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k is a base field, $\operatorname{char} k = 0$, $k \hookrightarrow \mathbb{C}$. We have a category $\mathrm{Spt}_{\mathbb{P}^1}(k)$. An object here is a gadget $\mathcal{E} = (\mathcal{E}_0, \mathcal{E}_1, \dots)$ where $\mathcal{E}_n \in \mathrm{Spc}_\bullet(k)$. $\mathcal{E}_n\colon \mathrm{Sm}/k \to \mathrm{Spc}_\bullet$ (category of pointed spaces) with $\varepsilon_n\colon \rceil_n \wedge \mathbb{P}^1 \to \mathcal{E}_{n+1}$. $\mathrm{Spt}_{\mathbb{P}^1}(k)$ is a stable model category; ${}_sWE_{\mathbb{A}^1}$ are weak equivalences (isomorphisms on the bi-graded $\mathbb{A}^1$-homotopy sheaves $\pi^{\mathbb{A}^1}_{a,b}$ for all a, b). Then we have $\mathrm{SH}(k) = \mathrm{Ho}(\mathrm{Spt}_{\mathbb{P}^1}(k))$; a triangulated $\otimes$-category. From our embedding $k \hookrightarrow \mathbb{C}$ we get a realisation functor $\Re_B\colon \mathrm{SH}(k) \to \mathrm{SH} = H_0(\mathrm{Spt}_{S^2})$.

For $\mathcal{E} \in \mathrm{SH}(k)$ we have $\pi_n(\mathcal{E}, \mathbb{Z}/N) = \pi_{n,0}(\mathcal{E}/N)(\bar{k})$, where $\mathcal{E}/N = cone(\mathcal{E} \xrightarrow{\cdot N} \mathcal{E})$. Similarly, if $E \in \mathrm{SH}$, we get $\pi_n(E, \mathbb{Z}/N) = \pi_n(E/N)$. $\Re_B\colon \pi_n(\mathcal{E}, \mathbb{Z}/N) \to \pi_n(\Re_B\mathcal{E}; \mathbb{Z}/N)$. $\mathrm{SH}^{\mathrm{eff}}(k)$ is a localising subcategory of $\mathrm{SH}(k)$ generated by $\Sigma^\infty_{\mathbb{P}^1} X_+$ for all $X \in \mathrm{Sm}/k$.

Теорема 0.1. For $\mathcal{E}\colon \mathrm{SH}^{\mathrm{eff}}(k)$ the realisation map $\Re_B\colon \pi_n(\mathcal{E}, \mathbb{Z}/N) \to \pi_n(\Re_B\mathcal{E}, \mathbb{Z}/N)$ is an isomorphism.

Note that this is a generalisation of the Suslin–Voevodsky theorem:

Теорема 0.2 (Suslin–Voevodsky, 1994)**.** If X is a finite type scheme over $k = \bar{k} \subseteq \mathbb{C}$, then

$$H^{Sus}_n(X; \mathbb{Z}/N) \cong H^{sing}_n(X(\mathbb{C}); \mathbb{Z}/N).$$

By definition the right hand side is $H_n(C^{Sus}_*(X) \otimes \mathbb{Z}/N) = \mathrm{Hom}_{DM^{\mathrm{eff}}(k)}(\mathbb{Z}(0), M(X)/N[n])$.

Recall that $C^{Sus}_m(X)$ is a free abelian group spanned by the subschemes $W \subseteq X \times \Delta^m$ such that W is integral, finite, and surjective over Δ^n. It is made into a complex by the maps $\partial = \sum_{i=0}^m (-1)^i d_i\colon C^{Sus}_m(X) \to C^{Sus}_{m-1}(X)$.

Corollary (of Theorem): if $X \in \mathrm{Sm}/k$, then $\pi_n(\Sigma^\infty_{\mathbb{P}^1} X_+; \mathbb{Z}/N) \cong \pi_n(\Sigma^\infty X(\mathbb{C})_+; \mathbb{Z}/N)$.

Key ingredients:

0. Suslin–Voevodsky theorem;

1. Voevodsky's slice tower

Consider $i_n\colon \Sigma^n_{\mathbb{P}^1} \mathrm{SH}^{\mathrm{eff}}(k) \hookrightarrow \mathrm{SH}(k)$ for $n \in \mathbb{Z}$. This inclusion map admits a right adjoint r_n (Neeman).

Let us define a truncation functor $f_n\colon \mathrm{SH}(k) \to \mathrm{SH}(k)$ as $f_n = i_n \circ r_n \to \mathrm{id}$. We have the maps $f_{n+1} \to f_n$. Hence we get a 'slice tower': if $\mathcal{E} \in \mathrm{SH}(k)$, then

$$\begin{array}{ccccccc} \dots \longrightarrow & f_{n+1}\mathcal{E} & \longrightarrow & f_n\mathcal{E} & \longrightarrow \dots \longrightarrow & \mathcal{E} \\ & & & \downarrow & & \\ & & & s_n\mathcal{E} & & \end{array}$$

Теорема 0.3 (Pelaez–Voevodsky,Röndigs–Østvær)**.** There exists a unique $\pi^M_n\mathcal{E} \in DM(k) \xrightarrow{EM_{\mathbb{A}^1}} \mathrm{SH}(k)$ with a canonical isomorphism $s_n\mathcal{E} \cong \Sigma^n_{\mathbb{P}^1} EM_{\mathcal{A}^1}(\pi^M_n\mathcal{E})$.

$s_n\mathcal{E}$ is called the 'nth slice' of $\mathcal{E}$.

This construction is a kind of analog of the classical Postnikov tower for SH. There you take a spectrum E and define $E\langle n\rangle$ — $(n-1)$-connective cover of E which has zero homotopy groups in degrees less than n. Again we have $i_n\colon \Sigma^n \mathrm{SH}^{\mathrm{eff}} \hookrightarrow \mathrm{SH}$. In this case $E\langle n\rangle = i_n r_n E$. Then we define $E\langle n+1\rangle \to E\langle n\rangle \to \Sigma^n EM(\pi_n E)$.

Note that f_n, s_n are exact functors. Hence $f_n\mathcal{E}/N = f_n(\mathcal{E}/N)$ and $s_n\mathcal{E}/N = s_n(\mathcal{E}/N)$.

Strategy for the proof: a) use Suslin–Voevodsky's theorem to show that for $q \geq 0$ we have $\pi_{n,0}(s_q\mathcal{E};\mathbb{Z}/N)(\overline{k}) \cong \pi_n(\Re_B s_q\mathcal{E}, \mathbb{Z}/N)$ (note that $s_q\mathcal{E}$ is effective for $q > 0$);

show that the spectral sequences b)

$$E_2^{p,q}(AH) = \pi_{p+q,0}(s_{-q}\mathcal{E}, \mathbb{Z}/N)(\overline{k}) \Rightarrow \pi_{p+1,0}(\mathcal{E}, \mathbb{Z}/N)(\overline{k})$$

and c)

$$E_2^{p,q}(ReAH) = \pi_{p+1}(\Re_B s_{-1}\mathcal{E}, \mathbb{Z}/N) \Rightarrow \pi_{p+q}(\Re_B\mathcal{E}; \mathbb{Z}/N)$$

are convergent and bounded for $\mathcal{E} \in \mathrm{SH}_{fin}(k)$.

Part (a): we interpret Suslin–Voevodsky theorem to say that

$$\pi_{n,0}(EM_{\mathbb{A}^1}(M(X)); \mathbb{Z}/N) = \mathrm{Hom}_{DM(k)}(\mathbb{Z}(0), (M(X)/N)[n])(\overline{k}) = H_n^{sing}(M(X); \mathbb{Z}/N),$$

and $H_n^{sing}(M(X); \mathbb{Z}/N) \to H_n^{sing}(X(\mathbb{C}); \mathbb{Z}/N)$ is iso by S–V. Then $\Re_B$ maps $\pi_{n,0}(EM_{\mathbb{A}^1}(M(X)); \mathbb{Z}/N)$ to

$$\pi_n(\Re_B EM_{\mathbb{A}^1}(M(X)); \mathbb{Z}/N) \cong H_n^{sing}(X(\mathbb{C}), \mathbb{Z}/N),$$

and this is an isomorphism. Note that $DM^{\mathrm{eff}}(k)$ is generated as a localising category by $M(X)$. It follows that for all $M \in DM^{\mathrm{eff}}$ the realisation map $\Re_B\colon \pi_{n,0}(EM_{\mathbb{A}^1}(M); \mathbb{Z}/N)(\overline{k}) \to \pi_n(\Re_B EM_{\mathbb{A}^1}(M); \mathbb{Z}/N)$ is an isomorphism. Recall that $s_q\mathcal{E} = EM_{\mathbb{A}^1}(\pi_q^M\mathcal{E}(q)[2q])$, so we proved (a).

Part (c): if $\mathcal{E}$ is sufficiently connected (say, N-connected), then $\Re_B(f_q\mathcal{E})$ is $(q + \mathrm{const})$-connected ($(q+N)$-connected). If $\mathcal{E}$ is finite, then $\mathcal{E} = f_{-q}\mathcal{E}$ for some q large enough. So what does it mean for $\mathcal{E}$ to be N-connected? We say that $\mathcal{E}$ is *topologically N-connected* if $\pi_{m+q,q}\mathcal{E} = 0$ for all $m \leq N$, $q \in \mathbb{Z}$.

Sketch of the proof: Morel's connectedness theorem says $\Sigma_{\mathrm{top}}^{N+1}\mathrm{SH}(k)$ (full subcategory of topologically N-connected objects) is generated by $S^m \wedge \Sigma_{\mathbb{P}^1}^\infty X_+$ for $m \geq N+1$. Hence $\Sigma_{\mathbb{P}^1}^1\mathrm{SH}^{\mathrm{eff}}(k)$ is generated by $S^a \wedge \mathbb{G}_m^{\wedge m} \wedge \Sigma_{\mathbb{P}^1}\infty X_+$ for $m \geq q$. Then $f_q\mathcal{E}$ is going to be a cell complex build out of $S^q \wedge \mathbb{G}_m^b \wedge \Sigma_{\mathbb{P}^1}^\infty X_+$ for $a \geq N+1$, $b \geq q$. Hence the realisaiton of it gives you something like $S^{a+b} \wedge \Sigma^\infty X(\mathbb{C})_+$ for $a + b \geq q + N + 1$.

Now take a finite spectrum $\mathcal{E}$ (we may assume that it is effective)

$$\cdots \to f_q\mathcal{E} \to \cdots \to f_1\mathcal{E} \to f_0\mathcal{E} = \mathcal{E}.$$

Let us fix a field F finitely generated over k; we need $F = k(X)$ for some variety X. We look at $\pi_{a,b}(f_q\mathcal{E})(F)$; it is zero for sufficiently large $q \geq q(a, b, \mathcal{E}, F)$. We can always assume that $b = 0$ (by shifting). Assume also that $\mathcal{E}$ is topologically (-1)-connected. Necessary assumption: the cohomological dimension of k is finite: $n_0 = c.d.(k) < \infty$. If $tr.deg_k F = d$, then $c.d(F) \leq n_0 + d$. If L/F is finite, then $c.d(L) = c.d(F)$.

Лемма 0.4. There is an integer $d(\mathcal{E})$ such that $\pi_{m+d,d}(\mathcal{E})_{\mathbb{Q}} = 0$ for all $d > d(\mathcal{E})$ and all $m \in \mathbb{Z}$. For example, if $\mathcal{E} = \Sigma_{\mathbb{P}^1}\infty X_+$ for X/k smooth projective, then $d(\mathcal{E}) = \dim_k X$.

Доказательство. Since $\mathcal{E}$ is finite, we reduce the problem to the case $\mathcal{E} = \Sigma_{\mathbb{P}^1}\infty X_+$ for X/k a smooth projective variety of dimension $d = \dim_k X$. Here we use a theorem by Cisinski–Deglise: $c.d.(k) < \infty$ implies $\mathrm{SH}(k)_{\mathbb{Q}} \cong DM(k)_{\mathbb{Q}}$. (Note that $I^n/I^{n+1} = H^n(k, \mathbb{Z}/2)$, therefore $I^n/I^{n+1} = 0$ for $n > c.d.(k)$. On the other hand, $\bigcap_{n\geq 0} I^n = 0$ by Arason–Pfister.) It follows that

$$\begin{aligned}
\pi_{a+b,b,b}(\Sigma_{\mathbb{P}^1}^\infty X_+)_{\mathbb{Q}}(k) &= \mathrm{Hom}_{DM(k)}(\mathbb{Z}(b)[a], M(X))_{\mathbb{Q}} \\
&= \mathrm{Hom}_{DM(k)}(M(X)(-d)[-2d], \mathbb{Z}(-b)[-a])_{\mathbb{Q}} \\
&= \mathrm{Hom}_{DM(k)}(M(X), \mathbb{Z}(d-b)[2d-a])_{\mathbb{Q}} \\
&= H^{2d-a}(X, \mathbb{Q}(d-b)),
\end{aligned}$$

and if $d - b < 0$, this equals zero. Then we pass from k to F, from X to X_F, and work in $DM(F)$. We showed that $\pi_{a,b}(\Sigma^\infty X)(F) = 0$ for $b > \dim_k X$. □

We need a concrete model for $f_q\mathcal{E}$, $\mathcal{E} \in \mathrm{SH}^{\mathrm{eff}}(k)$. Consider $E = \Omega^\infty_{\mathbb{P}^1}\mathcal{E}$ — a presheaf of spectra on Sm/k. Take $b \geq 0$. $(\pi_{a,b}\mathcal{E})(F)$ is related to $(\pi_{a,b}E)(F)$, which is the Nisnevich sheaf associated to the presheaf $U \mapsto \pi_{a-b}(E(U_+ \wedge \mathbb{G}_m^{\wedge b}))$. We have an inclusion functor $\Sigma^n_{\mathbb{P}^1}\mathrm{SH}_{S^1}(k) \to \mathrm{SH}_{S^1}(k)$, its right adjoint r_n, put $f_n = i_n \circ r_n$, and get the slice tower

$$\cdots \to f_{n+1}E \to f_nE \to \cdots \to f_0E = E.$$

It turns out than $f_nE = \Omega^\infty_{\mathbb{P}^1}f_n\mathcal{E}$.

In fact, E satisfies two nice properties:

1. Nisnevich excision;
2. $\mathbb{A}^1$-homotopy invariance: $E(X) \to E(X \times \mathbb{A}^1)$ is iso.

Then we get a nice model for $(f_\bullet E)(X)$: a homotopy coniveau tower $f_nE(X) = \mathrm{Tot}(m \mapsto E^{(n)}(X,m))$ (Bloch cycle complexes applied to a presheaf of spectra). Here Tot is a kind of geometric realisation. What is $E^{(n)}(X,m)$? Take $W \subseteq X \times \Delta^m$ and consider $E(X \times \Delta^m)$ restricted to $X \times \Delta^m \setminus W$. The fiber of that is $E^W(X \times \Delta^m)$. Now we take a colimit of those where $\mathrm{codim}_{X\times F}(W \cap X \times F \geq n$ for any face $F \subseteq \Delta^m$, and get $E^{(n)}(X,m)$. For any map $m' \to m$ in the order category we get a map $E^{(n)}(X,m) \to E^{(n)}(X,m')$.

Take $X = \mathrm{Spec}\,F$; now we look at the closed subsets of Δ^m. Suppose we have a simplicial spectrum $m \mapsto E_m$. Then we have a spectral sequence

$$E^1_{a,b} = \pi_bE_a \Rightarrow \pi_{a+b}\,\mathrm{Tot}(m \mapsto E_m).$$

In particular, in our case $E_m = E^{(n)}(F,m)$ is 0 if $m < n$ (then we have to take $W \subseteq \Delta^m$ with $\mathrm{codim}\,W = n > m$, hence $W = \varnothing$, and its fiber is just a point). Hence we get a surjection $\pi_bE^{(n)}(F,n) \twoheadrightarrow Fil_n\pi_{b+n}f_nE(X) \subseteq \pi_{b+n}f_nE(X)$. We assumed the spectrum was (-1)-connected, so essentially this filtration is going to be finite, and we can use induction.

Fix $r \in \mathbb{Z}$ and look at contributions to $\pi_rf_nE(X)$ from $\pi_{r-m}E^{(n)}(F,m)$. Recall that $\pi_{r-m}E^{(n)}(F,m)$ is a colimit over good $W \subseteq \Delta^m$ of $\pi_{r-m}E^W(\Delta^m_F)$. For a fixed W we can pass to the direct sum over all generic points $X_i \in W$ such that $\mathrm{codim}_{\Delta^m} X_i = n$ of $\pi_{r-n}E^{X_i}(\Delta^m_F)$. This passage is an inclusion map. Hence for a point x of codimension n we have $\pi_{r-m}E^x(\Delta^m) = \pi_{r-m}\mathrm{Hom}(S^{2n,n}_{F(x)}, E) = (\pi_{r-m+2n,n}E)(F(x))$. The degree here gets smaller as m gets bigger, and for minimum $m = n$ we have $(\pi_{r+n,n}E)(F(x))$. This contributes to $Fil^n(\pi_rf_nE)(F)$, and the map $\bigoplus_x(\pi_{r+n,n}E)(F(x)) \to Fil^n(\pi_rf_nE)(F)$ is surjective. Let us map that to $\pi_rE(F)$. Now we are going to construct $\rho_n\colon Fil^n(\pi_rf_nE)(F) \to \pi_rE(F)$.

The idea is to use that spectral sequence plus an explicit bound (depending on $d(E)$, $c.d.(F)$, r) such that $Fil^n \to 0$ in $\pi_rE(F)$; then replace E with f_nE, etc., to show that $Fil^n = 0$.

Suppose $\alpha \in (\pi_{r+n,n}E)(F(x))$ for a closed point $x = (x_0, \ldots, x_n)$, $x_i \in F(x)$. Consider the symbol $[-x_1/x_0, \ldots, -x_n/x_0] \in K^{MW}_n(F(x)) = \pi_{-n,-n}(\mathbb{S})(F(x))$. Cupping with this element gives us a map $(\pi_{r+n,n}E)(F(x)) \xrightarrow{\cup} (\pi_{r,0}E)(F(x))$. We have a canonically defined norm map $(\pi_{r,0}E)(F(x)) \to (\pi_{r,0}E)(F)$. The image of α here is exactly what we need.

Let us define

$$Fil^n_{MW}\pi_{r,0}E(F) = \mathrm{im}((\pi_{r+n,n}E)(F(x)) \otimes K^{MW}_n(F(x)) \to (\pi_{r,0}E)(F))$$

(summing over all closed points $x \in \Delta^n_F$). We've shown that

$$\rho_n(Fil^n_{sing}\pi_{r,0}f_nE(F)) \subseteq Fil^n_{MW}\pi_{r,0}E(F).$$

The task is to find a bound n_0 so that $Fil^n_{MW}\pi_rE(F) = 0$ for $n > n_0$.

If $n > d(E)$, we know that $(\pi_{r+n,n}E)(F(x))$ is torsion, say,

$$(\pi_{r+n,n}E)(F(x)) = \bigcup_N (\pi_{r+n,n}E)(F(x))_{N-tors}.$$

Consider an exact sequence

$$0 \to I^{n+1}(F(x)) \to K_n^{MW}(F(x)) \to K_n^M(F(x)) \to 0.$$

We know that $I^{n+1}(F(x)) = 0$ for $n \geq cd(F(x)) = cd(F) = cd(k) + trdeg_k F$.

$\bigcup_N (\pi_{r+n,n} E)(F(x)) \otimes K_n^M(F(x))/N \to \pi_{r,0} E(F)$ factors through $Fil^n_{MW}(\dots)$. We use Bloch–Kato and get that the left-hand side is $H^n(F(x), \mu_N^{\otimes n}) = 0$, so its image is zero.