*.

Denote by Θ the set of all triads, $|\Theta| = 27 \cdot 10$. Then the trilinear form F takes the values

$$F(v^\lambda, v^\mu, v^\nu) = \begin{cases} \pm 1 & \text{if } (\lambda, \mu, \nu) \in \Theta, \\ 0 & \text{otherwise,} \end{cases}$$

and we need only specify the signs.

We fix the sign in such a way that $F(v^{\lambda_0}, v^{\mu_0}, v^{\nu_0}) = 1$ for the *distinguished triad*

$$(\lambda_0, \mu_0, \nu_0) = \left(\begin{smallmatrix}000001\\0\end{smallmatrix}, \begin{smallmatrix}012221\\1\end{smallmatrix}, \begin{smallmatrix}234321\\2\end{smallmatrix}\right).$$

Then for any other triad (λ, μ, ν), the sum

$$\lambda + \mu + \nu = \lambda_0 + \mu_0 + \nu_0 = \begin{smallmatrix}246543\\3\end{smallmatrix}$$

is orthogonal to the fundamental roots $\alpha_1, \dots, \alpha_6$. This means that for any fundamental reflection $w_\alpha \in W(\mathrm{E}_6)$, one has the following alternative:

$\circ$ either $w_\alpha(\lambda, \mu, \nu) = (\lambda, \mu, \nu)$

$\circ$ or precisely two of the weights λ, μ, ν are moved by the fundamental reflection w_α, and in opposite directions, say,

$$w_\alpha(\lambda) = \lambda + \alpha, \qquad w_\alpha(\mu) = \mu - \alpha, \qquad w_\alpha(\nu) = \nu.$$

Looking at the signed base $\pm v^\lambda$ with respect to the action of the *extended* Weyl group, we see that, under the action of $w_\alpha(1)$, either the triple v^λ, v^μ, v^ν is invariant or it is transformed to *another* triple with exactly *one* sign change. Thus, we get the following result stated in [162, § 3] in a slightly different language.

Lemma. *In the invariant trilinear form F, one has*

$$F(v^\lambda, v^\mu, v^\nu) = \mathrm{sign}(w),$$

where w is the shortest element of the Weyl group $W(\mathrm{E}_6)$ satisfying the relation

$$w(\lambda_0, \mu_0, \nu_0) = (\lambda, \mu, \nu).$$

For practical purposes, it is easier to implement this algorithm recursively, rather than by looking at all elements of the Weyl group. Namely, at each step we just add the triads obtained by one fundamental reflection from the triads generated in the previous step, changing the sign every time a new triad is generated.

The actual code may look as follows. First, we specify a distinguished triad *together with a sign*:

```
distriple={Sort[Part[minimalE6insideE7,1,13,27]],1}
```

Now we proceed as follows: we pass over all fundamental roots and over all permutations of the triad (λ, μ, ν) and verify if any new triads occur as we go along. At the next step, we apply the same procedure to all triads obtained in the previous step. For example, the distinguished triad generates the following two:

$$w_{\alpha_1}(\lambda_0, \mu_0, \nu_0) = \left(\begin{matrix}000001\\0\end{matrix}, \begin{matrix}112221\\1\end{matrix}, \begin{matrix}134321\\2\end{matrix}\right), w_{\alpha_6}(\lambda_0, \mu_0, \nu_0) = \left(\begin{matrix}000011\\0\end{matrix}, \begin{matrix}012211\\1\end{matrix}, \begin{matrix}234321\\2\end{matrix}\right).$$

Both of them arise with negative sign, etc.

The list `triples` of all triads together with their signs may be generated as follows. Again, since this list is calculated exactly once, we are not concerned about efficiency of this code.

```
triples=Block[{trip={distriple},newtriple},
                        For[i=1,i<=Length[trip],Outer[
            If[MemberQ[minimalE6insideE7,trip[[i,#1[[1]]]]-#2]&&
                MemberQ[minimalE6insideE7,trip[[i,#1[[2]]]]+#2],
            newtriple=Join[Sort[
{trip[[i,#1[[1]]]]-#2,trip[[i,#1[[2]]]]+#2,trip[[i,#1[[3]]]]}],
                                        {-trip[[i,4]]}];
            If[!MemberQ[trip,newtriple],
                              AppendTo[trip,newtriple]]]&,
            Permutations[1,2,3],Most[rootbaseE7],1];i++],
                                        Return[trip]]
```

Since at every step we *sort* the triads obtained, this function returns the list Θ_0 of 45 *unordered* triads $\{\lambda, \mu, \nu\}$. To get the list Θ of all 270 triads, we need to apply to every member of the list Θ_0 all permutations of its first three parts.

The list Θ_0 of unordered triads is exactly what we need to specify the cubic form. By summing over the *ordered* triads, we get an extra coefficient 6, causing trouble in characteristics 2 and 3.

Table 11 consists of two parts. First, there is the list of triads written as monomials of the cubic form. It is generated by a search for the corresponding weights on the list `minimalE6insideE7`. Second, there is the list of its partial derivatives. They were calculated by an ingenuous application of `D`.

In Tables 14 and 17 we give cubic forms and their partial derivatives for the A_5-numbering and the D_5-numbering of weights. To construct these tables, as in Sec. 9, we simply converted the list of unordered triads to the list of monomials by looking at their positions on the list `minimalE6branchA5` or, respectively, on the list `minimalE6branchD5`, rather than on the list `minimalE6insideE7`, as before.

In addition to the papers quoted in the text so far, in the bibliography we also list some further papers where direct calculations with exceptional Chevalley groups in minimal representations are performed – especially with those of types E_6 and/or E_7. In particular, we quote papers containing *explicit* calculations of *some* of the structure constants, root elements, invariants, equations, orbits, etc. Mostly we cite papers containing computer-aided calculations, but also some of the *most important* papers, where such explicit calculations are applied to problems of STRUCTURE THEORY[2]: description of subgroups, decompositions, conjugacy classes, generation, K-theory, etc. Our bibliography is *highly* selective and deliberately makes no claim to completeness. With very few exceptions we do not cite papers devoted to the representation theory of exceptional groups as such, to their applications in algebraic geometry or number theory, to automorphic functions, combinatorial geometries, cohomology, etc., as well as innumerable articles published in physical journals.

TABLE 1. Positive roots of E_6

1	$\substack{10000\\0} = (0,-1,1,0,0,0,0)$	$\substack{00000\\1} = (1,1,1,1,0,0,0)$
	$\substack{01000\\0} = (0,0,-1,1,0,0,0)$	$\substack{00100\\0} = (0,0,0,-1,1,0,0)$
	$\substack{00010\\0} = (0,0,0,0,-1,1,0)$	$\substack{00001\\0} = (0,0,0,0,0,-1,1)$
2	$\substack{11000\\0} = (0,-1,0,1,0,0,0)$	$\substack{00100\\1} = (1,1,1,0,1,0,0)$
	$\substack{01100\\0} = (0,0,-1,0,1,0,0)$	$\substack{00110\\0} = (1,1,0,1,1,0,0)$
	$\substack{00011\\0} = (0,0,0,0,-1,0,1)$	
3	$\substack{11100\\0} = (0,-1,0,0,1,0,0)$	$\substack{01100\\1} = (1,1,0,1,1,0,0)$
	$\substack{00110\\1} = (1,1,1,0,0,1,0)$	$\substack{01110\\0} = (0,0,-1,0,0,1,0)$
	$\substack{00111\\0} = (0,0,0,-1,0,0,1)$	
4	$\substack{11100\\1} = (1,0,1,1,1,0,0)$	$\substack{11110\\0} = (0,-1,0,0,0,1,0)$
	$\substack{01110\\1} = (1,1,0,1,0,1,0)$	$\substack{00111\\1} = (1,1,1,0,0,0,1)$
	$\substack{01111\\0} = (0,0,-1,0,0,0,1)$	
5	$\substack{11110\\1} = (1,0,1,1,0,1,0)$	$\substack{11111\\0} = (0,-1,0,0,0,0,1)$
	$\substack{01210\\1} = (1,1,0,0,1,1,0)$	$\substack{01111\\1} = (1,1,0,1,0,0,1)$

[2]The first-named author started this work, as well as paper [163], in early 1990s, mostly at Milano and Bielefeld, partly at Cambridge and Bar-Ilan. I am grateful to my Italian and German friends, especially to Lino Di Martino and Anthony Bak, for their unfailing support over the hard years. The primary motivation for these calculations was my joint papers with Di Martino and Eugene Plotkin. These papers urged me to develop algorithms for explicit calculations in exceptional groups and to tabulate necessary data. I had a privilege to thoroughly discuss these issues with some of the best experts in exceptional groups, including Michael Aschbacher, Roger Carter, Arjeh Cohen, Bruce Cooperstein, Igor Frenkel, and Gary Seitz. This discussion helped me to clarify many aspects, which are otherwise hard to extract from the existing literature. Over the last years a new incentive to return to these issues and to finalize them has come from the papers of my students, where remarkable progress in some of these directions has been obtained; in particular, see [7–9], [13, 15, 19, 127]. I am deeply grateful to all of them. N.V.

6	$\substack{11210\\1} = (1,0,1,0,1,1,0)$	$\substack{11111\\1} = (1,0,1,1,0,0,1)$
	$\substack{01211\\1} = (1,1,0,0,1,0,1)$	
7	$\substack{12210\\1} = (1,0,0,1,1,1,0)$	$\substack{11211\\1} = (1,0,1,0,1,0,1)$
	$\substack{01221\\1} = (1,1,0,0,0,1,1)$	
8	$\substack{12211\\1} = (1,0,0,1,1,0,1)$	$\substack{11221\\1} = (1,0,1,0,0,1,1)$
9	$\substack{12221\\1} = (1,0,0,1,0,1,1)$	
10	$\substack{12321\\1} = (1,0,0,0,1,1,1)$	
11	$\substack{12321\\2} = (2,1,1,1,1,1,1)$	

Table 2. Positive roots of E_7

1	$\substack{100000\\0} = (0,-1,1,0,0,0,0,0)$	$\substack{000000\\1} = (1,1,1,1,0,0,0,0)$
	$\substack{010000\\0} = (0,0,-1,1,0,0,0,0)$	$\substack{001000\\0} = (0,0,0,-1,1,0,0,0)$
	$\substack{000100\\0} = (0,0,0,0,-1,1,0,0)$	$\substack{000010\\0} = (0,0,0,0,0,-1,1,0)$
	$\substack{000001\\0} = (0,0,0,0,0,0,-1,1)$	
2	$\substack{110000\\0} = (0,-1,0,1,0,0,0,0)$	$\substack{001000\\1} = (1,1,1,0,1,0,0,0)$
	$\substack{011000\\0} = (0,0,-1,0,1,0,0,0)$	$\substack{001100\\0} = (0,0,0,-1,0,1,0,0)$
	$\substack{000110\\0} = (0,0,0,0,-1,0,1,0)$	$\substack{000011\\0} = (0,0,0,0,0,-1,0,1)$
3	$\substack{111000\\0} = (0,-1,0,0,1,0,0,0)$	$\substack{011000\\1} = (1,1,0,1,1,0,0,0)$
	$\substack{001100\\1} = (1,1,1,0,0,1,0,0)$	$\substack{011100\\0} = (0,0,-1,0,0,1,0,0)$
	$\substack{001110\\0} = (0,0,0,-1,0,0,1,0)$	$\substack{000111\\0} = (0,0,0,0,-1,0,0,1)$
4	$\substack{111000\\1} = (1,0,1,1,1,0,0,0)$	$\substack{111100\\0} = (0,-1,0,0,0,1,0,0)$
	$\substack{011100\\1} = (1,1,0,1,0,1,0,0)$	$\substack{001110\\1} = (1,1,1,0,0,0,1,0)$
	$\substack{011110\\0} = (0,0,-1,0,0,0,1,0)$	$\substack{001111\\0} = (0,0,0,-1,0,0,0,1)$
5	$\substack{111100\\1} = (1,0,1,1,0,1,0,0)$	$\substack{111110\\0} = (0,-1,0,0,0,0,1,0)$
	$\substack{012100\\1} = (1,1,0,0,1,1,0,0)$	$\substack{011110\\1} = (1,1,0,1,0,0,1,0)$
	$\substack{001111\\1} = (1,1,1,0,0,0,0,1)$	$\substack{011111\\0} = (0,0,-1,0,0,0,0,1)$

6	$\genfrac{}{}{0pt}{}{112100}{1} = (1,0,1,0,1,1,0,0)$	$\genfrac{}{}{0pt}{}{111110}{1} = (1,0,1,1,0,0,1,0)$
	$\genfrac{}{}{0pt}{}{111111}{0} = (0,-1,0,0,0,0,0,1)$	$\genfrac{}{}{0pt}{}{012110}{1} = (1,1,0,0,1,0,1,0)$
	$\genfrac{}{}{0pt}{}{011111}{1} = (1,1,0,1,0,0,0,1)$	
7	$\genfrac{}{}{0pt}{}{122100}{1} = (1,0,0,1,1,1,0,0)$	$\genfrac{}{}{0pt}{}{112110}{1} = (1,0,1,0,1,0,1,0)$
	$\genfrac{}{}{0pt}{}{111111}{1} = (1,0,1,1,0,0,0,1)$	$\genfrac{}{}{0pt}{}{012210}{1} = (1,1,0,0,0,1,1,0)$
	$\genfrac{}{}{0pt}{}{012111}{1} = (1,1,0,0,1,0,0,1)$	
8	$\genfrac{}{}{0pt}{}{122110}{1} = (1,0,0,1,1,0,1,0)$	$\genfrac{}{}{0pt}{}{112210}{1} = (1,0,1,0,0,1,1,0)$
	$\genfrac{}{}{0pt}{}{112111}{1} = (1,0,1,0,1,0,0,1)$	$\genfrac{}{}{0pt}{}{012211}{1} = (1,1,0,0,0,1,0,1)$
9	$\genfrac{}{}{0pt}{}{122210}{1} = (1,0,0,1,0,1,1,0)$	$\genfrac{}{}{0pt}{}{122111}{1} = (1,0,0,1,1,0,0,1)$
	$\genfrac{}{}{0pt}{}{112211}{1} = (1,0,1,0,0,1,0,1)$	$\genfrac{}{}{0pt}{}{012221}{1} = (1,1,0,0,0,0,1,1)$
10	$\genfrac{}{}{0pt}{}{123210}{1} = (1,0,0,0,1,1,1,0)$	$\genfrac{}{}{0pt}{}{122211}{1} = (1,0,0,1,0,1,0,1)$
	$\genfrac{}{}{0pt}{}{112221}{1} = (1,0,1,0,0,0,1,1)$	
11	$\genfrac{}{}{0pt}{}{123210}{2} = (2,1,1,1,1,1,1,0)$	$\genfrac{}{}{0pt}{}{123211}{1} = (1,0,0,0,1,1,0,1)$
	$\genfrac{}{}{0pt}{}{122221}{1} = (1,0,0,1,0,0,1,1)$	
12	$\genfrac{}{}{0pt}{}{123211}{2} = (2,1,1,1,1,1,0,1)$	$\genfrac{}{}{0pt}{}{123221}{1} = (1,0,0,0,1,0,1,1)$
13	$\genfrac{}{}{0pt}{}{123221}{2} = (2,1,1,1,1,0,1,1)$	$\genfrac{}{}{0pt}{}{123321}{1} = (1,0,0,0,0,1,1,1)$
14	$\genfrac{}{}{0pt}{}{123321}{2} = (2,1,1,1,0,1,1,1)$	
15	$\genfrac{}{}{0pt}{}{124321}{2} = (2,1,1,0,1,1,1,1)$	
16	$\genfrac{}{}{0pt}{}{134321}{2} = (2,1,0,1,1,1,1,1)$	
17	$\genfrac{}{}{0pt}{}{234321}{2} = (2,0,1,1,1,1,1,1)$	

TABLE 3. Weights of $V(\varpi_1)$

8	$\frac{1}{3}\binom{45642}{3}$	$=$	$\genfrac{}{}{0pt}{}{10000}{0}$	$=$	$(1,-\frac{1}{3},\frac{2}{3},\frac{2}{3},\frac{2}{3},\frac{2}{3},\frac{2}{3})$
7	$\frac{1}{3}\binom{15642}{3}$	$=$	$\genfrac{}{}{0pt}{}{\bar{1}1000}{0}$	$=$	$(1,\frac{2}{3},-\frac{1}{3},\frac{2}{3},\frac{2}{3},\frac{2}{3},\frac{2}{3})$
6	$\frac{1}{3}\binom{12642}{3}$	$=$	$\genfrac{}{}{0pt}{}{0\bar{1}100}{0}$	$=$	$(1,\frac{2}{3},\frac{2}{3},-\frac{1}{3},\frac{2}{3},\frac{2}{3},\frac{2}{3})$
5	$\frac{1}{3}\binom{12342}{3}$	$=$	$\genfrac{}{}{0pt}{}{00\bar{1}10}{1}$	$=$	$(1,\frac{2}{3},\frac{2}{3},\frac{2}{3},-\frac{1}{3},\frac{2}{3},\frac{2}{3})$

4	$\frac{1}{3}\binom{12342}{0}$	$=$	$\begin{matrix}00010\\ \bar{1}\end{matrix}$	$=$	$(0,-\frac{1}{3},-\frac{1}{3},-\frac{1}{3},-\frac{1}{3},\frac{2}{3},\frac{2}{3})$
	$\frac{1}{3}\binom{12312}{3}$	$=$	$\begin{matrix}000\bar{1}1\\ 1\end{matrix}$	$=$	$(1,\frac{2}{3},\frac{2}{3},\frac{2}{3},\frac{2}{3},-\frac{1}{3},\frac{2}{3})$
3	$\frac{1}{3}\binom{12312}{0}$	$=$	$\begin{matrix}001\bar{1}1\\ \bar{1}\end{matrix}$	$=$	$(0,-\frac{1}{3},-\frac{1}{3},-\frac{1}{3},\frac{2}{3},-\frac{1}{3},\frac{2}{3})$
	$\frac{1}{3}\binom{1231\bar{1}}{3}$	$=$	$\begin{matrix}0000\bar{1}\\ 1\end{matrix}$	$=$	$(1,\frac{2}{3},\frac{2}{3},\frac{2}{3},\frac{2}{3},\frac{2}{3},-\frac{1}{3})$
2	$\frac{1}{3}\binom{12012}{0}$	$=$	$\begin{matrix}01\bar{1}01\\ 0\end{matrix}$	$=$	$(0,-\frac{1}{3},-\frac{1}{3},\frac{2}{3},-\frac{1}{3},-\frac{1}{3},\frac{2}{3})$
	$\frac{1}{3}\binom{1231\bar{1}}{0}$	$=$	$\begin{matrix}0010\bar{1}\\ \bar{1}\end{matrix}$	$=$	$(0,-\frac{1}{3},-\frac{1}{3},-\frac{1}{3},\frac{2}{3},\frac{2}{3},-\frac{1}{3})$
1	$\frac{1}{3}\binom{1\bar{1}012}{0}$	$=$	$\begin{matrix}1\bar{1}001\\ 0\end{matrix}$	$=$	$(0,-\frac{1}{3},\frac{2}{3},-\frac{1}{3},-\frac{1}{3},-\frac{1}{3},\frac{2}{3})$
	$\frac{1}{3}\binom{1201\bar{1}}{0}$	$=$	$\begin{matrix}01\bar{1}1\bar{1}\\ 0\end{matrix}$	$=$	$(0,-\frac{1}{3},-\frac{1}{3},\frac{2}{3},-\frac{1}{3},\frac{2}{3},-\frac{1}{3})$
0	$\frac{1}{3}\binom{\overline{21}012}{0}$	$=$	$\begin{matrix}\bar{1}0001\\ 0\end{matrix}$	$=$	$(0,\frac{2}{3},-\frac{1}{3},-\frac{1}{3},-\frac{1}{3},-\frac{1}{3},\frac{2}{3})$
	$\frac{1}{3}\binom{1\bar{1}01\bar{1}}{0}$	$=$	$\begin{matrix}1\bar{1}01\bar{1}\\ 0\end{matrix}$	$=$	$(0,-\frac{1}{3},\frac{2}{3},-\frac{1}{3},-\frac{1}{3},\frac{2}{3},-\frac{1}{3})$
	$\frac{1}{3}\binom{120\overline{21}}{0}$	$=$	$\begin{matrix}010\bar{1}0\\ 0\end{matrix}$	$=$	$(0,-\frac{1}{3},-\frac{1}{3},\frac{2}{3},\frac{2}{3},-\frac{1}{3},-\frac{1}{3})$
−1	$\frac{1}{3}\binom{\overline{21}01\bar{1}}{0}$	$=$	$\begin{matrix}\bar{1}001\bar{1}\\ 0\end{matrix}$	$=$	$(0,\frac{2}{3},-\frac{1}{3},-\frac{1}{3},-\frac{1}{3},\frac{2}{3},-\frac{1}{3})$
	$\frac{1}{3}\binom{1\bar{1}0\overline{21}}{0}$	$=$	$\begin{matrix}1\bar{1}1\bar{1}0\\ 0\end{matrix}$	$=$	$(0,-\frac{1}{3},\frac{2}{3},-\frac{1}{3},\frac{2}{3},-\frac{1}{3},-\frac{1}{3})$
−2	$\frac{1}{3}\binom{\overline{21}0\overline{21}}{0}$	$=$	$\begin{matrix}\bar{1}01\bar{1}0\\ 0\end{matrix}$	$=$	$(0,\frac{2}{3},-\frac{1}{3},-\frac{1}{3},\frac{2}{3},-\frac{1}{3},-\frac{1}{3})$
	$\frac{1}{3}\binom{1\overline{1321}}{0}$	$=$	$\begin{matrix}10\bar{1}00\\ 1\end{matrix}$	$=$	$(0,-\frac{1}{3},\frac{2}{3},\frac{2}{3},-\frac{1}{3},-\frac{1}{3},-\frac{1}{3})$
−3	$\frac{1}{3}\binom{\overline{21321}}{0}$	$=$	$\begin{matrix}\bar{1}1\bar{1}00\\ 1\end{matrix}$	$=$	$(0,\frac{2}{3},-\frac{1}{3},\frac{2}{3},-\frac{1}{3},-\frac{1}{3},-\frac{1}{3})$
	$\frac{1}{3}\binom{1\overline{1321}}{\bar{3}}$	$=$	$\begin{matrix}10000\\ \bar{1}\end{matrix}$	$=$	$(-1,-\frac{4}{3},-\frac{1}{3},-\frac{1}{3},-\frac{1}{3},-\frac{1}{3},-\frac{1}{3})$
−4	$\frac{1}{3}\binom{\overline{21321}}{\bar{3}}$	$=$	$\begin{matrix}\bar{1}1000\\ \bar{1}\end{matrix}$	$=$	$(-1,-\frac{1}{3},-\frac{4}{3},-\frac{1}{3},-\frac{1}{3},-\frac{1}{3},-\frac{1}{3})$
	$\frac{1}{3}\binom{\overline{24321}}{0}$	$=$	$\begin{matrix}0\bar{1}000\\ 1\end{matrix}$	$=$	$(0,\frac{2}{3},\frac{2}{3},-\frac{1}{3},-\frac{1}{3},-\frac{1}{3},-\frac{1}{3})$
−5	$\frac{1}{3}\binom{\overline{24321}}{\bar{3}}$	$=$	$\begin{matrix}0\bar{1}100\\ \bar{1}\end{matrix}$	$=$	$(-1,-\frac{1}{3},-\frac{1}{3},-\frac{4}{3},-\frac{1}{3},-\frac{1}{3},-\frac{1}{3})$
−6	$\frac{1}{3}\binom{\overline{24621}}{\bar{3}}$	$=$	$\begin{matrix}00\bar{1}10\\ 0\end{matrix}$	$=$	$(-1,-\frac{1}{3},-\frac{1}{3},-\frac{1}{3},-\frac{4}{3},-\frac{1}{3},-\frac{1}{3})$

−7	$\frac{1}{3}\begin{pmatrix}\overline{24651}\\ \overline{3}\end{pmatrix}$	$=$	$\begin{smallmatrix}000\bar{1}1\\ 0\end{smallmatrix}$	$=$	$(-1,-\frac{1}{3},-\frac{1}{3},-\frac{1}{3},-\frac{1}{3},-\frac{4}{3},-\frac{1}{3})$
−8	$\frac{1}{3}\begin{pmatrix}\overline{24654}\\ \overline{3}\end{pmatrix}$	$=$	$\begin{smallmatrix}0000\bar{1}\\ 0\end{smallmatrix}$	$=$	$(-1,-\frac{1}{3},-\frac{1}{3},-\frac{1}{3},-\frac{1}{3},-\frac{1}{3},-\frac{4}{3})$

TABLE 4. Weights of $V(\varpi_1)$ as roots of E_7

1	$\begin{smallmatrix}000001\\ 0\end{smallmatrix}=(0,0,0,0,0,0,-1,1)$	
2	$\begin{smallmatrix}000011\\ 0\end{smallmatrix}=(0,0,0,0,0,-1,0,1)$	
3	$\begin{smallmatrix}000111\\ 0\end{smallmatrix}=(0,0,0,0,-1,0,0,1)$	
4	$\begin{smallmatrix}001111\\ 0\end{smallmatrix}=(0,0,0,-1,0,0,0,1)$	
5	$\begin{smallmatrix}001111\\ 1\end{smallmatrix}=(1,1,1,0,0,0,0,1)$	$\begin{smallmatrix}011111\\ 0\end{smallmatrix}=(0,0,-1,0,0,0,0,1)$
6	$\begin{smallmatrix}111111\\ 0\end{smallmatrix}=(0,-1,0,0,0,0,0,1)$	$\begin{smallmatrix}011111\\ 1\end{smallmatrix}=(1,1,0,1,0,0,0,1)$
7	$\begin{smallmatrix}111111\\ 1\end{smallmatrix}=(1,0,1,1,0,0,0,1)$	$\begin{smallmatrix}012111\\ 1\end{smallmatrix}=(1,1,0,0,1,0,0,1)$
8	$\begin{smallmatrix}112111\\ 1\end{smallmatrix}=(1,0,1,0,1,0,0,1)$	$\begin{smallmatrix}012211\\ 1\end{smallmatrix}=(1,1,0,0,0,1,0,1)$
9	$\begin{smallmatrix}122111\\ 1\end{smallmatrix}=(1,0,0,1,1,0,0,1)$	$\begin{smallmatrix}112211\\ 1\end{smallmatrix}=(1,0,1,0,0,1,0,1)$
	$\begin{smallmatrix}012221\\ 1\end{smallmatrix}=(1,1,0,0,0,0,1,1)$	
10	$\begin{smallmatrix}122211\\ 1\end{smallmatrix}=(1,0,0,1,0,1,0,1)$	$\begin{smallmatrix}112221\\ 1\end{smallmatrix}=(1,0,1,0,0,0,1,1)$
11	$\begin{smallmatrix}123211\\ 1\end{smallmatrix}=(1,0,0,0,1,1,0,1)$	$\begin{smallmatrix}122221\\ 1\end{smallmatrix}=(1,0,0,1,0,0,1,1)$
12	$\begin{smallmatrix}123211\\ 2\end{smallmatrix}=(2,1,1,1,1,1,0,1)$	$\begin{smallmatrix}123221\\ 1\end{smallmatrix}=(1,0,0,0,1,0,1,1)$
13	$\begin{smallmatrix}123221\\ 2\end{smallmatrix}=(2,1,1,1,1,0,1,1)$	$\begin{smallmatrix}123321\\ 1\end{smallmatrix}=(1,0,0,0,0,1,1,1)$
14	$\begin{smallmatrix}123321\\ 2\end{smallmatrix}=(2,1,1,1,0,1,1,1)$	
15	$\begin{smallmatrix}124321\\ 2\end{smallmatrix}=(2,1,1,0,1,1,1,1)$	
16	$\begin{smallmatrix}134321\\ 2\end{smallmatrix}=(2,1,0,1,1,1,1,1)$	
17	$\begin{smallmatrix}234321\\ 2\end{smallmatrix}=(2,0,1,1,1,1,1,1)$	

Table 5. Structure constants of $G(\mathrm{E}_6, R)$

	1 0 0 0 0 0 1 0 0 0 0 0 1 0 0 0 0 0 1 0 0 0 0 0 1 0 0 0 0 0	0 1 0 0 0 0 0 1 0 0 0 1 0 1 0 0 0 1 1 1 0 0 0 0 1 1 0 0 0 0	0 1 0 0 0 0 0 1 1 0 0 1 1 0 1 0 1 1 1 1 1 0 0 1 1 1 0 0 0 0	0 1 1 0 0 0 1 0 1 1 0 1 1 1 0 1 1 1 1 1 1 0 1 1 1 1 0 0 0 1	0 1 1 0 0 0 1 0 1 1 1 1 1 1 1 1 1 1 2 1 1 1 1 1 1 1 0 1 0 1	1 1 0 1 1 1 1 1 1 1 1 1 1 2 1 2 1 2 2 2 1 1 1 1 1 0 1 1 0 1	0 1 1 1 1 1 1 1 1 1 1 2 1 2 2 2 2 2 2 3 2 1 2 2 2 1 1 1 1 1	1 2 2 3 2 1
100000	0 0 + 0 0	0 0 0 + 0	0 0 + 0 +	0 0 0 + 0	+ 0 0 + +	0 0 + 0 0	+ 0 0 0 0	0
010000	0 0 0 + 0	0 0 0 + +	0 + 0 0 +	+ 0 + 0 0	+ 0 + 0 0	0 0 0 0 0	0 0 0 0 +	0
001000	− 0 0 + 0	0 0 + 0 +	0 0 0 + 0	+ 0 0 0 +	0 0 0 0 0	+ 0 0 0 +	0 0 + 0 0	0
000100	0 − − 0 +	0 − 0 0 0	+ 0 0 0 0	0 0 0 + 0	0 + 0 0 +	0 + 0 0 0	0 0 0 + 0	0
000010	0 0 0 − 0	+ 0 − − 0	0 − − 0 0	0 − 0 0 0	0 0 0 0 0	0 0 + 0 +	0 + 0 0 0	0
000001	0 0 0 0 −	0 0 0 0 −	0 0 0 − −	0 0 − − 0	0 − 0 − 0	− 0 0 − 0	0 0 0 0 0	0
101000	0 0 0 + 0	0 0 + 0 +	0 0 0 + 0	+ 0 0 0 +	0 0 0 − 0	0 0 − 0 0	− 0 0 0 0	0
010100	0 0 − 0 +	0 − 0 0 0	+ 0 0 0 −	0 0 − 0 0	− 0 − 0 0	0 0 0 0 0	0 0 0 + 0	0
001100	− − 0 0 +	0 0 0 0 0	+ 0 0 − 0	0 0 0 0 −	0 + 0 0 0	0 + 0 0 0	0 0 − 0 0	0
000110	0 − − 0 0	+ − 0 0 0	0 0 − 0 0	0 − 0 0 0	0 0 0 0 −	0 − 0 0 0	0 + 0 0 0	0
000011	0 0 0 − 0	0 0 − − 0	0 − − 0 0	0 − 0 0 0	0 0 0 − 0	− 0 0 − 0	0 0 0 0 0	0
101100	0 − 0 0 +	0 0 0 0 0	+ 0 0 − 0	0 0 0 − −	0 0 0 0 −	0 0 0 0 0	+ 0 0 0 0	0
011100	− 0 0 0 +	0 0 0 0 +	+ 0 0 0 0	+ 0 − 0 0	0 0 − 0 0	0 0 0 0 0	0 0 − 0 0	0
010110	0 0 − 0 0	+ − 0 + 0	0 + 0 0 0	0 0 0 0 0	+ 0 + 0 0	0 0 0 0 0	0 + 0 0 0	0
001110	− − 0 0 0	+ 0 + 0 0	0 0 0 0 0	0 − 0 0 +	0 0 0 0 0	0 − 0 0 −	0 0 0 0 0	0
000111	0 − − 0 0	0 − 0 0 0	0 0 − 0 0	0 − 0 + 0	0 + 0 0 0	0 0 0 − 0	0 0 0 0 0	0
111100	0 0 0 0 +	0 0 0 0 +	+ 0 0 0 +	+ 0 0 0 0	+ 0 0 0 0	0 0 0 0 0	+ 0 0 0 0	0
101110	0 − 0 0 0	+ 0 + 0 0	0 0 + 0 0	0 0 0 0 +	0 0 0 0 +	0 0 + 0 0	0 0 0 0 0	0
011110	− 0 0 − 0	+ 0 0 0 0	0 + 0 0 0	− 0 0 0 0	0 0 + 0 0	0 0 0 0 −	0 0 0 0 0	0
010111	0 0 − 0 0	0 − 0 + 0	0 + 0 0 −	0 0 − 0 0	0 0 0 0 0	0 0 0 − 0	0 0 0 0 0	0
001111	− − 0 0 0	0 0 + 0 0	0 0 0 − 0	0 − 0 0 0	0 + 0 0 0	+ 0 0 0 0	0 0 0 0 0	0
111110	0 0 0 − 0	+ 0 0 − 0	0 0 0 0 0	− 0 0 0 0	− 0 0 0 0	0 0 + 0 0	0 0 0 0 0	0
101111	0 − 0 0 0	0 0 + 0 0	0 0 + − 0	0 0 0 − 0	0 0 0 − 0	0 0 0 0 0	0 0 0 0 0	0
011210	− 0 0 0 0	+ + 0 0 0	+ 0 0 0 0	0 0 0 0 0	0 0 + 0 0	0 + 0 0 0	0 0 0 0 0	0
011111	− 0 0 − 0	0 0 0 0 +	0 + 0 0 0	0 0 − 0 0	0 0 0 0 0	+ 0 0 0 0	0 0 0 0 0	0
111210	0 0 − 0 0	+ 0 0 0 0	+ 0 0 0 0	0 0 0 0 0	− 0 0 0 −	0 0 0 0 0	0 0 0 0 0	0
111111	0 0 0 − 0	0 0 0 − +	0 0 0 0 +	0 0 0 0 0	0 0 0 − 0	0 0 0 0 0	0 0 0 0 0	0
011211	− 0 0 0 −	0 + 0 0 0	0 0 0 0 0	0 0 − 0 0	0 − 0 0 0	0 0 0 0 0	0 0 0 0 0	0
112210	0 0 0 0 0	+ 0 0 0 0	+ 0 0 0 0	+ 0 0 0 +	0 0 0 0 0	0 0 0 0 0	0 0 0 0 0	0
111211	0 0 − 0 −	0 0 0 0 0	0 0 0 0 +	0 0 0 + 0	0 0 0 0 0	0 0 0 0 0	0 0 0 0 0	0
011221	− 0 0 0 0	0 + 0 0 0	0 − 0 0 0	0 − 0 0 0	0 0 0 0 0	0 0 0 0 0	0 0 0 0 0	0
112211	0 0 0 0 −	0 0 0 0 −	0 0 0 − 0	0 0 0 0 0	0 0 0 0 0	0 0 0 0 0	0 0 0 0 0	0
111221	0 0 − 0 0	0 0 0 + 0	0 0 + 0 0	0 0 0 0 0	0 0 0 0 0	0 0 0 0 0	0 0 0 0 0	0
112221	0 0 0 − 0	0 0 − 0 0	0 0 0 0 0	0 0 0 0 0	0 0 0 0 0	0 0 0 0 0	0 0 0 0 0	0
112321	0 − 0 0 0	0 0 0 0 0	0 0 0 0 0	0 0 0 0 0	0 0 0 0 0	0 0 0 0 0	0 0 0 0 0	0
122321	0 0 0 0 0	0 0 0 0 0	0 0 0 0 0	0 0 0 0 0	0 0 0 0 0	0 0 0 0 0	0 0 0 0 0	0

Table 6. Structure constants of $V(\varpi_1)$

	0000001	0000011	0000111	0001111	0011111	0101111	1011111	0111111	1111111	0112111	1112111	0112211	1122111	1112211	0112221	1122211	1112221	1123211	1122221	1223211	1123221	1223221	1123321	1223321	1224321	1234321	2234321
100000	0	0	0	0	0	+	0	+	0	+	0	+	0	0	+	0	0	0	0	0	0	0	0	0	0	+	0
010000	0	0	0	+	0	+	+	0	0	0	0	0	0	0	0	0	0	+	0	0	+	0	+	0	0	0	0
001000	0	0	0	+	+	0	0	0	0	0	+	0	0	+	0	0	+	0	0	0	0	0	0	0	+	0	0
000100	0	0	+	0	0	0	0	+	+	0	0	0	0	0	0	+	0	0	+	0	0	0	0	+	0	0	0
000010	0	+	0	0	0	0	0	0	0	+	+	0	+	0	0	0	0	0	0	0	+	+	0	0	0	0	0
000001	+	0	0	0	0	0	0	0	0	0	0	+	0	+	0	+	0	+	0	+	0	0	0	0	0	0	0
101000	0	0	0	+	+	0	0	0	0	−	0	−	0	0	−	0	0	0	0	0	0	0	0	0	+	0	0
010100	0	0	+	0	0	−	−	0	0	0	0	0	0	0	0	+	0	0	+	0	0	0	−	0	0	0	0
001100	0	0	+	0	−	0	0	0	+	0	0	0	0	−	0	0	−	0	0	0	0	0	0	+	0	0	0
000110	0	+	0	0	0	0	0	−	−	0	0	0	+	0	0	0	0	0	−	0	0	+	0	0	0	0	0
000011	+	0	0	0	0	0	0	0	0	−	−	0	−	0	0	0	0	+	0	+	0	0	0	0	0	0	0
101100	0	0	+	0	−	0	0	−	0	0	0	+	0	0	+	0	0	0	0	0	0	0	0	+	0	0	0
011100	0	0	+	+	0	0	−	0	0	0	0	0	0	−	0	0	−	0	0	0	0	0	−	0	0	0	0
010110	0	+	0	0	0	+	+	0	0	0	0	0	+	0	0	0	0	0	−	0	−	0	0	0	0	0	0
001110	0	+	0	0	+	0	0	0	−	0	−	0	0	0	0	0	+	0	0	0	0	+	0	0	0	0	0
000111	+	0	0	0	0	0	0	+	+	0	0	0	−	0	0	−	0	0	0	+	0	0	0	0	0	0	0
111100	0	0	+	+	0	+	0	0	0	0	0	+	0	0	+	0	0	0	0	0	0	0	−	0	0	0	0
101110	0	+	0	0	+	0	0	+	0	+	0	0	0	0	−	0	0	0	0	0	0	+	0	0	0	0	0
011110	0	+	0	−	0	0	+	0	0	0	−	0	0	0	0	0	+	0	0	0	−	0	0	0	0	0	0
010111	+	0	0	0	0	−	−	0	0	0	0	0	−	0	0	−	0	−	0	0	0	0	0	0	0	0	0
001111	+	0	0	0	−	0	0	0	+	0	+	0	0	+	0	0	0	0	0	+	0	0	0	0	0	0	0
111110	0	+	0	−	0	−	0	0	0	+	0	0	0	0	−	0	0	0	0	0	−	0	0	0	0	0	0
101111	+	0	0	0	−	0	0	−	0	−	0	−	0	0	0	0	0	0	0	+	0	0	0	0	0	0	0
011210	0	+	+	0	0	0	+	0	+	0	0	0	0	0	0	0	+	0	+	0	0	0	0	0	0	0	0
011111	+	0	0	+	0	0	−	0	0	0	+	0	0	+	0	0	0	−	0	0	0	0	0	0	0	0	0
111210	0	+	+	0	0	−	0	−	0	0	0	0	0	0	−	0	0	0	+	0	0	0	0	0	0	0	0
111111	+	0	0	+	0	+	0	0	0	−	0	−	0	0	0	0	0	−	0	0	0	0	0	0	0	0	0
011211	+	0	−	0	0	0	−	0	−	0	0	0	0	+	0	+	0	0	0	0	0	0	0	0	0	0	0
112210	0	+	+	+	+	0	0	0	0	0	0	0	0	0	−	0	−	0	0	0	0	0	0	0	0	0	0
111211	+	0	−	0	0	+	0	+	0	0	0	−	0	0	0	+	0	0	0	0	0	0	0	0	0	0	0
011221	+	+	0	0	0	0	−	0	−	0	−	0	−	0	0	0	0	0	0	0	0	0	0	0	0	0	0
112211	+	0	−	−	−	0	0	0	0	0	0	−	0	−	0	0	0	0	0	0	0	0	0	0	0	0	0
111221	+	+	0	0	0	+	0	+	0	+	0	0	−	0	0	0	0	0	0	0	0	0	0	0	0	0	0
112221	+	+	0	−	−	0	0	0	0	+	+	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
112321	+	+	+	0	−	0	0	−	−	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
122321	+	+	+	+	0	+	+	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

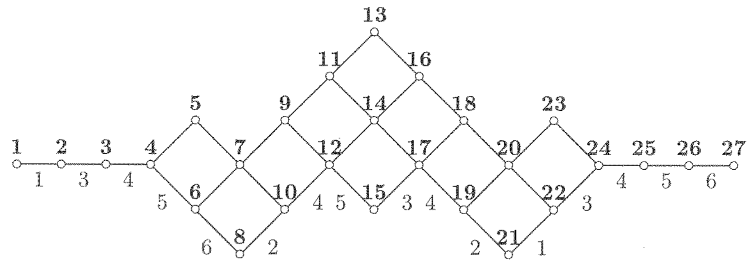
Table 7. Matrix of signs for $V(\varpi_1)$

	0000001	0000011	0000111	0001111	0011111	0101111	1011111	0111111	1111111	0112111	1112111	0112211	1122111	1112211	0112221	1122211	1112221	1123211	1122221	1223211	1123221	1223221	1123321	1223321	1224321	1234321	2234321
0000001	0	+	+	+	+	+	+	+	+	+	+	+	+	+	0	+	0	+	0	+	0	0	0	0	0	0	0
0000011	+	0	+	+	+	+	+	+	+	+	+	0	+	0	+	0	+	0	+	0	+	+	0	0	0	0	0
0000111	+	+	0	+	+	+	+	+	+	0	0	+	0	+	−	+	−	0	−	0	0	0	+	+	0	0	0
0001111	+	+	+	0	+	+	+	0	0	+	+	−	0	−	+	0	+	+	0	0	−	0	−	0	+	0	0
0011111	+	+	+	+	0	0	0	+	+	−	−	+	0	+	−	0	−	0	0	+	0	−	0	−	−	0	0
0101111	+	+	+	+	0	0	+	+	0	−	0	+	+	0	−	−	0	−	+	0	+	0	+	0	0	+	0
1011111	+	+	+	+	0	+	0	0	+	0	−	0	−	+	0	+	−	+	−	0	−	0	−	0	0	0	+
0111111	+	+	+	0	+	+	0	0	+	+	0	−	−	0	+	+	0	0	−	−	0	+	0	+	0	−	0
1111111	+	+	+	0	+	0	+	+	0	0	+	0	+	−	0	−	+	0	+	+	0	−	0	−	0	0	−
0112111	+	+	0	+	−	−	0	+	0	0	+	+	−	0	−	0	0	+	0	+	−	−	0	0	+	+	0
1112111	+	+	0	+	−	0	−	0	+	+	0	0	+	+	0	0	−	−	0	−	+	+	0	0	−	0	+
0112211	+	0	+	−	+	+	0	−	0	+	0	0	0	+	+	−	0	+	0	+	0	0	−	−	−	−	0
1122111	+	+	0	0	0	+	−	−	+	−	+	0	0	0	0	+	0	+	−	+	−	−	0	0	0	−	−
1112211	+	0	+	−	+	0	+	0	−	0	+	+	0	0	0	+	+	−	0	−	0	0	+	+	+	0	−
0112221	0	+	−	+	−	−	0	+	0	−	0	+	0	0	0	0	+	0	−	0	+	+	−	−	−	−	0
1122211	+	0	+	0	0	−	+	+	−	0	0	−	+	+	0	0	0	+	+	+	0	0	−	−	0	+	+
1112221	0	+	−	+	−	0	−	0	+	0	−	0	0	+	+	0	0	0	+	0	−	−	+	+	+	0	−
1123211	+	0	0	+	0	−	+	0	0	+	−	+	+	−	0	+	0	0	0	+	+	0	+	0	−	−	−
1122221	0	+	−	0	0	+	−	−	+	0	0	0	−	0	−	+	+	0	0	0	+	+	−	−	0	+	+
1223211	+	0	0	0	+	0	0	−	+	+	−	+	+	−	0	+	0	+	0	0	0	+	0	+	+	+	+
1123221	0	+	0	−	0	+	−	0	0	−	+	0	−	0	+	0	−	+	+	0	0	+	+	0	−	−	−
1223221	0	+	0	0	−	0	0	+	−	−	+	0	−	0	+	0	−	0	+	+	+	0	0	+	+	+	+
1123321	0	0	+	−	0	+	−	0	0	0	0	−	0	+	−	−	+	+	−	0	+	0	0	+	−	−	−
1223321	0	0	+	0	−	0	0	+	−	0	0	−	0	+	−	−	+	0	−	+	0	+	+	0	+	+	+
1224321	0	0	0	+	−	0	0	0	0	+	−	−	0	+	−	0	+	−	0	+	−	+	−	+	0	+	+
1234321	0	0	0	0	0	+	0	−	0	+	0	−	−	0	−	+	0	−	+	+	−	+	−	+	+	0	+
2234321	0	0	0	0	0	0	+	0	−	0	+	0	−	−	0	+	−	−	+	+	−	+	−	+	+	+	0

Table 8. Numberings of weights: natural, A_5, and D_5

1	234321 2	234321 2	234321 2
2	134321 2	134321 2	134321 2
3	124321 2	124321 2	124321 2
4	123321 2	123321 2	123321 2
5	123321 1	123221 2	123321 1
6	123221 2	123211 2	123221 2
7	123221 1	123321 1	123221 1
8	123211 2	123221 1	123211 2
9	122221 1	122221 1	122221 1
10	123211 1	123211 1	123211 1
11	112221 1	112221 1	112221 1
12	122211 1	122211 1	122211 1
13	012221 1	012221 1	112211 1
14	112211 1	112211 1	122111 1
15	122111 1	122111 1	112111 1
16	012211 1	012211 1	111111 1
17	112111 1	112111 1	111111 0
18	012111 1	012111 1	012221 1
19	111111 1	111111 1	012211 1
20	011111 1	011111 1	012111 1
21	111111 0	001111 1	011111 1
22	011111 0	111111 0	011111 0
23	001111 1	011111 0	001111 1
24	001111 0	001111 0	001111 0
25	000111 0	000111 0	000111 0
26	000011 0	000011 0	000011 0
27	000001 0	000001 0	000001 0

TABLE 9. Matrix of signs for the natural numbering of weights



	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27
1	0	+	+	+	−	+	−	+	+	−	−	+	0	−	−	0	+	0	−	0	+	0	0	0	0	0	0
2	+	0	+	+	−	+	−	+	+	−	0	+	−	0	−	−	0	+	0	−	0	+	0	0	0	0	0
3	+	+	0	+	−	+	−	+	0	−	+	0	−	+	0	−	−	+	0	0	0	0	−	+	0	0	0
4	+	+	+	0	+	+	0	+	−	0	+	−	−	+	0	−	0	0	−	+	0	0	−	0	+	0	0
5	−	−	−	+	0	0	+	0	−	+	+	−	−	+	0	−	0	0	0	0	−	+	0	−	+	0	0
6	+	+	+	+	0	0	+	+	+	0	−	0	+	0	−	0	+	−	−	+	0	0	−	0	0	+	0
7	−	−	−	0	+	+	0	0	+	+	−	0	+	0	−	0	+	−	0	0	−	+	0	−	0	+	0
8	+	+	+	+	0	+	0	0	0	+	0	+	0	−	+	+	−	+	+	−	0	0	+	0	0	0	+
9	+	+	0	−	−	+	+	0	0	0	+	+	−	0	−	0	0	0	+	−	−	+	0	0	−	+	0
10	−	−	−	0	+	0	+	+	0	0	0	+	0	−	+	+	−	+	0	0	+	−	0	+	0	0	+
11	−	0	+	+	+	−	−	0	+	0	0	0	+	+	0	0	−	0	+	0	−	0	−	+	−	+	0
12	+	+	0	−	−	0	0	+	+	+	0	0	0	+	+	−	0	0	−	+	+	−	0	0	+	0	+
13	0	−	−	−	−	+	+	0	−	0	+	0	0	0	0	+	0	−	0	+	0	−	−	+	−	+	0
14	−	0	+	+	+	0	0	−	0	−	+	+	0	0	0	+	+	0	−	0	+	0	+	−	+	0	+
15	−	−	0	0	0	−	−	+	−	+	0	+	0	0	0	0	+	−	+	−	−	+	0	0	0	+	+
16	0	−	−	−	−	0	0	+	0	+	0	−	+	+	0	0	0	+	0	−	0	+	+	−	+	0	+
17	+	0	−	0	0	+	+	−	0	−	−	0	0	+	+	0	0	+	+	0	−	0	−	+	0	+	+
18	0	+	+	0	0	−	−	+	0	+	0	0	−	0	−	+	+	0	0	+	0	−	−	+	0	+	+
19	−	0	0	−	0	−	0	+	+	0	+	−	0	−	+	0	+	0	0	+	+	0	+	0	+	+	+
20	0	−	0	+	0	+	0	−	−	0	0	+	+	0	−	−	0	+	+	0	0	+	+	0	+	+	+
21	+	0	0	0	−	0	−	0	−	+	−	+	0	+	−	0	−	0	+	0	0	+	0	+	+	+	+
22	0	+	0	0	+	0	+	0	+	−	0	−	−	0	+	+	0	−	0	+	+	0	0	+	+	+	+
23	0	0	−	−	0	−	0	+	0	0	−	0	−	+	0	+	−	−	+	+	0	0	0	+	+	+	+
24	0	0	+	0	−	0	−	0	0	+	+	0	+	−	0	−	+	+	0	0	+	+	+	0	+	+	+
25	0	0	0	+	+	0	0	0	−	0	−	+	−	+	0	+	0	0	+	+	+	+	+	+	0	+	+
26	0	0	0	0	0	+	+	0	+	0	+	0	+	0	+	0	+	+	+	+	+	+	+	+	+	0	+
27	0	0	0	0	0	0	0	+	0	+	0	+	0	+	+	+	+	+	+	+	+	+	+	+	+	+	0

Table 10. Root elements for the natural numbering of weights

$$e_{\substack{10000\\0}} = e_{1,2} + e_{11,13} + e_{14,16} + e_{17,18} + e_{19,20} + e_{21,22}$$
$$e_{\substack{00000\\1}} = e_{4,5} + e_{6,7} + e_{8,10} + e_{19,21} + e_{20,22} + e_{23,24}$$
$$e_{\substack{01000\\0}} = e_{2,3} + e_{9,11} + e_{12,14} + e_{15,17} + e_{20,23} + e_{22,24}$$
$$e_{\substack{00100\\0}} = e_{3,4} + e_{7,9} + e_{10,12} + e_{17,19} + e_{18,20} + e_{24,25}$$
$$e_{\substack{00010\\0}} = e_{4,6} + e_{5,7} + e_{12,15} + e_{14,17} + e_{16,18} + e_{25,26}$$
$$e_{\substack{00001\\0}} = e_{6,8} + e_{7,10} + e_{9,12} + e_{11,14} + e_{13,16} + e_{26,27}$$
$$e_{\substack{11000\\0}} = e_{1,3} - e_{9,13} - e_{12,16} - e_{15,18} + e_{19,23} + e_{21,24}$$
$$e_{\substack{00100\\1}} = -e_{3,5} + e_{6,9} + e_{8,12} - e_{17,21} - e_{18,22} + e_{23,25}$$
$$e_{\substack{01100\\0}} = e_{2,4} - e_{7,11} - e_{10,14} + e_{15,19} - e_{18,23} + e_{22,25}$$
$$e_{\substack{00110\\0}} = e_{3,6} - e_{5,9} + e_{10,15} - e_{14,19} - e_{16,20} + e_{24,26}$$
$$e_{\substack{00011\\0}} = e_{4,8} + e_{5,10} - e_{9,15} - e_{11,17} - e_{13,18} + e_{25,27}$$
$$e_{\substack{11100\\0}} = e_{1,4} + e_{7,13} + e_{10,16} - e_{15,20} - e_{17,23} + e_{21,25}$$
$$e_{\substack{01100\\1}} = -e_{2,5} - e_{6,11} - e_{8,14} - e_{15,21} + e_{18,24} + e_{20,25}$$
$$e_{\substack{00110\\1}} = -e_{3,7} - e_{4,9} + e_{8,15} + e_{14,21} + e_{16,22} + e_{23,26}$$
$$e_{\substack{01110\\0}} = e_{2,6} + e_{5,11} - e_{10,17} - e_{12,19} + e_{16,23} + e_{22,26}$$
$$e_{\substack{00111\\0}} = e_{3,8} - e_{5,12} - e_{7,15} + e_{11,19} + e_{13,20} + e_{24,27}$$
$$e_{\substack{11100\\1}} = -e_{1,5} + e_{6,13} + e_{8,16} + e_{15,22} + e_{17,24} + e_{19,25}$$
$$e_{\substack{11110\\0}} = e_{1,6} - e_{5,13} + e_{10,18} + e_{12,20} + e_{14,23} + e_{21,26}$$
$$e_{\substack{01110\\1}} = -e_{2,7} + e_{4,11} - e_{8,17} + e_{12,21} - e_{16,24} + e_{20,26}$$
$$e_{\substack{00111\\1}} = -e_{3,10} - e_{4,12} - e_{6,15} - e_{11,21} - e_{13,22} + e_{23,27}$$
$$e_{\substack{01111\\0}} = e_{2,8} + e_{5,14} + e_{7,17} + e_{9,19} - e_{13,23} + e_{22,27}$$
$$e_{\substack{11110\\1}} = -e_{1,7} - e_{4,13} + e_{8,18} - e_{12,22} - e_{14,24} + e_{19,26}$$
$$e_{\substack{11111\\0}} = e_{1,8} - e_{5,16} - e_{7,18} - e_{9,20} - e_{11,23} + e_{21,27}$$
$$e_{\substack{01210\\1}} = e_{2,9} + e_{3,11} + e_{8,19} + e_{10,21} + e_{16,25} + e_{18,26}$$
$$e_{\substack{01111\\1}} = -e_{2,10} + e_{4,14} + e_{6,17} - e_{9,21} + e_{13,24} + e_{20,27}$$
$$e_{\substack{11210\\1}} = e_{1,9} - e_{3,13} - e_{8,20} - e_{10,22} + e_{14,25} + e_{17,26}$$
$$e_{\substack{11111\\1}} = -e_{1,10} - e_{4,16} - e_{6,18} + e_{9,22} + e_{11,24} + e_{19,27}$$
$$e_{\substack{01211\\1}} = e_{2,12} + e_{3,14} - e_{6,19} - e_{7,21} - e_{13,25} + e_{18,27}$$
$$e_{\substack{12210\\1}} = -e_{1,11} - e_{2,13} + e_{8,23} + e_{10,24} + e_{12,25} + e_{15,26}$$
$$e_{\substack{11211\\1}} = e_{1,12} - e_{3,16} + e_{6,20} + e_{7,22} - e_{11,25} + e_{17,27}$$
$$e_{\substack{01221\\1}} = -e_{2,15} - e_{3,17} - e_{4,19} - e_{5,21} + e_{13,26} + e_{16,27}$$

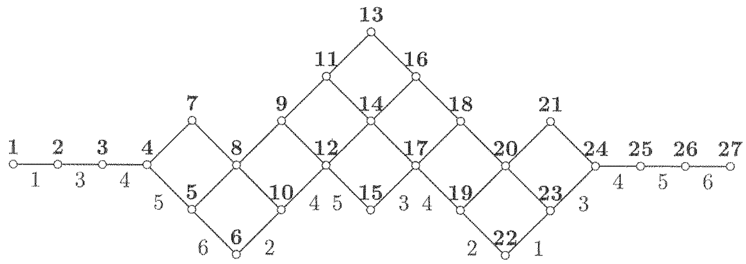
$$e_{\substack{12211\\1}} = -e_{1,14} - e_{2,16} - e_{6,23} - e_{7,24} - e_{9,25} + e_{15,27}$$
$$e_{\substack{11221\\1}} = -e_{1,15} + e_{3,18} + e_{4,20} + e_{5,22} + e_{11,26} + e_{14,27}$$
$$e_{\substack{12221\\1}} = e_{1,17} + e_{2,18} - e_{4,23} - e_{5,24} + e_{9,26} + e_{12,27}$$
$$e_{\substack{12321\\1}} = -e_{1,19} - e_{2,20} - e_{3,23} + e_{5,25} + e_{7,26} + e_{10,27}$$
$$e_{\substack{12321\\2}} = e_{1,21} + e_{2,22} + e_{3,24} + e_{4,25} + e_{6,26} + e_{8,27}$$

Table 11. Cubic form for the natural numbering of weights

$$\begin{aligned}
Q(x) = {} & x_1x_{13}x_{27} - x_1x_{16}x_{26} + x_1x_{18}x_{25} - x_1x_{20}x_{24} + x_1x_{22}x_{23} \\
& - x_2x_{11}x_{27} + x_2x_{14}x_{26} - x_2x_{17}x_{25} + x_2x_{19}x_{24} - x_2x_{21}x_{23} \\
& + x_3x_9x_{27} - x_3x_{12}x_{26} + x_3x_{15}x_{25} - x_3x_{19}x_{22} + x_3x_{20}x_{21} \\
& - x_4x_7x_{27} + x_4x_{10}x_{26} - x_4x_{15}x_{24} + x_4x_{17}x_{22} - x_4x_{18}x_{21} \\
& + x_5x_6x_{27} - x_5x_8x_{26} + x_5x_{15}x_{23} - x_5x_{17}x_{20} + x_5x_{18}x_{19} \\
& - x_6x_{10}x_{25} + x_6x_{12}x_{24} - x_6x_{14}x_{22} + x_6x_{16}x_{21} + x_7x_8x_{25} \\
& - x_7x_{12}x_{23} + x_7x_{14}x_{20} - x_7x_{16}x_{19} - x_8x_9x_{24} + x_8x_{11}x_{22} \\
& - x_8x_{13}x_{21} + x_9x_{10}x_{23} - x_9x_{14}x_{18} + x_9x_{16}x_{17} - x_{10}x_{11}x_{20} \\
& + x_{10}x_{13}x_{19} + x_{11}x_{12}x_{18} - x_{11}x_{15}x_{16} - x_{12}x_{13}x_{17} + x_{13}x_{14}x_{15}
\end{aligned}$$

$$\begin{aligned}
f_1(x) &= x_{13}x_{27} - x_{16}x_{26} + x_{18}x_{25} - x_{20}x_{24} + x_{22}x_{23} \\
f_2(x) &= -x_{11}x_{27} + x_{14}x_{26} - x_{17}x_{25} + x_{19}x_{24} - x_{21}x_{23} \\
f_3(x) &= x_9x_{27} - x_{12}x_{26} + x_{15}x_{25} - x_{19}x_{22} + x_{20}x_{21} \\
f_4(x) &= -x_7x_{27} + x_{10}x_{26} - x_{15}x_{24} + x_{17}x_{22} - x_{18}x_{21} \\
f_5(x) &= x_6x_{27} - x_8x_{26} + x_{15}x_{23} - x_{17}x_{20} + x_{18}x_{19} \\
f_6(x) &= x_5x_{27} - x_{10}x_{25} + x_{12}x_{24} - x_{14}x_{22} + x_{16}x_{21} \\
f_7(x) &= -x_4x_{27} + x_8x_{25} - x_{12}x_{23} + x_{14}x_{20} - x_{16}x_{19} \\
f_8(x) &= -x_5x_{26} + x_7x_{25} - x_9x_{24} + x_{11}x_{22} - x_{13}x_{21} \\
f_9(x) &= x_3x_{27} - x_8x_{24} + x_{10}x_{23} - x_{14}x_{18} + x_{16}x_{17} \\
f_{10}(x) &= x_4x_{26} - x_6x_{25} + x_9x_{23} - x_{11}x_{20} + x_{13}x_{19} \\
f_{11}(x) &= -x_2x_{27} + x_8x_{22} - x_{10}x_{20} + x_{12}x_{18} - x_{15}x_{16} \\
f_{12}(x) &= -x_3x_{26} + x_6x_{24} - x_7x_{23} + x_{11}x_{18} - x_{13}x_{17} \\
f_{13}(x) &= x_1x_{27} - x_8x_{21} + x_{10}x_{19} - x_{12}x_{17} + x_{14}x_{15} \\
f_{14}(x) &= x_2x_{26} - x_6x_{22} + x_7x_{20} - x_9x_{18} + x_{13}x_{15} \\
f_{15}(x) &= x_3x_{25} - x_4x_{24} + x_5x_{23} - x_{11}x_{16} + x_{13}x_{14} \\
f_{16}(x) &= -x_1x_{26} + x_6x_{21} - x_7x_{19} + x_9x_{17} - x_{11}x_{15} \\
f_{17}(x) &= -x_2x_{25} + x_4x_{22} - x_5x_{20} + x_9x_{16} - x_{12}x_{13} \\
f_{18}(x) &= x_1x_{25} - x_4x_{21} + x_5x_{19} - x_9x_{14} + x_{11}x_{12} \\
f_{19}(x) &= x_2x_{24} - x_3x_{22} + x_5x_{18} - x_7x_{16} + x_{10}x_{13} \\
f_{20}(x) &= -x_1x_{24} + x_3x_{21} - x_5x_{17} + x_7x_{14} - x_{10}x_{11} \\
f_{21}(x) &= -x_2x_{23} + x_3x_{20} - x_4x_{18} + x_6x_{16} - x_8x_{13} \\
f_{22}(x) &= x_1x_{23} - x_3x_{19} + x_4x_{17} - x_6x_{14} + x_8x_{11} \\
f_{23}(x) &= x_1x_{22} - x_2x_{21} + x_5x_{15} - x_7x_{12} + x_9x_{10} \\
f_{24}(x) &= -x_1x_{20} + x_2x_{19} - x_4x_{15} + x_6x_{12} - x_8x_9 \\
f_{25}(x) &= x_1x_{18} - x_2x_{17} + x_3x_{15} - x_6x_{10} + x_7x_8 \\
f_{26}(x) &= -x_1x_{16} + x_2x_{14} - x_3x_{12} + x_4x_{10} - x_5x_8 \\
f_{27}(x) &= x_1x_{13} - x_2x_{11} + x_3x_9 - x_4x_7 + x_5x_6
\end{aligned}$$

Table 12. Matrix of signs for the A_5-numbering of weights



	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27
1	0	+	+	+	+	+	−	−	+	−	−	+	0	−	−	0	+	0	−	0	0	+	0	0	0	0	0
2	+	0	+	+	+	+	−	−	+	−	0	+	−	0	−	−	0	+	0	−	0	0	+	0	0	0	0
3	+	+	0	+	+	+	−	−	0	−	+	0	−	+	0	−	−	+	0	0	−	0	0	+	0	0	0
4	+	+	+	0	+	+	+	0	−	0	+	−	−	+	0	−	0	0	−	+	−	0	0	0	+	0	0
5	+	+	+	+	0	+	0	+	+	0	−	0	+	0	−	0	+	−	−	+	−	0	0	0	0	+	0
6	+	+	+	+	+	0	0	0	0	+	0	+	0	−	+	+	−	+	+	−	+	0	0	0	0	0	+
7	−	−	−	+	0	0	0	+	−	+	+	−	−	+	0	−	0	0	0	0	0	−	+	−	+	0	0
8	−	−	−	0	+	0	+	0	+	+	−	0	+	0	−	0	+	−	0	0	0	−	+	−	0	+	0
9	+	+	0	−	+	0	−	+	0	0	+	+	−	0	−	0	0	0	+	−	0	−	+	0	−	+	0
10	−	−	−	0	0	+	+	+	0	0	0	+	0	−	+	+	−	+	0	0	0	+	−	+	0	0	+
11	−	0	+	+	−	0	+	−	+	0	0	0	+	+	0	0	−	0	+	0	−	−	0	+	−	+	0
12	+	+	0	−	0	+	−	0	+	+	0	0	0	+	+	−	0	0	−	+	0	+	−	0	+	0	+
13	0	−	−	−	+	0	−	+	−	0	+	0	0	0	0	+	0	−	0	+	−	0	−	+	−	+	0
14	−	0	+	+	0	−	+	0	0	−	+	+	0	0	0	+	+	0	−	0	+	+	0	−	+	0	+
15	−	−	0	0	−	+	0	−	−	+	0	+	0	0	0	0	+	−	+	−	0	−	+	0	0	+	+
16	0	−	−	−	0	+	−	0	0	+	0	−	+	+	0	0	0	+	0	−	+	0	+	−	+	0	+
17	+	0	−	0	+	−	0	+	0	−	−	0	0	+	+	0	0	+	+	0	−	−	0	+	0	+	+
18	0	+	+	0	−	+	0	−	0	+	0	0	−	0	−	+	+	0	0	+	−	0	−	+	0	+	+
19	−	0	0	−	−	+	0	0	+	0	+	−	0	−	+	0	+	0	0	+	+	+	0	0	+	+	+
20	0	−	0	+	+	−	0	0	−	0	0	+	+	0	−	−	0	+	+	0	+	0	+	0	+	+	+
21	0	0	−	−	−	+	0	0	0	0	−	0	−	+	0	+	−	−	+	+	0	0	0	+	+	+	+
22	+	0	0	0	0	0	−	−	−	+	−	+	0	+	−	0	−	0	+	0	0	0	+	+	+	+	+
23	0	+	0	0	0	0	+	+	+	−	0	−	−	0	+	+	0	−	0	+	0	+	0	+	+	+	+
24	0	0	+	0	0	0	−	−	0	+	+	0	+	−	0	−	+	+	0	0	+	+	+	0	+	+	+
25	0	0	0	+	0	0	+	0	−	0	−	+	−	+	0	+	0	0	+	+	+	+	+	+	0	+	+
26	0	0	0	0	+	0	0	+	+	0	+	0	+	0	+	0	+	+	+	+	+	+	+	+	+	0	+
27	0	0	0	0	0	+	0	0	0	+	0	+	0	+	+	+	+	+	+	+	+	+	+	+	+	+	0

TABLE 13. Root elements for the A_5-numbering of weights

$$
\begin{aligned}
e_{\substack{10000\\0}} &= e_{1,2} + e_{11,13} + e_{14,16} + e_{17,18} + e_{19,20} + e_{22,23}\\
e_{\substack{00000\\1}} &= e_{4,7} + e_{5,8} + e_{6,10} + e_{19,22} + e_{20,23} + e_{21,24}\\
e_{\substack{01000\\0}} &= e_{2,3} + e_{9,11} + e_{12,14} + e_{15,17} + e_{20,21} + e_{23,24}\\
e_{\substack{00100\\0}} &= e_{3,4} + e_{8,9} + e_{10,12} + e_{17,19} + e_{18,20} + e_{24,25}\\
e_{\substack{00010\\0}} &= e_{4,5} + e_{7,8} + e_{12,15} + e_{14,17} + e_{16,18} + e_{25,26}\\
e_{\substack{00001\\0}} &= e_{5,6} + e_{8,10} + e_{9,12} + e_{11,14} + e_{13,16} + e_{26,27}\\
e_{\substack{11000\\0}} &= e_{1,3} - e_{9,13} - e_{12,16} - e_{15,18} + e_{19,21} + e_{22,24}\\
e_{\substack{00100\\1}} &= -e_{3,7} + e_{5,9} + e_{6,12} - e_{17,22} - e_{18,23} + e_{21,25}\\
e_{\substack{01100\\0}} &= e_{2,4} - e_{8,11} - e_{10,14} + e_{15,19} - e_{18,21} + e_{23,25}\\
e_{\substack{00110\\0}} &= e_{3,5} - e_{7,9} + e_{10,15} - e_{14,19} - e_{16,20} + e_{24,26}\\
e_{\substack{00011\\0}} &= e_{4,6} + e_{7,10} - e_{9,15} - e_{11,17} - e_{13,18} + e_{25,27}\\
e_{\substack{11100\\0}} &= e_{1,4} + e_{8,13} + e_{10,16} - e_{15,20} - e_{17,21} + e_{22,25}\\
e_{\substack{01100\\1}} &= -e_{2,7} - e_{5,11} - e_{6,14} - e_{15,22} + e_{18,24} + e_{20,25}\\
e_{\substack{00110\\1}} &= -e_{3,8} - e_{4,9} + e_{6,15} + e_{14,22} + e_{16,23} + e_{21,26}\\
e_{\substack{01110\\0}} &= e_{2,5} + e_{7,11} - e_{10,17} - e_{12,19} + e_{16,21} + e_{23,26}\\
e_{\substack{00111\\0}} &= e_{3,6} - e_{7,12} - e_{8,15} + e_{11,19} + e_{13,20} + e_{24,27}\\
e_{\substack{11100\\1}} &= -e_{1,7} + e_{5,13} + e_{6,16} + e_{15,23} + e_{17,24} + e_{19,25}\\
e_{\substack{11110\\0}} &= e_{1,5} - e_{7,13} + e_{10,18} + e_{12,20} + e_{14,21} + e_{22,26}\\
e_{\substack{01110\\1}} &= -e_{2,8} + e_{4,11} - e_{6,17} + e_{12,22} - e_{16,24} + e_{20,26}\\
e_{\substack{00111\\1}} &= -e_{3,10} - e_{4,12} - e_{5,15} - e_{11,22} - e_{13,23} + e_{21,27}\\
e_{\substack{01111\\0}} &= e_{2,6} + e_{7,14} + e_{8,17} + e_{9,19} - e_{13,21} + e_{23,27}\\
e_{\substack{11110\\1}} &= -e_{1,8} - e_{4,13} + e_{6,18} - e_{12,23} - e_{14,24} + e_{19,26}\\
e_{\substack{11111\\0}} &= e_{1,6} - e_{7,16} - e_{8,18} - e_{9,20} - e_{11,21} + e_{22,27}\\
e_{\substack{01210\\1}} &= e_{2,9} + e_{3,11} + e_{6,19} + e_{10,22} + e_{16,25} + e_{18,26}\\
e_{\substack{01111\\1}} &= -e_{2,10} + e_{4,14} + e_{5,17} - e_{9,22} + e_{13,24} + e_{20,27}\\
e_{\substack{11210\\1}} &= e_{1,9} - e_{3,13} - e_{6,20} - e_{10,23} + e_{14,25} + e_{17,26}\\
e_{\substack{11111\\1}} &= -e_{1,10} - e_{4,16} - e_{5,18} + e_{9,23} + e_{11,24} + e_{19,27}\\
e_{\substack{01211\\1}} &= e_{2,12} + e_{3,14} - e_{5,19} - e_{8,22} - e_{13,25} + e_{18,27}\\
e_{\substack{12210\\1}} &= -e_{1,11} - e_{2,13} + e_{6,21} + e_{10,24} + e_{12,25} + e_{15,26}\\
e_{\substack{11211\\1}} &= e_{1,12} - e_{3,16} + e_{5,20} + e_{8,23} - e_{11,25} + e_{17,27}\\
e_{\substack{01221\\1}} &= -e_{2,15} - e_{3,17} - e_{4,19} - e_{7,22} + e_{13,26} + e_{16,27}
\end{aligned}
$$

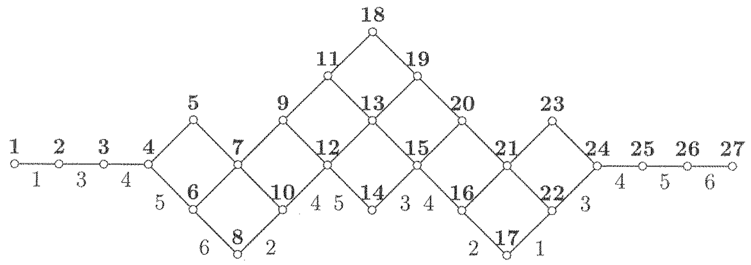
$$e_{\substack{12211\\1}} = -e_{1,14} - e_{2,16} - e_{5,21} - e_{8,24} - e_{9,25} + e_{15,27}$$
$$e_{\substack{11221\\1}} = -e_{1,15} + e_{3,18} + e_{4,20} + e_{7,23} + e_{11,26} + e_{14,27}$$
$$e_{\substack{12221\\1}} = e_{1,17} + e_{2,18} - e_{4,21} - e_{7,24} + e_{9,26} + e_{12,27}$$
$$e_{\substack{12321\\1}} = -e_{1,19} - e_{2,20} - e_{3,21} + e_{7,25} + e_{8,26} + e_{10,27}$$
$$e_{\substack{12321\\2}} = e_{1,22} + e_{2,23} + e_{3,24} + e_{4,25} + e_{5,26} + e_{6,27}$$

TABLE 14. Cubic form for the A_5-numbering of weights

$$\begin{aligned}
Q(x) = {} & x_1x_{13}x_{27} - x_1x_{16}x_{26} + x_1x_{18}x_{25} - x_1x_{20}x_{24} + x_1x_{21}x_{23} \\
& - x_2x_{11}x_{27} + x_2x_{14}x_{26} - x_2x_{17}x_{25} + x_2x_{19}x_{24} - x_2x_{21}x_{22} \\
& + x_3x_9x_{27} - x_3x_{12}x_{26} + x_3x_{15}x_{25} - x_3x_{19}x_{23} + x_3x_{20}x_{22} \\
& - x_4x_8x_{27} + x_4x_{10}x_{26} - x_4x_{15}x_{24} + x_4x_{17}x_{23} - x_4x_{18}x_{22} \\
& + x_5x_7x_{27} - x_5x_{10}x_{25} + x_5x_{12}x_{24} - x_5x_{14}x_{23} + x_5x_{16}x_{22} \\
& - x_6x_7x_{26} + x_6x_8x_{25} - x_6x_9x_{24} + x_6x_{11}x_{23} - x_6x_{13}x_{22} \\
& + x_7x_{15}x_{21} - x_7x_{17}x_{20} + x_7x_{18}x_{19} - x_8x_{12}x_{21} + x_8x_{14}x_{20} \\
& - x_8x_{16}x_{19} + x_9x_{10}x_{21} - x_9x_{14}x_{18} + x_9x_{16}x_{17} - x_{10}x_{11}x_{20} \\
& + x_{10}x_{13}x_{19} + x_{11}x_{12}x_{18} - x_{11}x_{15}x_{16} - x_{12}x_{13}x_{17} + x_{13}x_{14}x_{15}
\end{aligned}$$

$$\begin{aligned}
f_1(x) &= x_{13}x_{27} - x_{16}x_{26} + x_{18}x_{25} - x_{20}x_{24} + x_{21}x_{23} \\
f_2(x) &= -x_{11}x_{27} + x_{14}x_{26} - x_{17}x_{25} + x_{19}x_{24} - x_{21}x_{22} \\
f_3(x) &= x_9x_{27} - x_{12}x_{26} + x_{15}x_{25} - x_{19}x_{23} + x_{20}x_{22} \\
f_4(x) &= -x_8x_{27} + x_{10}x_{26} - x_{15}x_{24} + x_{17}x_{23} - x_{18}x_{22} \\
f_5(x) &= x_7x_{27} - x_{10}x_{25} + x_{12}x_{24} - x_{14}x_{23} + x_{16}x_{22} \\
f_6(x) &= -x_7x_{26} + x_8x_{25} - x_9x_{24} + x_{11}x_{23} - x_{13}x_{22} \\
f_7(x) &= x_5x_{27} - x_6x_{26} + x_{15}x_{21} - x_{17}x_{20} + x_{18}x_{19} \\
f_8(x) &= -x_4x_{27} + x_6x_{25} - x_{12}x_{21} + x_{14}x_{20} - x_{16}x_{19} \\
f_9(x) &= x_3x_{27} - x_6x_{24} + x_{10}x_{21} - x_{14}x_{18} + x_{16}x_{17} \\
f_{10}(x) &= x_4x_{26} - x_5x_{25} + x_9x_{21} - x_{11}x_{20} + x_{13}x_{19} \\
f_{11}(x) &= -x_2x_{27} + x_6x_{23} - x_{10}x_{20} + x_{12}x_{18} - x_{15}x_{16} \\
f_{12}(x) &= -x_3x_{26} + x_5x_{24} - x_8x_{21} + x_{11}x_{18} - x_{13}x_{17} \\
f_{13}(x) &= x_1x_{27} - x_6x_{22} + x_{10}x_{19} - x_{12}x_{17} + x_{14}x_{15} \\
f_{14}(x) &= x_2x_{26} - x_5x_{23} + x_8x_{20} - x_9x_{18} + x_{13}x_{15} \\
f_{15}(x) &= x_3x_{25} - x_4x_{24} + x_7x_{21} - x_{11}x_{16} + x_{13}x_{14} \\
f_{16}(x) &= -x_1x_{26} + x_5x_{22} - x_8x_{19} + x_9x_{17} - x_{11}x_{15} \\
f_{17}(x) &= -x_2x_{25} + x_4x_{23} - x_7x_{20} + x_9x_{16} - x_{12}x_{13} \\
f_{18}(x) &= x_1x_{25} - x_4x_{22} + x_7x_{19} - x_9x_{14} + x_{11}x_{12} \\
f_{19}(x) &= x_2x_{24} - x_3x_{23} + x_7x_{18} - x_8x_{16} + x_{10}x_{13} \\
f_{20}(x) &= -x_1x_{24} + x_3x_{22} - x_7x_{17} + x_8x_{14} - x_{10}x_{11} \\
f_{21}(x) &= x_1x_{23} - x_2x_{22} + x_7x_{15} - x_8x_{12} + x_9x_{10} \\
f_{22}(x) &= -x_2x_{21} + x_3x_{20} - x_4x_{18} + x_5x_{16} - x_6x_{13} \\
f_{23}(x) &= x_1x_{21} - x_3x_{19} + x_4x_{17} - x_5x_{14} + x_6x_{11} \\
f_{24}(x) &= -x_1x_{20} + x_2x_{19} - x_4x_{15} + x_5x_{12} - x_6x_9 \\
f_{25}(x) &= x_1x_{18} - x_2x_{17} + x_3x_{15} - x_5x_{10} + x_6x_8 \\
f_{26}(x) &= -x_1x_{16} + x_2x_{14} - x_3x_{12} + x_4x_{10} - x_6x_7 \\
f_{27}(x) &= x_1x_{13} - x_2x_{11} + x_3x_9 - x_4x_8 + x_5x_7
\end{aligned}$$

TABLE 15. Matrix of signs for the D_5-numbering of weights



	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27
1	0	+	+	+	−	+	−	+	+	−	−	+	−	−	+	−	+	0	0	0	0	0	0	0	0	0	0
2	+	0	+	+	−	+	−	+	+	−	0	+	0	−	0	0	0	−	−	+	−	+	0	0	0	0	0
3	+	+	0	+	−	+	−	+	0	−	+	0	+	0	−	0	0	−	−	+	0	0	−	+	0	0	0
4	+	+	+	0	+	+	0	+	−	0	+	−	+	0	0	−	0	−	−	0	+	0	−	0	+	0	0
5	−	−	−	+	0	0	+	0	−	+	+	−	+	0	0	0	−	−	−	0	0	+	0	−	+	0	0
6	+	+	+	+	0	0	+	+	+	0	−	0	0	−	+	−	0	+	0	−	+	0	−	0	0	+	0
7	−	−	−	0	+	+	0	0	+	+	−	0	0	−	+	0	−	+	0	−	0	+	0	−	0	+	0
8	+	+	+	+	0	+	0	0	0	+	0	+	−	+	−	+	0	0	+	+	−	0	+	0	0	0	+
9	+	+	0	−	−	+	+	0	0	0	+	+	0	−	0	+	−	−	0	0	−	+	0	0	−	+	0
10	−	−	−	0	+	0	+	+	0	0	0	+	−	+	−	0	+	0	+	+	0	−	0	+	0	0	+
11	−	0	+	+	+	−	−	0	+	0	0	0	+	0	−	+	−	+	0	0	0	0	−	+	−	+	0
12	+	+	0	−	−	0	0	+	+	+	0	0	+	+	0	−	+	0	−	0	+	−	0	0	+	0	+
13	−	0	+	+	+	0	0	−	0	−	+	+	0	0	+	−	+	0	+	0	0	0	+	−	+	0	+
14	−	−	0	0	0	−	−	+	−	+	0	+	0	0	+	+	−	0	0	−	−	+	0	0	0	+	+
15	+	0	−	0	0	+	+	−	0	−	−	0	+	+	0	+	−	0	0	+	0	0	−	+	0	+	+
16	−	0	0	−	0	−	0	+	+	0	+	−	−	+	+	0	+	0	0	0	+	0	+	0	+	+	+
17	+	0	0	0	−	0	−	0	−	+	−	+	+	−	−	+	0	0	0	0	0	+	0	+	+	+	+
18	0	−	−	−	−	+	+	0	−	0	+	0	0	0	0	0	0	0	+	−	+	−	−	+	−	+	0
19	0	−	−	−	−	0	0	+	0	+	0	−	+	0	0	0	0	+	0	+	−	+	+	−	+	0	+
20	0	+	+	0	0	−	−	+	0	+	0	0	0	−	+	0	0	−	+	0	+	−	−	+	0	+	+
21	0	−	0	+	0	+	0	−	−	0	0	+	0	−	0	+	0	+	−	+	0	+	+	0	+	+	+
22	0	+	0	0	+	0	+	0	+	−	0	−	0	+	0	0	+	−	+	−	+	0	0	+	+	+	+
23	0	0	−	−	0	−	0	+	0	0	−	0	+	0	−	+	0	−	+	−	+	0	0	+	+	+	+
24	0	0	+	0	−	0	−	0	0	+	+	0	−	0	+	0	+	+	−	+	0	+	+	0	+	+	+
25	0	0	0	+	+	0	0	0	−	0	−	+	+	0	0	+	+	−	+	0	+	+	+	+	0	+	+
26	0	0	0	0	0	+	+	0	+	0	+	0	0	+	+	+	+	+	0	+	+	+	+	+	+	0	+
27	0	0	0	0	0	0	0	+	0	+	0	+	+	+	+	+	+	0	+	+	+	+	+	+	+	+	0

TABLE 16. Root elements for the D_5-numbering of weights

$$e_{\substack{10000\\0}} = e_{1,2} + e_{11,18} + e_{13,19} + e_{15,20} + e_{16,21} + e_{17,22}$$
$$e_{\substack{00000\\1}} = e_{4,5} + e_{6,7} + e_{8,10} + e_{16,17} + e_{21,22} + e_{23,24}$$
$$e_{\substack{01000\\0}} = e_{2,3} + e_{9,11} + e_{12,13} + e_{14,15} + e_{21,23} + e_{22,24}$$
$$e_{\substack{00100\\0}} = e_{3,4} + e_{7,9} + e_{10,12} + e_{15,16} + e_{20,21} + e_{24,25}$$
$$e_{\substack{00010\\0}} = e_{4,6} + e_{5,7} + e_{12,14} + e_{13,15} + e_{19,20} + e_{25,26}$$
$$e_{\substack{00001\\0}} = e_{6,8} + e_{7,10} + e_{9,12} + e_{11,13} + e_{18,19} + e_{26,27}$$
$$e_{\substack{11000\\0}} = e_{1,3} - e_{9,18} - e_{12,19} - e_{14,20} + e_{16,23} + e_{17,24}$$
$$e_{\substack{00100\\1}} = -e_{3,5} + e_{6,9} + e_{8,12} - e_{15,17} - e_{20,22} + e_{23,25}$$
$$e_{\substack{01100\\0}} = e_{2,4} - e_{7,11} - e_{10,13} + e_{14,16} - e_{20,23} + e_{22,25}$$
$$e_{\substack{00110\\0}} = e_{3,6} - e_{5,9} + e_{10,14} - e_{13,16} - e_{19,21} + e_{24,26}$$
$$e_{\substack{00011\\0}} = e_{4,8} + e_{5,10} - e_{9,14} - e_{11,15} - e_{18,20} + e_{25,27}$$
$$e_{\substack{11100\\0}} = e_{1,4} + e_{7,18} + e_{10,19} - e_{14,21} - e_{15,23} + e_{17,25}$$
$$e_{\substack{01100\\1}} = -e_{2,5} - e_{6,11} - e_{8,13} - e_{14,17} + e_{20,24} + e_{21,25}$$
$$e_{\substack{00110\\1}} = -e_{3,7} - e_{4,9} + e_{8,14} + e_{13,17} + e_{19,22} + e_{23,26}$$
$$e_{\substack{01110\\0}} = e_{2,6} + e_{5,11} - e_{10,15} - e_{12,16} + e_{19,23} + e_{22,26}$$
$$e_{\substack{00111\\0}} = e_{3,8} - e_{5,12} - e_{7,14} + e_{11,16} + e_{18,21} + e_{24,27}$$
$$e_{\substack{11100\\1}} = -e_{1,5} + e_{6,18} + e_{8,19} + e_{14,22} + e_{15,24} + e_{16,25}$$
$$e_{\substack{11110\\0}} = e_{1,6} - e_{5,18} + e_{10,20} + e_{12,21} + e_{13,23} + e_{17,26}$$
$$e_{\substack{01110\\1}} = -e_{2,7} + e_{4,11} - e_{8,15} + e_{12,17} - e_{19,24} + e_{21,26}$$
$$e_{\substack{00111\\1}} = -e_{3,10} - e_{4,12} - e_{6,14} - e_{11,17} - e_{18,22} + e_{23,27}$$
$$e_{\substack{01111\\0}} = e_{2,8} + e_{5,13} + e_{7,15} + e_{9,16} - e_{18,23} + e_{22,27}$$
$$e_{\substack{11110\\1}} = -e_{1,7} - e_{4,18} + e_{8,20} - e_{12,22} - e_{13,24} + e_{16,26}$$
$$e_{\substack{11111\\0}} = e_{1,8} - e_{5,19} - e_{7,20} - e_{9,21} - e_{11,23} + e_{17,27}$$
$$e_{\substack{01210\\1}} = e_{2,9} + e_{3,11} + e_{8,16} + e_{10,17} + e_{19,25} + e_{20,26}$$
$$e_{\substack{01111\\1}} = -e_{2,10} + e_{4,13} + e_{6,15} - e_{9,17} + e_{18,24} + e_{21,27}$$
$$e_{\substack{11210\\1}} = e_{1,9} - e_{3,18} - e_{8,21} - e_{10,22} + e_{13,25} + e_{15,26}$$
$$e_{\substack{11111\\1}} = -e_{1,10} - e_{4,19} - e_{6,20} + e_{9,22} + e_{11,24} + e_{16,27}$$
$$e_{\substack{01211\\1}} = e_{2,12} + e_{3,13} - e_{6,16} - e_{7,17} - e_{18,25} + e_{20,27}$$
$$e_{\substack{12210\\1}} = -e_{1,11} - e_{2,18} + e_{8,23} + e_{10,24} + e_{12,25} + e_{14,26}$$
$$e_{\substack{11211\\1}} = e_{1,12} - e_{3,19} + e_{6,21} + e_{7,22} - e_{11,25} + e_{15,27}$$
$$e_{\substack{01221\\1}} = -e_{2,14} - e_{3,15} - e_{4,16} - e_{5,17} + e_{18,26} + e_{19,27}$$

$$e_{\substack{12211\\1}} = -e_{1,13} - e_{2,19} - e_{6,23} - e_{7,24} - e_{9,25} + e_{14,27}$$
$$e_{\substack{11221\\1}} = -e_{1,14} + e_{3,20} + e_{4,21} + e_{5,22} + e_{11,26} + e_{13,27}$$
$$e_{\substack{12221\\1}} = e_{1,15} + e_{2,20} - e_{4,23} - e_{5,24} + e_{9,26} + e_{12,27}$$
$$e_{\substack{12321\\1}} = -e_{1,16} - e_{2,21} - e_{3,23} + e_{5,25} + e_{7,26} + e_{10,27}$$
$$e_{\substack{12321\\2}} = e_{1,17} + e_{2,22} + e_{3,24} + e_{4,25} + e_{6,26} + e_{8,27}$$

TABLE 17. Cubic form for the D_5-numbering of weights

$$\begin{aligned}
Q(x) = {} & x_1x_{18}x_{27} - x_1x_{19}x_{26} + x_1x_{20}x_{25} - x_1x_{21}x_{24} + x_1x_{22}x_{23} \\
& - x_2x_{11}x_{27} + x_2x_{13}x_{26} - x_2x_{15}x_{25} + x_2x_{16}x_{24} - x_2x_{17}x_{23} \\
& + x_3x_9x_{27} - x_3x_{12}x_{26} + x_3x_{14}x_{25} - x_3x_{16}x_{22} + x_3x_{17}x_{21} \\
& - x_4x_7x_{27} + x_4x_{10}x_{26} - x_4x_{14}x_{24} + x_4x_{15}x_{22} - x_4x_{17}x_{20} \\
& + x_5x_6x_{27} - x_5x_8x_{26} + x_5x_{14}x_{23} - x_5x_{15}x_{21} + x_5x_{16}x_{20} \\
& - x_6x_{10}x_{25} + x_6x_{12}x_{24} - x_6x_{13}x_{22} + x_6x_{17}x_{19} + x_7x_8x_{25} \\
& - x_7x_{12}x_{23} + x_7x_{13}x_{21} - x_7x_{16}x_{19} - x_8x_9x_{24} + x_8x_{11}x_{22} \\
& - x_8x_{17}x_{18} + x_9x_{10}x_{23} - x_9x_{13}x_{20} + x_9x_{15}x_{19} - x_{10}x_{11}x_{21} \\
& + x_{10}x_{16}x_{18} + x_{11}x_{12}x_{20} - x_{11}x_{14}x_{19} - x_{12}x_{15}x_{18} + x_{13}x_{14}x_{18}
\end{aligned}$$

$$\begin{aligned}
f_1(x) &= x_{18}x_{27} - x_{19}x_{26} + x_{20}x_{25} - x_{21}x_{24} + x_{22}x_{23} \\
f_2(x) &= -x_{11}x_{27} + x_{13}x_{26} - x_{15}x_{25} + x_{16}x_{24} - x_{17}x_{23} \\
f_3(x) &= x_9x_{27} - x_{12}x_{26} + x_{14}x_{25} - x_{16}x_{22} + x_{17}x_{21} \\
f_4(x) &= -x_7x_{27} + x_{10}x_{26} - x_{14}x_{24} + x_{15}x_{22} - x_{17}x_{20} \\
f_5(x) &= x_6x_{27} - x_8x_{26} + x_{14}x_{23} - x_{15}x_{21} + x_{16}x_{20} \\
f_6(x) &= x_5x_{27} - x_{10}x_{25} + x_{12}x_{24} - x_{13}x_{22} + x_{17}x_{19} \\
f_7(x) &= -x_4x_{27} + x_8x_{25} - x_{12}x_{23} + x_{13}x_{21} - x_{16}x_{19} \\
f_8(x) &= -x_5x_{26} + x_7x_{25} - x_9x_{24} + x_{11}x_{22} - x_{17}x_{18} \\
f_9(x) &= x_3x_{27} - x_8x_{24} + x_{10}x_{23} - x_{13}x_{20} + x_{15}x_{19} \\
f_{10}(x) &= x_4x_{26} - x_6x_{25} + x_9x_{23} - x_{11}x_{21} + x_{16}x_{18} \\
f_{11}(x) &= -x_2x_{27} + x_8x_{22} - x_{10}x_{21} + x_{12}x_{20} - x_{14}x_{19} \\
f_{12}(x) &= -x_3x_{26} + x_6x_{24} - x_7x_{23} + x_{11}x_{20} - x_{15}x_{18} \\
f_{13}(x) &= x_2x_{26} - x_6x_{22} + x_7x_{21} - x_9x_{20} + x_{14}x_{18} \\
f_{14}(x) &= x_3x_{25} - x_4x_{24} + x_5x_{23} - x_{11}x_{19} + x_{13}x_{18} \\
f_{15}(x) &= -x_2x_{25} + x_4x_{22} - x_5x_{21} + x_9x_{19} - x_{12}x_{18} \\
f_{16}(x) &= x_2x_{24} - x_3x_{22} + x_5x_{20} - x_7x_{19} + x_{10}x_{18} \\
f_{17}(x) &= -x_2x_{23} + x_3x_{21} - x_4x_{20} + x_6x_{19} - x_8x_{18} \\
f_{18}(x) &= x_1x_{27} - x_8x_{17} + x_{10}x_{16} - x_{12}x_{15} + x_{13}x_{14} \\
f_{19}(x) &= -x_1x_{26} + x_6x_{17} - x_7x_{16} + x_9x_{15} - x_{11}x_{14} \\
f_{20}(x) &= x_1x_{25} - x_4x_{17} + x_5x_{16} - x_9x_{13} + x_{11}x_{12} \\
f_{21}(x) &= -x_1x_{24} + x_3x_{17} - x_5x_{15} + x_7x_{13} - x_{10}x_{11} \\
f_{22}(x) &= x_1x_{23} - x_3x_{16} + x_4x_{15} - x_6x_{13} + x_8x_{11} \\
f_{23}(x) &= x_1x_{22} - x_2x_{17} + x_5x_{14} - x_7x_{12} + x_9x_{10} \\
f_{24}(x) &= -x_1x_{21} + x_2x_{16} - x_4x_{14} + x_6x_{12} - x_8x_9 \\
f_{25}(x) &= x_1x_{20} - x_2x_{15} + x_3x_{14} - x_6x_{10} + x_7x_8 \\
f_{26}(x) &= -x_1x_{19} + x_2x_{13} - x_3x_{12} + x_4x_{10} - x_5x_8 \\
f_{27}(x) &= x_1x_{18} - x_2x_{11} + x_3x_9 - x_4x_7 + x_5x_6
\end{aligned}$$

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