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NORMALIZER OF THE CHEVALLEY GROUP OF TYPE E_6

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Dedicated to the centenary of the birth of Dmitriĭ Konstantinovich Faddeev

Abstract. We consider the simply connected Chevalley group $G(\mathrm{E}_6, R)$ of type E_6 in a 27-dimensional representation. The main goal is to establish that the following four groups coincide: the normalizer of the Chevally group $G(\mathrm{E}_6, R)$ itself, the normalizer of its elementary subgroup $E(\mathrm{E}_6, R)$, the transporter of $E(\mathrm{E}_6, R)$ in $G(\mathrm{E}_6, R)$, and the extended Chevalley group $\overline{G}(\mathrm{E}_6, R)$. This is true over an arbitrary commutative ring R, all normalizers and transporters being taken in $\mathrm{GL}(27, R)$. Moreover, $\overline{G}(\mathrm{E}_6, R)$ is characterized as the stabilizer of a system of quadrics. This result is classically known over algebraically closed fields; in the paper it is established that the corresponding scheme over $\mathbb{Z}$ is smooth, which implies that the above characterization is valid over an arbitrary commutative ring. As an application of these results, we explicitly list equations a matrix $g \in \mathrm{GL}(27, R)$ must satisfy in order to belong to $\overline{G}(\mathrm{E}_6, R)$. These results are instrumental in a subsequent paper of the authors, where overgroups of exceptional groups in minimal representations will be studied.

The easiest way to define the general orthogonal group is to realize it as the stabilizer of a quadric. In the present paper we prove a similar characteristic free geometric characterization of the normalizer of the simply connected Chevalley group $G_{\mathrm{sc}}(\mathrm{E}_6, R)$ as the stabilizer of the intersection of 27 quadrics in the 27-dimensional space, and prove that it coincides with the normalizer of the elementary group $E_{\mathrm{sc}}(\mathrm{E}_6, R)$.

§1. Introduction

In [17], the second author initiated a generalization of the results by the first author and Viktor Petrov [13, 14, 58] on overgroups of classical groups to the overgroups of the exceptional groups $E(\mathrm{E}_6, R)$ and $E(\mathrm{E}_7, R)$ in minimal representations. It turned out that one of the first steps necessary to implement the localization proof in the spirit of the above papers is an *explicit* calculation of the normalizer of these groups in the corresponding general linear group $\mathrm{GL}(27, R)$ or $\mathrm{GL}(56, R)$. In the present paper we completely solve this problem for the group $E(\mathrm{E}_6, R)$. Unfortunately, for the group E_7 we could only get the corresponding result under an additional assumption $2 \in R^*$.

More precisely, in §§2 and 3, in the ring of integer polynomials $\mathbb{Z}[x_1, \dots, x_{27}]$ we explicitly construct an ideal I generated by 27 quadratic forms $f_1, \dots, f_{27}$, which has the following property. Let $\mathrm{Fix}_R(I)$ denote the set of R-linear transformations preserving the ideal I; see §5 for the precise definitions. Our first main goal in the present paper is the proof of the following result. Here we denote by G the affine group scheme such that $G(R) = \mathrm{Fix}_R(I)$.

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Theorem 1. *We have the isomorphism* $G \cong \overline{G}_{\mathrm{sc}}(\mathrm{E}_6, -)$ *of affine group schemes over* $\mathbb{Z}$.

This result can be viewed as an explicit description of equations defining the *extended* simply connected Chevalley–Demazure group scheme $\overline{G}_{\mathrm{sc}}(\Phi, -)$ of type $\Phi = \mathrm{E}_6$. This scheme was constructed in [36]; see also [5, 6, 8] and §1 below. For $\Phi = \mathrm{E}_6$, the easiest way to visualize the scheme $\overline{G}_{\mathrm{sc}}(\Phi, -)$ is to think of it as the Levi factor of the parabolic subscheme of type P_7 in $G_{\mathrm{sc}}(\mathrm{E}_7, -)$, where $G_{\mathrm{sc}}(\Phi, -)$ is the usual simply connected Chevalley–Demazure group scheme of type Φ. See [57] for the scheme-theoretic definition of parabolic subgroups and their Levi factors, and see [8] for the above identification itself.

Our results are closely related to the description of $G_{\mathrm{sc}}(\mathrm{E}_6, R)$ as the stabilizer of a cubic form f on $V = V(\varpi_1)$. The system of quadratic forms under study is precisely the set of all first order partial derivatives of the form f, parametrized by the weights of the dual module $V^* = V(\varpi_6)$.

Let E, F be two subgroups of a group G. Recall that the *transporter* of E to F is the group

$$\mathrm{Tran}_G(E, F) = \{g \in G \mid E^g \le F\}.$$

Actually, we mostly use this notation in the case where $E \le F$, and then

$$\mathrm{Tran}_G(E, F) = \{g \in G \mid [g, E] \le F\}.$$

Now we are all set to state the main result of the present paper. Observe that all normalizers and transporters here are taken in the general linear group $\mathrm{GL}(27, R)$.

Theorem 2. *Let* R *be any commutative ring. Then*

$$N(E(\mathrm{E}_6, R)) = N(G(\mathrm{E}_6, R)) = \mathrm{Tran}(E(\mathrm{E}_6, R), G(\mathrm{E}_6, R)) = G(R).$$

Observe that Eiichi Abe and James Hurley [27] established that the corresponding *centralizers* coincide, but that is a much easier result. Theorem 2 is proved by roughly the same method as Theorem 3 in [13], as a combination of the following three ingredients: normality of the elementary subgroup $E(\mathrm{E}_6, R)$ in the Chevalley group $G(\mathrm{E}_6, R)$, established by Giovanni Taddei, Theorem 1, and explicit matrix calculations in the 27-dimensional representation, similar to the calculations of levels and centralizers carried out in [17] and [9]. On the other hand, the proof of Theorem 1 presented below follows the outline proposed in [78, 79], which is now standard in the theory of affine group schemes, and consists of the following three stages.

• Verification of the fact that $G = \mathrm{Fix}(I)$ is an affine group scheme defined over $\mathbb{Z}$.

• Verification of the fact that G_K coincides with the normalizer of $G_{\mathrm{sc}}(\mathrm{E}_6, K)$ for an *algebraically closed* field K.

• Verification of the fact that G_K is smooth, or, in other words, calculation of the dimension of the Lie algebra of G_K, viewed as an affine group scheme over K.

The second item is nontrivial, but, *essentially*, well known. There are several possible approaches to the proof of this coincidence.

• On one hand, it is possible to directly use the classification of simple algebraic groups, or a characterization of an algebraic group in terms of its building. These approaches were employed by Michael Aschbacher in [29]–[31] and by Bruce Cooperstein in the paper [42] devoted to the case where $\Phi = \mathrm{E}_7$.

• On the other hand, the coincidence in question follows immediately from the description of the maximal connected subgroups in classical groups, obtained by Eugene Dynkin (in characteristic 0) and by Gary Seitz and others (regardless of the characteristic). A proof in this style was reproduced in [15].

• Finally, in [73] for the case of E_6 we sketched an elementary proof, which does not involve anything beyond elementary multilinear algebra. This proof is very similar in spirit to the proof of the Ree–Dieudonné identification theorem for classical Chevalley groups, as presented in [52]. However, due to the lack of space, many details were omitted in [73], in particular the modifications necessary to cover characteristics 2 and 3, as well as a proof that G_K is smooth.

The remaining items are essentially *exercises* in the theory of affine group schemes; see, e.g., [77]. As a matter of fact, the third item requires rather serious computations. But these calculations are quite standard and closely follow the proofs of Theorems 3.4, 5.5 and 6.5 in the paper [79] by William Waterhouse.

This paper is organized as follows. In §2 we recall the basic definitions pertaining to Chevalley groups. In §3 we construct an invariant cubic form and a system of quadrics, and in §4 we prove that this form is indeed invariant. The core of the paper is §§5–8, which are devoted directly to the proofs of Theorems 1 and 2. After that, in §9 we explicitly list the equations satisfied by a matrix in $\overline{G}_{\mathrm{sc}}(\mathrm{E}_6, R)$.

§2. Extended Chevalley group of type E_6

The present paper is wrapped into a certain context. It is impossible to recall here all the necessary notions, and we do not try to do this.

• All definitions pertaining to root systems, Weyl groups, weights, and representations can be found in [3, 4, 22]. Weight diagrams are discussed thoroughly in [69, 60, 73, 74, 7], where many further references can be found.

• We list some classical general references on Chevalley groups and their representations [2, 20, 38], on algebraic groups [19, 21], on linear groups [49], and on affine group schemes [77].

• All necessary definitions and specific facts pertaining to Chevalley groups over rings can be found, e.g., in [1, 7], [9]–[12], [24]–[28], [50, 57, 60, 68, 69], [72]–[76].

• Extended Chevalley groups were introduced by Claude Chevalley himself [23] in the adjoint case and by Berman and Moody [36] in the simply connected case. Various constructions of extended groups, and complete proofs of all facts concerning the interrelation of the extended and usual Chevalley groups can be found in [5, 6, 8].

• Invariant tensors for Chevalley groups of types E_6 and E_7 in minimal representations are discussed thoroughly in [11, 16], [29]–[32], [39]–[42], [44]–[48], [63]–[67], [73, 74], and we often turn to these papers for a hint.

In the above papers one can also find many further references. Below we briefly recall some basic notation to be used in the sequel.

Let Φ be a reduced irreducible root system of rank l (in the main body of the paper we assume that $\Phi = \mathrm{E}_6$), and let P be a lattice intermediate between the root lattice $Q(\Phi)$ and the weight lattice $P(\Phi)$. We fix an order on Φ and denote by $\Pi = \{\alpha_1, \dots, \alpha_l\}$, Φ^+, and Φ^- the corresponding sets of fundamental, positive, and negative roots. Our numbering of roots follows [3]. By δ we denote the maximal root of the system Φ with respect to this ordering. For example, for $\Phi = \mathrm{E}_6$ we have $\delta = \begin{smallmatrix}12321\\2\end{smallmatrix}$. Denote by $P(\Phi)_{++}$ the set of dominant weights with respect to this order. Recall that it consists of all nonnegative integral linear combinations of the fundamental weights $\varpi_1, \dots, \varpi_l$. We denote by $W = W(\Phi)$ the Weyl group of the root system Φ.

Next, let R be a commutative ring with 1. It is classically known that, starting with this data, one can construct the *Chevalley group* $G = G_P(\Phi, R)$, which is the group of R-points of an affine group scheme $G = G_P(\Phi, -)$ known as the *Chevalley–Demazure scheme*. For the type of problems in question, it suffices to limit ourselves to the simply connected (alias, universal) groups, for which $P = P(\Phi)$. For the simply connected groups we usually omit any reference to P and simply write $G(\Phi, R)$ or, when we wish to stress that the group in question is simply connected, $G_{\mathrm{sc}}(\Phi, R)$. The adjoint group, for which $P = Q(\Phi)$, is denoted by $G_{\mathrm{ad}}(\Phi, R)$.

Fix a split maximal torus $T(\Phi, R)$ in $G(\Phi, R)$ and a parametrization of the unipotent root subgroups X_α, $\alpha \in \Phi$, elementary with respect to this torus. Let $x_\alpha(\xi)$ be the elementary unipotent root corresponding to $\alpha \in \Phi$ and $\xi \in R$ in this parametrization. The group $X_\alpha = \{x_\alpha(\xi), \xi \in R\}$ is called simply an (elementary) *root subgroup*, and the group $E(\Phi, R) = \langle X_\alpha, \alpha \in \Phi \rangle$ generated by all elementary root subgroups is called the (absolute) *elementary subgroup* of the Chevalley group $G(\Phi, R)$.

As a matter of fact, apart from the usual Chevalley group, we also consider *extended* Chevalley groups $\overline{G}(\Phi, R)$, which play the same role with respect to $G(\Phi, R)$ as the general linear group $\mathrm{GL}(n, R)$ plays with respect to the special linear group $\mathrm{SL}(n, R)$. *Adjoint* extended groups were constructed in the original paper [23] by Chevalley. It is somewhat harder to construct *simply connected* extended groups, because here one must increase the dimension of the maximal torus. A unified elementary construction was proposed only by Berman and Moody in [36]. However, for the case of the group of type E_6 we are busy with, it is natural to interpret the group $\overline{G}_{\mathrm{sc}}(\mathrm{E}_7, K)$ as a subgroup of the usual Chevalley group $G_{\mathrm{sc}}(\mathrm{E}_7, K)$,

$$\overline{G}_{\mathrm{sc}}(\mathrm{E}_6, K) = G_{\mathrm{sc}}(\mathrm{E}_6, K) \cdot T_{\mathrm{sc}}(\mathrm{E}_7, K).$$

A priori, the elementary Chevalley group $E(\Phi, R)$ depends on the choice of a maximal torus $T(\Phi, R)$. However, the main result of the paper [71] by Giovanni Taddei, which plays a crucial role in the proof of our Theorem 2, asserts that for $\mathrm{rk}(\Phi) \geq 2$ this is not the case. For classical groups, similar results had been proved earlier by Andrei Suslin and Viacheslav Kopeiko; see [33, 34, 51, 70, 73] for the history of this result, other proofs and generalizations.

Lemma 1. *For* $\mathrm{rk}(\Phi) \geq 2$*, the elementary subgroup* $E(\Phi, R)$ *is normal in the extended Chevalley group* $\overline{G}(\Phi, R)$ *for any commutative ring* R*.*

Formally, [71] establishes only the normality of the elementary subgroup in the usual Chevalley group $G(\Phi, R)$, but normality in $\overline{G}(\Phi, R)$ can be proved easily by the same method. Moreover, in the papers [72] by Leonid Vaserstein and [51] by Roozbeh Hazrat and the first author [51], the following stronger results can be found, each implying, *in particular*, Lemma 1 in the above form.

Lemma 2. *For* $\mathrm{rk}(\Phi) \geq 2$*, the elementary subgroup* $E(\Phi, R)$ *is characteristic in the Chevalley group* $G(\Phi, R)$ *for any commutative ring* R*.*

Lemma 3. *Let* $\mathrm{rk}(\Phi) \geq 2$*, and if* $\Phi = \mathrm{B}_2, \mathrm{G}_2$*, then assume additionally that* R *does not have a residue field* $\mathbb{F}_2$ *of two elements. Then the group* $E(\Phi, R)$ *can be characterized as the largest perfect subgroup of* $G(\Phi, R)$*.*

In the majority of existing constructions, the Chevalley group $G = G(\Phi, R)$ arises together with an action on a *Weyl module* $V = V(\omega)$, for a dominant weight ω. Denote by $\Lambda = \Lambda(\omega)$ the *multiset* of weights of the module $V = V(\omega)$ *with multiplicity*. In the present paper we consider the group $G(\mathrm{E}_6, R)$ in the minimal representation with the highest weight ϖ_1. This is a microweight representation, so that the multiplicities of all weights equal 1. Fix an admissible base v^λ, $\lambda \in \Lambda$, of the module V. We conceive a vector $a \in V$, $a = \sum v^\lambda a_\lambda$, as a *column* of coordinates $a = (a_\lambda)$, $\lambda \in \Lambda$.

In Figure 1 we reproduce the weight diagram of the representation (E_6, ϖ_1), together with the *natural* numbering of weights, used in the sequel. In this numbering, the weights are listed in accordance with the order determined by the fundamental root system Π. We refer the reader to [11] for a list of weights in the Dynkin form and in the hyperbolic form, as well as for other common numberings. Recall that, in a weight diagram, two weights are joined by an edge if their difference is a *fundamental* root. A weight graph

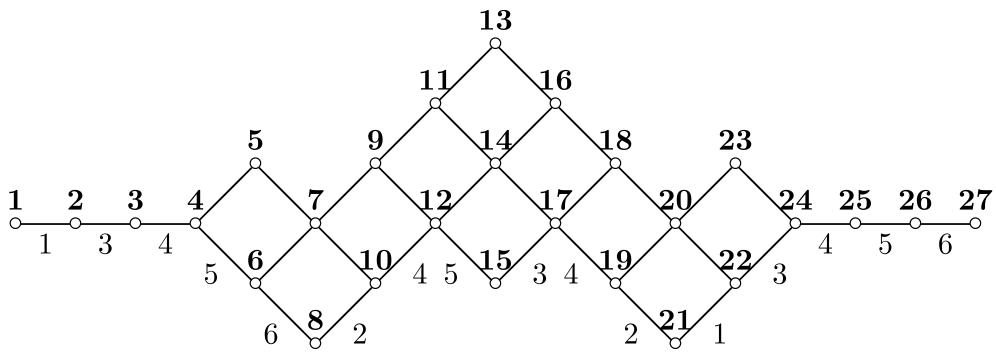


FIGURE 1. (E_6, ϖ_1).

is constructed in precisely the same way, but now two weights are joined by an edge if their difference is a *positive* root. In the sequel we denote by $d(\lambda,\mu)$ the distance between weights λ and μ in the weight graph. In other words, $d(\lambda,\mu)=0$ if $\lambda=\mu$; $d(\lambda,\mu)=1$ if $\lambda-\mu\in\Phi$; and finally, $d(\lambda,\mu)=2$ if $\lambda\neq\mu$ and $\lambda-\mu\notin\Phi$.

In [69, 61, 73, 74, 7], many further details can be found as to how the action of root unipotents $x_\alpha(\xi)$, $w_\alpha(\varepsilon)$, $h_\alpha(\varepsilon)$, the signs of structure constants, the shape and signs of equations, etc. can be read off this diagram. Formally, all these things are not at all necessary to understand the proofs expounded in the present paper. Actually, in developing the proofs presented in §§3, 4, 7–9, we made essential use of weight diagrams.

Over a field, or in general, over a semilocal ring, the extended Chevalley group $\overline{G}(E_6,R)$ is generated by the usual Chevalley group $G(E_6,R)$ and weight elements $h_{\varpi_1}(\varepsilon)$, $\varepsilon\in R^*$. In the natural numbering of weights, the element $h_{\varpi_1}(\varepsilon^{-1})$ acts on the module $V(\varpi_1)$ as follows:

$$h_{\varpi_1}(\varepsilon^{-1})=\mathrm{diag}(\varepsilon^{-1},1,1,1,1,1,1,1,1,1,1,1,\varepsilon,1,1,\varepsilon,1,\varepsilon,1,\varepsilon,1,\varepsilon,\varepsilon,\varepsilon,\varepsilon,\varepsilon,\varepsilon).$$

Observe that the exponent of ε increases by 1 each time we cross an edge marked α_1.

§3. CONSTRUCTION OF A SYSTEM OF QUADRICS

In the present section we construct the ideal I. It is generated by 27 quadrics in 27 variables, which will be constructed as the first partial derivatives of a G-invariant cubic form on V. Here are some of the basic references for this section: [29]–[31], [37], [39]–[42], [44, 48, 50], [63]–[67]. Further details and references can be found in [11, 60, 73, 74].

Let $V=V(\varpi_1)$ be the 27-dimensional module for the Chevalley group $G=G_{\mathrm{sc}}(\Phi,R)$ of type E_6. Then there exists a trilinear form $F: V\times V\times V\longrightarrow R$ such that G is the full isometry group of F, or in other words, G coincides with the group of all $g\in\mathrm{GL}(V)$ such that $F(gu,gv,gw)=F(u,v,w)$ for all $u,v,w\in V$. The similarities of the form F, i.e., the transformations g such that $F(gu,gv,gw)=\lambda F(u,v,w)$ for an appropriate scalar $\lambda\in R^*$, constitute the *extended* Chevalley group $\overline{G}=\overline{G}(E_6,R)$.

This form was discovered by Leonard Dickson in 1901 (!), was used by Elie Cartan in his geometric analysis of the real Lie group, and was studied by Claude Chevalley and Schaeffer in 1950–1951. A particularly elementary and elegant construction of this form was proposed by Hans Freudenthal in 1952. As a matter of fact, Freudenthal's construction gives not the trilinear form F itself, but rather the associated *cubic* form Q. Clearly, V can be identified with the 27-dimensional R-module $M(3,R)^3$. Now we define the value of the cubic form Q at an element $(a,b,c)\in M(3,R)^3$ as follows:

$$Q((a,b,c))=\det(a)+\det(b)+\det(c)-\mathrm{tr}(abc).$$

It can be proved (see [29, 30]) that over a field the (well-defined) isometry group of the cubic form Q coincides with the isometry group of its complete polarization F. In fact, Aschbacher used a different construction of the form, not in terms of $3\,\mathrm{A}_2$ as above, but rather in terms of A_5 (the spirit of this construction is expressed by the partition $27 = 6 + 15 + 6$), but the resulting forms are equivalent. This result does not depend on the characteristic and can be extended to all commutative rings (see [73, §6]).

Remark. Should the form Q be nondegenerate, equality of the isometry groups would break in characteristics 2 and 3. But the form Q has a rather strong degeneration. The fact that Q must be degenerate is clear because the semisimple part of the isometry group of a nondegenerate cubic form over a field of characteristic 0 is *finite*. To be more precise, Aschbacher considered 3-forms that are triples consisting of a cubic form Q, its partial polarization T, linear in the first argument and quadratic in the second argument, and its complete polarization F; see [29]–[31]. To get a *genuine* theory over rings, the notion of a 3-form must be generalized in the spirit of Bak's theory of quadratic forms over rings; see the references in [49, 34, 58]. This was done in 2001 in the paper [35], which, unfortunately, is still unpublished.

The cubic form Q can be interpreted also as the norm form of an exceptional 27-dimensional Jordan algebra; see [44]. As such it was studied by Hans Freudenthal, Tonny Springer, Ferdinand Veldkamp, Nathan Jacobson, and others. This interpretation is closely related to the realization of the Chevalley group of type E_6 as the *structure group* of the *split* exceptional Jordan algebra. Over fields, see the paper [67] where this result was stated in a slightly different, but essentially equivalent language of J-systems. Over rings, to be able to work also in the case where $2 \notin R^*$, one should consider *quadratic* Jordan algebras instead.

Recall the construction of the cubic form from [73, 74, 11], where the rule of signs can also be found. It is known that the monomials of the form correspond to the triads of weights and constitute a single Weyl orbit. Namely, a triple of distinct weights (λ, μ, ν) is called a *triad* provided the pairwise differences $\lambda - \mu$, $\lambda - \nu$, $\mu - \nu$ are not roots. In other words, the pairwise distances between these weights in the weight graph are all equal to 2. In the realization of the weights of the representation (E_6, ϖ_1) inside E_7, this means precisely that the roots λ, μ, ν are pairwise orthogonal. In the terminology of [29], a triple (λ, μ, ν) of weights is a triad if and only if v^λ, v^μ, and v^ν generate a *special plane*.

Clearly, a triad is completely determined by any two of its elements. In other words, for any two weights λ, μ at distance 2 in the weight graph, there exists a *unique* weight $\nu = \lambda \circ \mu$ such that $(\lambda, \mu, \lambda \circ \mu)$ forms a triad.

Let Θ be the set of all triads, $|\Theta| = 27 \cdot 10$. Then the trilinear form F takes the following values: $F(v^\lambda, v^\mu, v^\nu) = \pm 1$ if $(\lambda, \mu, \nu) \in \Theta$, and $F(v^\lambda, v^\mu, v^\nu) = 0$ otherwise. The signs are determined by the condition that F is invariant under the action of the extended Weyl group $\widetilde{W}$. In the realization inside the unipotent radical of P_7, the model triad has the form

$$(\lambda_0, \mu_0, \nu_0) = \left({234321 \atop 2}, {012221 \atop 1}, {000001 \atop 0} \right),$$

and we set $F(\lambda_0, \mu_0, \nu_0) = 1$ for this triad. For any other triad (λ, μ, ν) the sum

$$\lambda + \mu + \nu = \lambda_0 + \mu_0 + \nu_0 = {246543 \atop 3}$$

is orthogonal to all fundamental roots $\alpha_1, \dots, \alpha_6$. Thus, for $w_\alpha \in W(\mathrm{E}_6)$ we have the following alternative: either $w_\alpha(\lambda, \mu, \nu) = (\lambda, \mu, \nu)$, or exactly two of the weights λ, μ, ν are moved by w_α. In that case they are moved in opposite directions, say

$$w_\alpha(\lambda) = \lambda + \alpha, \qquad w_\alpha(\mu) = \mu - \alpha, \qquad w_\alpha(\nu) = \nu.$$

Looking at this from the viewpoint of the signed base $\pm v^\lambda$, on which the *extended* Weyl group acts, we see that either the triple v^λ, v^μ, v^ν remains unchanged under the action of $w_\alpha(1)$, or it is moved to *another* triple, and this is accompanied by exactly *one* change of sign.

This shows how to calculate the sign of $F(v^\lambda, v^\mu, v^\nu)$. Namely, set

$$F(v^\lambda, v^\mu, v^\nu) = \operatorname{sgn}(w),$$

where w is a shortest element of the Weyl group $W(\mathrm{E}_6)$ such that

$$w(\lambda_0, \mu_0, \nu_0) = (\lambda, \mu, \nu).$$

For actual calculations, either those in the proof of the theorem below, or those done with the help of a computer (see [11]), this definition is more convenient than that reproduced in [74], where the sign is specified as $(-1)^{h(\lambda,\mu,\nu)}$, in terms of distances in the weight diagram.

Theorem 3. *Define a trilinear form on the* 27*-dimensional module $V = V(\varpi_1)$ by the formula*

$$F(x, y, z) = \sum \operatorname{sgn}(w) x_\lambda y_\mu z_\nu,$$

where the sum is taken over all triads $(\lambda, \mu, \nu) \in \Theta$, $x, y, z \in V$, and $w \in W(\mathrm{E}_6)$ is such that $w(\lambda_0, \mu_0, \nu_0) = (\lambda, \mu, \nu)$. Then the form F is invariant under the action of the elementary Chevalley group $E(\mathrm{E}_6, R)$. Under the action of the weight element $h_{\varpi_1}(\varepsilon)$ the form F is multiplied by ε^{-1}.

The cubic form is defined similarly, but to avoid the occurrence of the coefficient 6, which causes problems in characteristics 2 and 3, now we have to sum over the set Θ_0 of unordered triads $\{\lambda, \mu, \nu\}$ instead. Clearly, $|\Theta_0| = |\Theta|/6 = 45$. Now the value of the form Q at a vector $x = \sum x_\lambda v^\lambda$ is defined by the formula

$$Q(x) = \sum \operatorname{sgn}(w) x_\lambda x_\mu x_\nu,$$

where the sum is taken over $\{\lambda, \mu, \nu\} \in \Theta_0$, while w has the same meaning as above.

We reproduce the resulting cubic form Q as it was given in [11], with respect to the *natural* numbering of weights as in Figure 1:

$$\begin{aligned} Q(x) = {} & x_1x_{13}x_{27} - x_1x_{16}x_{26} + x_1x_{18}x_{25} - x_1x_{20}x_{24} + x_1x_{22}x_{23} \\ & - x_2x_{11}x_{27} + x_2x_{14}x_{26} - x_2x_{17}x_{25} + x_2x_{19}x_{24} - x_2x_{21}x_{23} \\ & + x_3x_9x_{27} - x_3x_{12}x_{26} + x_3x_{15}x_{25} - x_3x_{19}x_{22} + x_3x_{20}x_{21} \\ & - x_4x_7x_{27} + x_4x_{10}x_{26} - x_4x_{15}x_{24} + x_4x_{17}x_{22} - x_4x_{18}x_{21} \\ & + x_5x_6x_{27} - x_5x_8x_{26} + x_5x_{15}x_{23} - x_5x_{17}x_{20} + x_5x_{18}x_{19} \\ & - x_6x_{10}x_{25} + x_6x_{12}x_{24} - x_6x_{14}x_{22} + x_6x_{16}x_{21} + x_7x_8x_{25} \\ & - x_7x_{12}x_{23} + x_7x_{14}x_{20} - x_7x_{16}x_{19} - x_8x_9x_{24} + x_8x_{11}x_{22} \\ & - x_8x_{13}x_{21} + x_9x_{10}x_{23} - x_9x_{14}x_{18} + x_9x_{16}x_{17} - x_{10}x_{11}x_{20} \\ & + x_{10}x_{13}x_{19} + x_{11}x_{12}x_{18} - x_{11}x_{15}x_{16} - x_{12}x_{13}x_{17} + x_{13}x_{14}x_{15}. \end{aligned}$$

Now, the symmetric trilinear form F is obtained by polarization.

For further reference, we also reproduce the explicit values of the first order partial derivatives of this form, with respect to the same weight numbering:

$$\begin{aligned} f_1(x) &= x_{13}x_{27} - x_{16}x_{26} + x_{18}x_{25} - x_{20}x_{24} + x_{22}x_{23}, \\ f_2(x) &= -x_{11}x_{27} + x_{14}x_{26} - x_{17}x_{25} + x_{19}x_{24} - x_{21}x_{23}, \\ f_3(x) &= x_9x_{27} - x_{12}x_{26} + x_{15}x_{25} - x_{19}x_{22} + x_{20}x_{21}, \\ f_4(x) &= -x_7x_{27} + x_{10}x_{26} - x_{15}x_{24} + x_{17}x_{22} - x_{18}x_{21}, \end{aligned}$$

$$\begin{aligned}
f_5(x) &= x_6x_{27} - x_8x_{26} + x_{15}x_{23} - x_{17}x_{20} + x_{18}x_{19},\\
f_6(x) &= x_5x_{27} - x_{10}x_{25} + x_{12}x_{24} - x_{14}x_{22} + x_{16}x_{21},\\
f_7(x) &= -x_4x_{27} + x_8x_{25} - x_{12}x_{23} + x_{14}x_{20} - x_{16}x_{19},\\
f_8(x) &= -x_5x_{26} + x_7x_{25} - x_9x_{24} + x_{11}x_{22} - x_{13}x_{21},\\
f_9(x) &= x_3x_{27} - x_8x_{24} + x_{10}x_{23} - x_{14}x_{18} + x_{16}x_{17},\\
f_{10}(x) &= x_4x_{26} - x_6x_{25} + x_9x_{23} - x_{11}x_{20} + x_{13}x_{19},\\
f_{11}(x) &= -x_2x_{27} + x_8x_{22} - x_{10}x_{20} + x_{12}x_{18} - x_{15}x_{16},\\
f_{12}(x) &= -x_3x_{26} + x_6x_{24} - x_7x_{23} + x_{11}x_{18} - x_{13}x_{17},\\
f_{13}(x) &= x_1x_{27} - x_8x_{21} + x_{10}x_{19} - x_{12}x_{17} + x_{14}x_{15},\\
f_{14}(x) &= x_2x_{26} - x_6x_{22} + x_7x_{20} - x_9x_{18} + x_{13}x_{15},\\
f_{15}(x) &= x_3x_{25} - x_4x_{24} + x_5x_{23} - x_{11}x_{16} + x_{13}x_{14},\\
f_{16}(x) &= -x_1x_{26} + x_6x_{21} - x_7x_{19} + x_9x_{17} - x_{11}x_{15},\\
f_{17}(x) &= -x_2x_{25} + x_4x_{22} - x_5x_{20} + x_9x_{16} - x_{12}x_{13},\\
f_{18}(x) &= x_1x_{25} - x_4x_{21} + x_5x_{19} - x_9x_{14} + x_{11}x_{12},\\
f_{19}(x) &= x_2x_{24} - x_3x_{22} + x_5x_{18} - x_7x_{16} + x_{10}x_{13},\\
f_{20}(x) &= -x_1x_{24} + x_3x_{21} - x_5x_{17} + x_7x_{14} - x_{10}x_{11},\\
f_{21}(x) &= -x_2x_{23} + x_3x_{20} - x_4x_{18} + x_6x_{16} - x_8x_{13},\\
f_{22}(x) &= x_1x_{23} - x_3x_{19} + x_4x_{17} - x_6x_{14} + x_8x_{11},\\
f_{23}(x) &= x_1x_{22} - x_2x_{21} + x_5x_{15} - x_7x_{12} + x_9x_{10},\\
f_{24}(x) &= -x_1x_{20} + x_2x_{19} - x_4x_{15} + x_6x_{12} - x_8x_9,\\
f_{25}(x) &= x_1x_{18} - x_2x_{17} + x_3x_{15} - x_6x_{10} + x_7x_8,\\
f_{26}(x) &= -x_1x_{16} + x_2x_{14} - x_3x_{12} + x_4x_{10} - x_5x_8,\\
f_{27}(x) &= x_1x_{13} - x_2x_{11} + x_3x_9 - x_4x_7 + x_5x_6.
\end{aligned}$$

§4. Proof of Theorem 3

We start with two easy lemmas necessary for the proof of Theorem 3.

Lemma 4. *The form F is symmetric in all arguments.*

Proof. It suffices to verify that the form F is symmetric on the triples v^λ, v^μ, v^ν of weight elements. Since the set Θ of triads is invariant under the action of the symmetric group S_3, we see that $F(v^\lambda, v^\mu, v^\nu)$ vanishes (or not) if and only if the same happens after an arbitrary permutation of the arguments. Thus, it only remains to check that for any triad $(\lambda,\mu,\nu)\in\Theta$ the sign of $F(v^\lambda, v^\mu, v^\nu)$ is preserved as well. By the very construction, the form F is invariant under the action of the extended Weyl group $W(\mathrm{E}_6)$, which is transitive on triads. This means that without loss of generality we may assume that $(\lambda,\mu,\nu) = (\lambda_0,\mu_0,\nu_0)$. However, from Figure 1 we see that, both under the permutation of λ_0 and μ_0 keeping ν_0 and under the permutation of μ_0 and ν_0 keeping λ_0, the sign of $F(v^\lambda, v^\mu, v^\nu)$ is inverted 8 times, and thus it remains unchanged. $\square$

Remark. In this proof, it would be more natural to work not with the distinguished triad (λ_0,μ_0,ν_0), but rather with the triad represented by the triple of roots

$$(\lambda,\mu,\nu) = \left(\begin{smallmatrix}012221\\1\end{smallmatrix}, \begin{smallmatrix}112211\\1\end{smallmatrix}, \begin{smallmatrix}122111\\1\end{smallmatrix}\right)$$

in the realization of V as an internal Chevalley module in E_7. For this triple, both under the permutation of v^λ and v^μ and under the permutation of v^μ and v^μ, there are exactly two sign changes.

Lemma 5. *Let $d(\lambda,\mu)=2$. Then there is no root $\alpha\in\Phi$ such that $\lambda+\alpha,\mu+\alpha\in\Lambda$.*

Proof. By the transitivity of the Weyl group $W(\mathrm{E}_6)$ on the pairs of weights at the same distance in the weight graph, there is no loss of generality in assuming that $\lambda=\lambda_0$ is the highest weight of the module V, whereas $\mu=\nu_0$ is the lowest weight. But then $\lambda+\alpha\in\Lambda$ implies that $\alpha\in\Phi^-$, while $\mu+\alpha\in\Lambda$ implies that $\alpha\in\Phi^+$. This is impossible, because $\Phi^+\cap\Phi^-=\varnothing$. □

Proof of Theorem 3. Recall that the form F is invariant with respect to the action of the extended Weyl group $\widetilde{W}(\mathrm{E}_6)$ and is symmetric, whereas the elementary group $E(\mathrm{E}_6,R)$ is generated by the root elements $x=x_\alpha(\xi)$, $\alpha\in\pm\Pi$, $\xi\in R$, corresponding to the fundamental and negative fundamental roots. Thus, to prove Theorem 3 it suffices to verify that

$$F(xv^\lambda,xv^\mu,xv^\nu)=F(v^\lambda,v^\mu,v^\nu)$$

for one triple of weights (λ,μ,ν) from any orbit of the group $\widetilde{W}(\mathrm{E}_6)\times S_3$ on such triples. The analysis is naturally split into the following five cases: i) at least two of the weights λ,μ,ν are equal; ii) all pairwise distances among the weights λ,μ,ν are equal to 1; iii) $d(\lambda,\mu)=2$ while $d(\lambda,\nu)=d(\mu,\nu)=1$; iv) $d(\lambda,\mu)=d(\lambda,\nu)=2$ and $d(\mu,\nu)=1$; v) $(\lambda,\mu,\nu)\in\Theta$. Out of all these cases, only iv) is not quite straightforward.

Since F is linear in each argument, we get

$$\begin{aligned}F(xv^\lambda,xv^\mu,xv^\nu)=F(v^\lambda,v^\mu,v^\nu)&\\ &+F(v^{\lambda+\alpha},v^\mu,v^\nu)+F(v^\lambda,v^{\mu+\alpha},v^\nu)+F(v^\lambda,v^\mu,v^{\nu+\alpha})\\ &+F(v^{\lambda+\alpha},v^{\mu+\alpha},v^\nu)+F(v^{\lambda+\alpha},v^\mu,v^{\nu+\alpha})+F(v^\lambda,v^{\mu+\alpha},v^{\nu+\alpha})\\ &+F(v^{\lambda+\alpha},v^{\mu+\alpha},v^{\nu+\alpha}),\end{aligned}$$

with the agreement that, whenever a base vector v^ρ where ρ is not a weight occurs as an argument, the corresponding summand should be disregarded. (For instance, if $\lambda+\alpha\notin\Lambda$, then all the summands where $v^{\lambda+\alpha}$ occurs are in fact absent.) We must show that the extra terms on the right-hand side of this formula always sum to 0. In other words, we must show that all the summands, except the first, are either simply absent (those corresponding to the indices that are not weights), or are equal to 0 (those corresponding to the triples of weights other than triads), or finally, they pairwise cancel.

i) If two of the weights λ,μ,ν are equal, say $\lambda=\mu$, then all weights $\lambda,\mu,\lambda+\alpha,\mu+\alpha$ are at pairwise distances at most 1. This means that all summands on the right-hand side of the formula are equal to 0.

ii) If the pairwise distances among the weights λ,μ,ν are equal to 1, then among any three of the six weights (or not weights!) $\lambda,\mu,\nu,\lambda+\alpha,\mu+\alpha,\nu+\alpha$ at least two are taken from the first or the second triple. Thus, there are no triads among these six weights, and consequently all summands on the right-hand side are again equal to 0.

iii) If $d(\lambda,\mu)=2$ while $d(\lambda,\nu)=d(\mu,\nu)=1$, then both the left-hand side and the right-hand side are equal to 0. Indeed, by assumption we have

$$F(xv^\lambda,xv^\mu,xv^\nu)=F(v^{\lambda+\alpha},v^\mu,v^\nu)=F(v^\lambda,v^{\mu+\alpha},v^\nu)=0.$$

Next, since ν is a unique weight forming a triad together with λ,μ, we have that $F(v^\lambda,v^\mu,\,v^{\nu+\alpha})=0$. On the other hand,

$$F(v^{\lambda+\alpha},v^\mu,v^{\nu+\alpha})=F(v^\lambda,v^{\mu+\alpha},v^{\nu+\alpha})=0.$$

Indeed, if $\lambda+\alpha, \nu+\alpha$ or $\mu+\alpha, \nu+\alpha$ are weights, then $d(\lambda+\alpha, \nu+\alpha) = d(\mu+\alpha, \nu+\alpha) = 1$. Finally,

$$F(v^{\lambda+\alpha}, v^{\mu+\alpha}, v^{\nu}) = F(v^{\lambda+\alpha}, v^{\mu+\alpha}, v^{\nu+\alpha}) = 0,$$

because, by Lemma 5, the sums $\lambda+\alpha$ and $\mu+\alpha$ cannot both be weights.

iv) If $d(\lambda, \mu) = d(\lambda, \nu) = 2$, whereas $d(\mu, \nu) = 1$, then $F(v^{\lambda+\alpha}, v^{\mu}, v^{\nu}) = 0$. As in the preceding case, we have

$$F(v^{\lambda+\alpha}, v^{\mu+\alpha}, v^{\nu}) = F(v^{\lambda+\alpha}, v^{\mu}, v^{\nu+\alpha}) = F(v^{\lambda+\alpha}, v^{\mu+\alpha}, v^{\nu+\alpha}) = 0,$$

because $\lambda+\alpha$ and $\mu+\alpha$ or, respectively, $\lambda+\alpha$ and $\nu+\alpha$ cannot both be weights. Thus, it only remains to take care of the summands $F(v^{\lambda}, v^{\mu+\alpha}, v^{\nu})$ and $F(v^{\lambda}, v^{\mu}, v^{\nu+\alpha})$. These summands can be nonzero, but since

$$(v^{\lambda}, v^{\mu}, v^{\nu+\alpha}) = w_{\alpha}(1)(v^{\lambda}, v^{\mu+\alpha}, v^{\nu}),$$

they occur with opposite signs, by the very definition of the form F.

v) Finally, let $(\lambda, \mu, \nu) \in \Theta$. In this case,

$$d(\lambda+\alpha, \mu) = d(\lambda, \mu+\alpha) = d(\lambda, \nu+\alpha) = 1,$$

whence

$$F(v^{\lambda+\alpha}, v^{\mu}, v^{\nu}) = F(v^{\lambda}, v^{\mu+\alpha}, v^{\nu}) = F(v^{\lambda}, v^{\mu}, v^{\nu+\alpha}) = 0.$$

On the other hand, as in case iii), all other summands on the right-hand side are equal to 0 by Lemma 5.

To verify the last claim, we observe that, relative to the action of $h_{\varpi_1}(\varepsilon)$, the following two types of summands occur in the form F.

- The summands of the form $\pm x_{\varpi_1} x_{\mu} x_{\nu}$, where μ, ν are at distance 2 from ϖ_1. Under the action of $h_{\varpi_1}(\varepsilon)$, the coordinate x_{ϖ_1} is multiplied by ε, whereas the other two coordinates are multiplied by ε^{-1}.
- All other summands $\pm x_{\lambda} x_{\mu} x_{\nu}$, where $\lambda, \mu, \nu \neq \varpi_1$. Since the width of the set of weights at distance 1 from ϖ_1 is equal to 2, exactly one among the weights λ, μ, ν, say ν, is at distance 2 from ϖ_1. This means that under the action of $h_{\varpi_1}(\varepsilon)$ the coordinate x_{ν} is multiplied by ε^{-1}, and the other two coordinates remain unchanged.

Thus, under the action of $h_{\varpi_1}(\varepsilon)$, both types of summands are multiplied by ε^{-1}, as claimed. □

§5. Proof of Theorem 1: An outline

First, let $f_1, \dots, f_s$ be arbitrary polynomials in t variables with coefficients in R (in most real world applications, either $R = \mathbb{Z}$ or $R = \mathbb{Z}[1/2]$). We are interested in the linear changes of variables $g \in \mathrm{GL}(t, R)$ that preserve the condition that all these polynomials vanish simultaneously. In other words, we consider all $g \in \mathrm{GL}(t, R)$ preserving the ideal A of the ring $R[x_1, \dots, x_t]$ generated by $f_1, \dots, f_s$. This last condition means that for any polynomial $f \in A$ the polynomial $f \circ g$ obtained from f by the variable change g is again in A. It is well known (see, e.g., [43, Lemma 1] or [79, Proposition 1.4.1]) that the set $G(R) = \mathrm{Fix}_R(A) = \mathrm{Fix}_R(f_1, \dots, f_s)$ of all such linear-variable changes g forms a group. For any R-algebra S with 1 we can view $f_1, \dots, f_s$ as polynomials with coefficients in S. Thus, the group $G(S)$ is defined for all R-algebras. It is clear that $G(S)$ depends functorially on S. It is easy to provide examples showing that $S \mapsto G(S)$ may fail to be an affine group scheme over R. This is due to the fact that $G(R)$ is determined by *congruences*, rather than equations, in its matrix entries. However, in [79, Theorem 1.4.3 and further], a simple sufficient condition was found that guarantees that $S \mapsto G(S)$ is an affine group scheme. Denote by $R[x_1, \dots, x_t]_r$ the submodule of polynomials of degree at most r. For our purposes it suffices to invoke Corollary 1.4.6 of [79], pertaining to the case where $R = \mathbb{Z}$.

Lemma 6. *Let $f_1, \dots, f_s \in \mathbb{Z}[x_1, \dots, x_t]$ be polynomials of degree at most r, and let A be the ideal they generate. Then for the functor $S \mapsto \mathrm{Fix}_S(f_1, \dots, f_s)$ to be an affine group scheme, it suffices that the rank of the intersection $A \cap R[x_1, \dots, x_t]_r$ does not change under reduction modulo any prime $p \in \mathbb{Z}$.*

We apply this lemma to the case of the ideal $A = I$ in $\mathbb{Z}[x_\lambda]$, constructed in §2. For any commutative ring R we set $G(R) = \mathrm{Fix}_R(I)$.

Lemma 7. *The functor $R \mapsto G(R)$ is an affine group scheme defined over $\mathbb{Z}$.*

Proof. For any prime p the relations f_ω are independent modulo p. Indeed, specializing x_λ appropriately, we can guarantee that one of these relations takes the value 1, while all other relations vanish. For this, we can set $x_\lambda = x_\mu = 1$ for two weights of the representation π at distance 2 and $x_\nu = 0$ for all other weights. Now our claim follows from the fact that the product $x_\lambda x_\mu$ occurs in a unique relation f_ω and that every relation f_ω consists of such products. In fact, the product $x_\lambda x_\mu$ only occurs in the relation $f_{\lambda \circ \mu}$. □

To prove the main results of the present paper, we need to recall some further known facts. The following result essentially reduces the verification of the isomorphism of affine group schemes to the isomorphism of their groups of points over algebraically closed fields and the dual numbers over such fields. Recall that the algebra $K[\delta]$ of dual numbers over a field is isomorphic as a K-module to $K \oplus K\delta$, with multiplication given by $\delta^2 = 0$. The following lemma is Theorem 1.6.1 of [79]. We denote by G an affine group scheme over $\mathbb{Z}$, by G^0 its connected component of the identity, by $G(K)$ its group of K-points, by G_K the scheme obtained from G by a change of scalars, and by $\mathrm{Lie}(G_K)$ the Lie algebra of the scheme G_K. A certain awkwardness of the statement below is explained by the fact that, *a priori*, the schemes G and H are not assumed connected or smooth.

Lemma 8. *Let H and G be affine group schemes of finite type over $\mathbb{Z}$, where H is flat, and let $\phi : H \longrightarrow G$ be a morphism of group schemes. Assume that the following conditions are satisfied for any algebraically closed field K:*
- $\dim(H_K) \geq \dim_K(\mathrm{Lie}(G_K))$;
- *ϕ induces monomorphisms of the groups of points*

$$H(K) \longrightarrow G(K) \qquad \textit{and} \qquad H(K[\delta]) \longrightarrow G(K[\delta]);$$

- *the normalizer $\phi(H^0_K(K))$ in $G(K)$ is contained in $\phi(G(K))$.*

Then ϕ is an isomorphism of group schemes over $\mathbb{Z}$.

Observe that in our case the assumptions initially imposed on the schemes are satisfied automatically. All schemes we consider are of finite type, being subschemes of GL_n. Flatness follows from the fact that H is reduced and irreducible. Thus, we only need to verify the conditions of the lemma.

§6. The case of an algebraically closed field

The following lemma summarizes the well-known properties of the minimal representations of Chevalley groups of type E_6. Since π is a microweight representation, $\pi(G)$ is irreducible and tensor indecomposable. Since $\Lambda(\pi) = P(\Phi)$, the representation π is faithful.

Lemma 9. *Let K be an algebraically closed field. Viewed as a subgroup of $\mathrm{GL}(27, K)$, the algebraic group $\overline{G}_{\mathrm{sc}}(E_6, R)$ is irreducible and tensor indecomposable. Moreover, it is equal to its own normalizer.*

Since the representation π of the group $G = \overline{G}_{\mathrm{sc}}(\mathrm{E}_6, K)$ with the highest weight ϖ_1 is faithful, in the sequel we identify G with $\pi(G)$.

The claim concerning normalizers follows from the description of automorphisms of Chevalley groups over fields; see [20]. Recall that, in accordance with this description, every *algebraic* automorphism of the extended Chevalley group is a product of inner, central, and graph automorphisms. The usual Chevalley group has diagonal automorphisms as well, but in the extended Chevalley group they all become inner; this is precisely the reason behind the introduction of extended groups. On the other hand, all nonalgebraic automorphisms are field automorphisms.

The group of type E_6 has a unique nontrivial graph automorphism given by the *Weyl involution* $\alpha \mapsto -w_0(\alpha)$, where w_0 is the longest element of the Weyl group. Since the Weyl involution maps ϖ_1 to ϖ_6, and $V(\varpi_6) \ncong V(\varpi_1)$, this automorphism is not realized in $\mathrm{GL}(27, K)$. It is well known to be realized under the embedding

$$G(\mathrm{E}_6, K) \hookrightarrow G(\mathrm{E}_7, K) \hookrightarrow \mathrm{GL}(56, K),$$

where the second embedding is the representation with the highest weight ϖ_7.

In a classical 1952 paper, Eugene Dynkin described the maximal connected closed subgroups of simple algebraic groups over an algebraically closed field of characteristic 0 (or more precisely, reduced their description to the representation theory of simple algebraic groups). Gary Seitz [62] generalized this description to subgroups of classical algebraic groups over an arbitrary algebraically closed field (earlier results by Seitz himself and by Donna Testerman were used). One of the main results of [62] can be stated as follows. Let V be the vector representation of $\mathrm{SL}(V)$ and X be a simple algebraic subgroup of $\mathrm{SL}(V)$ such that the restriction $V|X$ of the module V to X is irreducible and tensor indecomposable. Then either X is maximal among the connected closed subgroups of $\mathrm{SL}(V)$, $\mathrm{Sp}(V)$, or $\mathrm{SO}(V)$, or else X together with a proper connected closed subgroup containing it is explicitly listed in Table 1 of [62]. Since $V(\varpi_1)^* = V(\varpi_6) \ncong V(\varpi_1)$, the minimal representation of the group of type E_6 is not orthogonal or symplectic. Thus, [62] immediately implies the following result.

Lemma 10. *Theorem* 1 *holds for any algebraically closed field.*

Proof. It suffices to prove that the connected components of the groups in question coincide. Since $\overline{G}(\mathrm{E}_6, K)$ coincides with its own normalizer, it follows that the group $G(K)$ is also connected. The fact that $\overline{G}(\mathrm{E}_6, K)$ stabilizes the requisite system of forms follows from Theorem 3. The inverse inclusion can be established as follows. In Table 1 of [62] the group of type E_6 occurs in the column X three times, but each time in the embedding $\mathrm{E}_6 < \mathrm{A}_{26}$ and never in the minimal representation. Formally, this only implies the maximality of $G(\mathrm{E}_6, K)$ in $\mathrm{SL}(27, K)$, rather than the maximality of $\overline{G}(\mathrm{E}_6, K)$ in $\mathrm{GL}(27, K)$. However, since $\det(h_{\varpi_1}(\varepsilon)) = \varepsilon^9$, for every cubic closed field the determinant of $h_{\varpi_1}(\varepsilon)$ can be arbitrary. Therefore, any connected closed subgroup that properly contains $\overline{G}(\mathrm{E}_6, K)$, contains $\mathrm{SL}(27, K)$ and matrices of an arbitrary nonzero determinant, thus coinciding with $\mathrm{GL}(27, K)$. □

§7. Dimension of the Lie algebra

In the present section we proceed with the proof of Theorem 1. Namely, here we prove that the affine group scheme G is smooth. This is one of the very few places where we need some serious calculation. Namely, we should estimate the dimension of the Lie algebra of this scheme. It is well known how to calculate the Lie algebra that stabilizes a system of forms; see, e.g., [54, pp. 256–258]. Of course, before the theory of group schemes emerged, in positive characteristic it had been impossible to extract any information concerning the group stabilizing the same system of forms. Philosophically, our calculation closely follows the work by William Waterhouse; see, e.g., [79], where almost exactly the same calculation was performed in Lemmas 3.2, 5.3 and 6.3. A similar calculation for the polyvector representation of GL_n was carried through in [15].

Let K be a field, as above. The Lie algebra $\mathrm{Lie}(G_K)$ of the affine group scheme G_K is most naturally interpreted as the kernel of the homomorphism $G(K[\delta]) \longrightarrow G(K)$ sending δ to 0; see [2, 21, 75]. Let G be a subscheme of GL_n. Then $\mathrm{Lie}(G_K)$ consists of all matrices of the form $e + z\delta$, where $z \in \mathrm{M}(n, K)$, satisfying the equations defining $G(K)$. In the next lemma we specialize this statement in the case where G is the stabilizer of a system of polynomials.

Lemma 11. *Let $f_1, \ldots, f_s \in K[x_1, \ldots, x_t]$. Then a matrix $e + z\delta$, where $z \in \mathrm{M}(t, K)$, belongs to $\mathrm{Lie}(\mathrm{Fix}_K(f_1, \ldots, f_s))$ if and only if*

$$\sum_{1 \le i,j \le t} z_{ij} x_i \frac{\partial f_h}{\partial x_j} = 0$$

for all $h = 1, \ldots, s$.

The following result is proved in exactly the same way as Lemma 5.3 in [79]. Clearly, the dimension that arises here is the dimension of the simple Lie algebra of type E_6 increased by 1. From the proof, it can also be seen which of the coefficients $z_{\lambda\mu}$ correspond to roots and which correspond to the Cartan subalgebra. The extra 1 corresponds to the additional toral summand. The Lie algebra we consider is in fact the Lie algebra of the *extended* Chevalley group, whose dimension is larger than that of the Chevalley group itself. Despite the technical character of this claim we call it a theorem, since obtaining this bound is one of the main steps in the passage from fields to arbitrary commutative rings.

Theorem 4. *For any field K, the dimension of the Lie algebra $\mathrm{Lie}(G)$ does not exceed 79.*

Proof. In our case both the variables and the equations can be indexed by the weight of the module V. Then the equations take the form

$$\sum_{\lambda,\mu} z_{\lambda\mu} x_\lambda \frac{\partial f_\nu}{\partial x_\mu} = 0, \qquad \nu \in \Lambda(V).$$

For subsequent calculations, we recall that the partial derivative $\frac{\partial f_\nu}{\partial x_\mu}$ takes the following values:

$$\frac{\partial f_\nu}{\partial x_\mu} = \begin{cases} \pm x_{\mu \circ \nu} & \text{if } \ d(\mu,\nu) = 2, \\ 0 & \text{otherwise.} \end{cases}$$

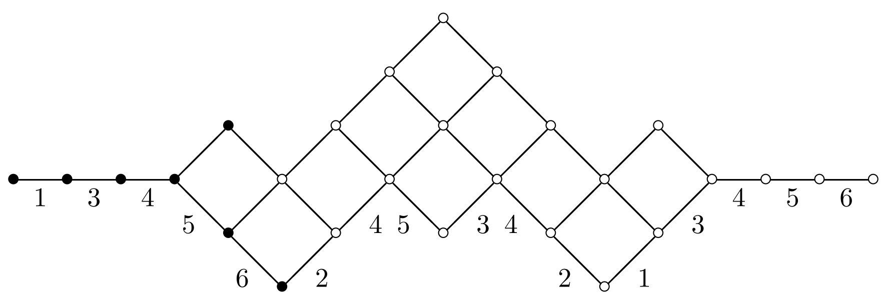
Here the sign could easily be specified, but it is not requisite for our purposes.

First, we verify the following claims.

- If $d(\lambda, \mu) \ge 2$, then $z_{\lambda\mu} = 0$. Indeed, in this case there exists a polynomial f_ν such that $\frac{\partial f_\nu}{\partial x_\mu} = x_\lambda$. However, the squares of the variables x_λ^2 do not occur in the defining equations. Thus, all the coefficients $z_{\lambda\mu}$ corresponding to such pairs (λ, μ) are equal to 0.
- If $d(\lambda, \mu) = d(\nu, \rho) = 1$ and $\lambda - \mu = \nu - \rho$, then $z_{\lambda\mu} = \pm z_{\nu\rho}$. Indeed, in this case $\lambda \circ \rho = \mu \circ \nu = \sigma$, and thus $\frac{\partial f_\sigma}{\partial x_\mu} = \pm x_\nu$ and $\frac{\partial f_\sigma}{\partial x_\rho} = \pm x_\lambda$. Since $d(\lambda, \nu) = 1$, the monomial $x_\lambda x_\nu$ does not occur in any of the defining equations. Therefore, the coefficient $\pm z_{\lambda\mu}$ must cancel with some other coefficient. Clearly, the only possibility for it is to cancel with the coefficient $\pm z_{\nu\rho}$, with which the monomial $x_\nu x_\lambda$ occurs in the same equation.
- If $d(\lambda, \mu) = d(\nu, \rho) = 1$ and $\lambda - \mu = \nu - \rho$, then for an appropriate choice of signs we have $z_{\rho\rho} = \pm z_{\lambda\lambda} \pm z_{\mu\mu} \pm z_{\nu\nu}$. Indeed, for the same σ as in the preceding item, the monomial $x_\lambda x_\rho = \pm x_\lambda \frac{\partial f_\sigma}{\partial x_\lambda}$ occurs in the defining equations, but precisely

in one such equation. Moreover, it occurs with the same coefficient as the monomial $x_\mu x_\nu = \pm x_\mu \frac{\partial f_\sigma}{\partial x_\mu}$, possibly up to sign. It remains only to calculate the coefficients with which these monomials occur in the equation indexed by σ. The monomial $x_\lambda x_\rho$ occurs once again, now in the form $x_\rho x_\lambda = \pm x_\rho \frac{\partial f_\sigma}{\partial x_\rho}$, as so does also the monomial $x_\mu x_\nu$, now in the form $x_\nu x_\mu = \pm x_\nu \frac{\partial f_\sigma}{\partial x_\nu}$. Equating the coefficients of these monomials, we see that $z_{\lambda\lambda} \pm z_{\rho\rho} = \pm z_{\mu\mu} \pm z_{\nu\nu}$.

It only remains to summarize what we have established. The first two items show that the matrix entries $z_{\lambda\mu}$ with $d(\lambda,\mu) = 2$ give no contribution whatsoever to the dimension of the Lie algebra, whereas the coefficients $z_{\lambda\mu}$ with $d(\lambda,\mu) = 1$ give the contribution equal to the number of roots of Φ. The third item allows us to express all coefficients $z_{\lambda\lambda}$ as linear combinations of coefficients $z_{\mu\mu}$, $\mu = \mu_1, \dots, \mu_t$, such that among the pairwise differences of the weights μ_i every fundamental root α_i occurs at least once. It is easily seen that the smallest number of such weights λ equals 7, and the following picture shows how they could be located in the weight diagram:



As has already been mentioned, the extra 1 corresponds to an additional toric summand. This gives the upper bound $79 = 72 + 7$. $\square$

§8. Proofs of Theorems 1 and 2

Now we are ready to finish the proofs of our main results.

Proof of Theorem 1. Consider the rational representation of algebraic groups $\pi : \overline{G}(\mathrm{E}_6, -) \longrightarrow \mathrm{GL}_{27}$ with the highest weight ϖ_1. This representation is faithful, and since a Chevalley group is generated by the elementary subgroup and torus, by Theorem 3 its image is contained in the scheme G. We wish to apply Lemma 8 to this morphism π to conclude that π is in fact a monomorphism of affine group schemes.

Indeed, for an arbitrary ring R the representation π is a monomorphism. This means that the second condition of Lemma 8 is satisfied. Clearly, $\dim(\overline{G}(\mathrm{E}_6, K)) = 79$, and Theorem 4 implies that $\dim_K(\mathrm{Lie}(G_K)) \leq 79$, so that the first condition of Lemma 8 is also fulfilled. Finally, the third condition of Lemma 8 follows from the fact that, by Lemma 9, the normalizer of $\overline{G}(\mathrm{E}_6, K)$ in $\mathrm{GL}(27, K)$ is contained in and in fact coincides with $G(K)$. This means that we can apply Lemma 8 and conclude that π establishes an isomorphism of $\overline{G}(\mathrm{E}_6, -)$ and G as affine group schemes over $\mathbb{Z}$. $\square$

Proof of Theorem 2. Clearly, $\overline{G}(\mathrm{E}_6, R) \leq N(G(\mathrm{E}_6, R))$. By Lemma 1 we have $\overline{G}(\mathrm{E}_6, R) \leq N(E(\mathrm{E}_6, R))$. On the other hand, both $N(E(\mathrm{E}_6, R))$ and $N(G(\mathrm{E}_6, R))$ are obviously contained in $\mathrm{Tran}(E(\mathrm{E}_6, R), G(\mathrm{E}_6, R))$. Thus, to finish the proof of the theorem, it suffices to check that $\mathrm{Tran}(E(\mathrm{E}_6, R), G(\mathrm{E}_6, R))$ is contained in $\overline{G}(\mathrm{E}_6, R)$.

Let $g \in \mathrm{GL}(27, R)$ belong to $\mathrm{Tran}(E(\mathrm{E}_6, R), G(\mathrm{E}_6, R))$. We pick any $\alpha \in \Phi$ and any $\xi \in R$. Then $h = gx_\alpha(1)g^{-1}$ lies in $G(\mathrm{E}_6, R)$, and thus $F(hu, hv, hw) = F(u, v, w)$ for all $u, v, w \in V$. Substituting (gu, gv, gw) for (u, v, w), we get

$$F(gx_\alpha(1)u, gx_\alpha(1)v, gx_\alpha(1)w) = F(gu, gv, gw).$$

Recalling that $x_\alpha(1) = e + e_\alpha$ and using the linearity of F in all arguments, we see that this identity expands as follows:

$$\begin{aligned} 0 = F(gu, gv, ge_\alpha w) &+ F(gu, ge_\alpha v, gw) + F(ge_\alpha u, gv, gw) \\ &+ F(gu, ge_\alpha v, ge_\alpha w) + F(ge_\alpha u, gv, ge_\alpha w) \\ &+ F(ge_\alpha u, ge_\alpha v, gw) + F(ge_\alpha u, ge_\alpha v, ge_\alpha w) \end{aligned}$$

for all $u, v, w \in V$.

i) Now, let $e_\alpha u = 0$. Applying the above condition to the vectors $(u, e_\alpha v, e_\alpha w)$ and using the fact that $e_\alpha^2 = 0$, we see that

$$F(gu, ge_\alpha v, ge_\alpha w) = 0 \qquad \text{if } e_\alpha u = 0.$$

ii) We continue to assume that $e_\alpha u = 0$ and apply the above condition to the triple (u, v, w), obtaining

$$F(gu, gv, ge_\alpha w) + F(gu, ge_\alpha v, gw) + F(gu, ge_\alpha v, ge_\alpha w) = 0.$$

However, as we have shown above, the third summand also vanishes.

Thus, $F(gu, gv, ge_\alpha w) + F(gu, ge_\alpha v, gw) = 0$ for all $v, w \in V$ and all $u \in V$ such that $e_\alpha u = 0$. Substituting $e_\alpha u$ for u into this relation, we get

$$F(ge_\alpha u, gv, ge_\alpha w) + F(ge_\alpha u, ge_\alpha v, gw) = 0 \qquad \text{for all} \quad u, v, w \in V.$$

iii) Next, let weights $\lambda, \mu \in \Lambda$ such that $d(\lambda, \mu) = 1$ and another weight $\nu \in \Lambda$ be fixed. We take $\alpha \in \Phi$ such that $\lambda - \alpha \in \Lambda$, $\mu - \alpha \in \Lambda$, and $\nu + \alpha \notin \Lambda$. Such a choice is possible: indeed, we can assume that $\lambda = \lambda_0$ is the highest weight, while $\mu = \lambda - \delta$, where $\delta \in \Phi$ is the maximal root. Then $\alpha = \alpha_1$ has the required properties for all $\nu \in \Lambda$ except those for which $\nu + \alpha_1 \in \Lambda$. But for these last weights it is easy to indicate a possible choice of α:

$$\begin{aligned} \alpha &= {}^{11000}_{\ \ 0} &&\text{for} \quad \nu = {}^{134321}_{\ \ \ 2}, \ {}^{011111}_{\ \ \ 1}, \ {}^{011111}_{\ \ \ 0}, \\ \alpha &= {}^{11100}_{\ \ 0} &&\text{for} \quad \nu = {}^{012111}_{\ \ \ 1}, \\ \alpha &= {}^{11110}_{\ \ 0} &&\text{for} \quad \nu = {}^{012211}_{\ \ \ 1}, \\ \alpha &= {}^{11111}_{\ \ 0} &&\text{for} \quad \nu = {}^{012221}_{\ \ \ 1}. \end{aligned}$$

Take $u = v^{\lambda-\alpha}$, $v = v^\nu$, and $w = v^{\mu-\alpha}$. By our choice of α we have $e_\alpha v^\nu = 0$, and now ii) amounts to the relation $F(ge_\lambda, ge_\nu, ge_\mu) = 0$. Since this is true for all $\nu \in \Lambda$, we see that

$$F(gv^\lambda, gv, gv^\mu) = 0 \quad \text{if} \quad d(\lambda, \mu) = 1.$$

In particular, here we can substitute $v = g^{-1}v^\nu$, $\nu \in \Lambda$, and get all equations on the pair of columns $g_{*\lambda}$ and $g_{*\mu}$.

iv) Thus, $F(gv^\lambda, gv^\mu, gv^\nu) = 0$ for the weights $\lambda, \mu, \nu \in \Lambda$ such that at least one of the pairwise distances between them equals 1. If at least one of these distances equals 0, for instance, if $\lambda = \mu$, then there exists a root α such that $\lambda + \alpha \notin \Lambda$, $\nu - \alpha \in \Lambda$. Now ii) implies that $F(gu, gv, ge_\alpha w) + F(gu, ge_\alpha v, gw) = 0$ for all $v, w \in V$ and all $u \in V$ such that $e_\alpha u = 0$. Taking $u = e^\lambda$, $v = e^\mu$, and $w = e^{\nu-\alpha}$, we see that $F(gv^\lambda, gv^\mu, gv^\nu)$ is equal to 0 in this case as well.

v) It only remains to consider the values $F(gv^\lambda, gv^\mu, gv^\nu)$ for $(\lambda,\mu,\nu)\in\Theta$. Set $k = F(gv^{\lambda_0}, gv^{\mu_0}, gv^{\nu_0})$ and take any root $\alpha\in\Pi$ such that w_α does not fix the triad (λ,μ,ν). We may assume that $w_\alpha(\lambda)=\lambda$, and $w_\alpha(\mu)=\mu+\alpha$, $w_\alpha(\nu)=\nu-\alpha$. Then $\lambda+a\notin\Lambda$ by Lemma 5. Next, we substitute $(v^\lambda, v^\mu, v^{\nu-\alpha})$ in the condition stated immediately before case 1 and use the result of iii). We obtain $F(gv^\lambda, gv^\mu, gv^\nu) = -F(gv^\lambda, gv^{\mu+\alpha}, gv^{\nu-\alpha})$. By applying such root reflections w_α consecutively, we can get any triad starting with the distinguished triad (λ_0,μ_0,ν_0). It follows that $F(gv^\lambda, gv^\mu, gv^\nu) = (-1)^{h(\lambda,\mu,\nu)}k$. On the other hand, the explicit formula for F, reproduced in the satetement of Theorem 3, implies that $F(v^\lambda, v^\mu, v^\nu) = (-1)^{h(\lambda,\mu,\nu)}$. Thus, $F(gv^\lambda, gv^\mu, gv^\nu) = kF(v^\lambda, v^\mu, v^\nu)$. Thus, we have shown that $F(gv^\lambda, gv^\mu, gv^\nu) = kF(v^\lambda, v^\mu, v^\nu) = 0$ for all weight triples $(\lambda,\mu,\nu)\notin\Theta$. By the three-linearity of the form F, we obtain $F(gu, gv, gw) = kF(u,v,w)$ for all $u,v,w\in V$. Replacing g by g^{-1}, we see that $k\in R^*$. We can conclude that g lies in the similarity group of the form F. □

§9. Equations made explicit

In the present section we give yet another, more explicit form to the equations that determine whether a matrix $g\in \mathrm{GL}(27,R)$ belongs to the normalizer of the Chevalley group $G(\mathrm{E}_6,R)$. It is precisely in this form that we intend to use these equations in our subsequent papers on overgroups of exceptional groups. In fact, Theorem 1 characterizes $\overline{G}(\mathrm{E}_6,R)$ as the largest *subgroup* in $\mathrm{GL}(27,R)$ that consists entirely of matrices whose columns satisfy a certain system of quadrics. However, this theorem does not answer the question as to when an *individual* matrix $g\in\mathrm{GL}(27,R)$ belongs to $\overline{G}(\mathrm{E}_6,R)$. It is clear that such equations on a matrix g should involve components of several columns, not merely one of them.

For this, we introduce the following notation for the *polarization* of a partial derivative of the cubic form F:

$$f_\lambda(x,y) = F(e_\lambda, x, y) = \sum \mathrm{sgn}(w) x_\mu y_{\lambda\circ\mu}.$$

Here the sum is taken over all weights ν such that $d(\lambda,\mu)=2$, and $w\in W(\mathrm{E}_6)$ is chosen in such a way that $w(\lambda_0,\mu_0,\nu_0) = (\lambda,\mu,\lambda\circ\mu)$. Not to incorporate Weyl group elements for distinct triples in the proof, in the sequel we use the familiar notation $\mathrm{sgn}(w) = (-1)^{h(\lambda,\mu,\lambda\circ\mu)}$.

The next result is an analog of [13, Proposition 4] and [14, Proposition 1]. Observe that now instead of the entries of the matrix g themselves, in equations we see rather quadratic forms in these elements. Thus, with respect to the entries of g and g^{-1} these equations are not quadratic, as was the case for classical groups, but *cubic*.

Theorem 5. *A matrix $g\in\mathrm{GL}(27,R)$ belongs to $N(G(\mathrm{E}_6,R))$ if and only if its entries satisfy the following equations.*

- *Equations on a pair of adjacent columns*:

$$f_\lambda(g_{*\mu}, g_{*\nu}) = 0$$

for all $\lambda,\mu,\nu\in\Lambda$ such that $d(\mu,\nu)\le 1$.

- *Equations on two pairs of nonadjacent columns*:

$$(-1)^{h(\mu\circ\nu,\mu,\nu)} g'_{\mu\circ\nu,\lambda} f_\rho(g_{*\sigma}, g_{*\tau}) = (-1)^{h(\sigma\circ\tau,\sigma,\tau)} g'_{\sigma\circ\tau,\rho} f_\lambda(g_{*\mu}, g_{*\nu})$$

for all $\lambda,\mu,\nu,\rho,\sigma,\tau\in\Lambda$ such that $d(\mu,\nu)=d(\sigma,\tau)=2$.

Proof. First, we verify that any matrix of $\overline{G}(\mathrm{E}_6,R)$ satisfies these equations. By the very definition, $g\in\overline{G}(\mathrm{E}_6,R)$ if and only if there exists $k(g)\in R^*$ such that $F(gu,gv,gw) = k(g)F(u,v,w)$. In its turn, this last condition is equivalent to the same condition in which u, v, w are base vectors v^λ, v^μ, v^ν, respectively, for all triples of weights $\lambda,\mu,\nu\in\Lambda$.

When $d(\mu,\nu)\leq 1$, our condition becomes $F(gv^\lambda, gv^\mu, gv^\nu)=0$ for all $\lambda\in\Lambda$, or, what is the same, $F(u, gv^\mu, gv^\nu)=0$ for all $u\in V$. This is equivalent to $F(v^\lambda, gv^\mu, gv^\nu)=0$, but these are precisely the equations on pairs of adjacent columns. If $d(\mu,\nu)=2$, our condition becomes $F(gu, gv^\mu, gv^\nu)=k(g)F(u, v^\mu, v^\nu)$ for all $u\in V$, and this is equivalent to $F(u, gv^\mu, gv^\nu)=k(g)F(g^{-1}u, v^\mu, v^\nu)$ for all $u\in V$. It suffices to impose this equation only for $u=v^\lambda$; in other words, $F(v^\lambda, gv^\mu, gv^\nu)=k(g)F(g^{-1}v^\lambda, v^\mu, v^\nu)$. We transform the right-hand side of this equation:

$$\begin{aligned}k(g)F(g^{-1}v^\lambda, v^\mu, v^\nu)&=k(g)F(g'_{*\lambda}, v^\mu, v^\nu)=k(g)\sum_{\kappa\in\Lambda}g'_{\kappa\lambda}F(v^\kappa, v^\mu, v^\nu)\\&=k(g)g'_{\mu\circ\nu,\lambda}F(v^{\mu\circ\nu}, v^\mu, v^\nu)=k(g)g'_{\mu\circ\nu,\lambda}(-1)^{h(\mu\circ\nu,\mu,\nu)}.\end{aligned}$$

At the same time, the left-hand side equals $f_\lambda(g_{*\mu}, g_{*\nu})$. To get rid of the factor $k(g)$, we take another triple of weights $\rho,\sigma,\tau\in\Lambda$ with $d(\sigma,\tau)=2$. By eliminating $k(g)$ from the resulting equations, we get precisely the equations on the pairs of nonadjacent columns.

Now, let H be an affine group scheme over $\mathbb{Z}$ defined by the equations imposed in the statement of the theorem. It suffices to establish the inclusion $H(R)\subseteq\overline{G}(\mathrm{E}_6,R)$ for the case of a local ring R. Let M be the maximal ideal of R. Observe that the equations on a pair of adjacent columns imply that $F(gv^\lambda, gv^\mu, gv^\nu)=0$ for all $\lambda,\mu,\nu\in\Lambda$ with $d(\mu,\nu)\leq 1$. It remains to find $k\in R^*$ such that $f_\lambda(g_{*\mu}, g_{*\nu})=k(-1)^{h(\mu\circ\nu,\mu,\nu)}g'_{\mu\circ\nu,\lambda}$ for all $\lambda,\mu,\nu\in\Lambda$ with $d(\mu,\nu)=2$.

First, we show that there exist $\lambda,\mu,\nu\in\Lambda$ such that $d(\mu,\nu)=2$ and

$$g'_{\mu\circ\nu,\lambda}f_\lambda(g_{*\mu}, g_{*\nu})\in R^*.$$

Indeed, suppose that $g'_{\mu\circ\nu,\lambda}f_\lambda(g_{*\mu}, g_{*\nu})\in M$ for all $\lambda,\mu,\nu\in\Lambda$ such that $d(\mu,\nu)=2$. Observe that there exist $\lambda,\mu,\nu\in\Lambda$ with $d(\mu,\nu)=2$ and $g'_{\mu\circ\nu,\lambda}\in R^*$: to see this, it suffices to fix μ, ν, and to vary λ. Furthermore, there exist $\rho,\sigma,\tau\in\Lambda$ such that $d(\sigma,\tau)=2$ and $f_\rho(g_{*\sigma}, g_{*\tau})\in R^*$. Indeed, otherwise, invoking equations on pairs of adjacent columns, we can conclude that $f_\rho(g_{*\sigma}, g_{*\tau})\in M$ for any fixed ρ,σ and *any* $\tau\in\Lambda$. But then by linearity it follows that $f_\rho(g_{*\sigma}, u)\in M$ for all $u\in V$. This means that $f_\rho(g_{*\sigma}, v^\kappa)=\pm g_{\rho\circ\kappa,\sigma}\in M$ for all $\rho,\kappa\in\Lambda$ such that $d(\rho,\kappa)=2$, and this is clearly impossible. Thus, $g'_{\mu\circ\nu,\lambda}f_\rho(g_{*\sigma}, g_{*\tau})\in R^*$ and $g'_{\sigma\circ\tau,\rho}f_\lambda(g_{*\mu}, g_{*\nu})\in M$, which contradicts the fact that $g\in H(R)$. Therefore, there exist $\lambda,\mu,\nu\in\Lambda$ with $d(\mu,\nu)=2$ such that $g'_{\mu\circ\nu,\lambda}f_\lambda(g_{*\mu}, g_{*\nu})\in R^*$. We set

$$k=(-1)^{h(\mu\circ\nu,\mu,\nu)}(g'_{\mu\circ\nu,\lambda})^{-1}f_\lambda(g_{*\mu}, g_{*\nu}).$$

With this notation, the equations on g can be rewritten in the form

$$f_\rho(g_{*\sigma}, g_{*\tau})=k(-1)^{h(\sigma\circ\tau,\sigma,\tau)}g'_{\sigma\circ\tau,\rho},$$

as claimed. □

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