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THE HIRONAKA THEOREM ON RESOLUTION OF SINGULARITIES
(Or: A proof we always wanted to understand)

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ABSTRACT. This paper is a handyman's manual for learning how to resolve the singularities of algebraic varieties defined over a field of characteristic zero by sequences of blowups.

Three objectives: Pleasant writing, easy reading, good understanding.

One topic: How to prove resolution of singularities in characteristic zero.

Statement to be proven (No-Tech): The solutions of a system of polynomial equations can be parametrized by the points of a manifold.

Statement to be proven (Low-Tech): The zero-set X of finitely many real or complex polynomials in n variables admits a resolution of its singularities (we understand by singularities the points where X fails to be smooth). The resolution is a surjective differentiable map ε from a manifold $\tilde{X}$ to X which is almost everywhere a diffeomorphism, and which has in addition some nice properties (e.g., it is a composition of especially simple maps which can be explicitly constructed). Said differently, ε *parametrizes* the zero-set X (see Figure 1).

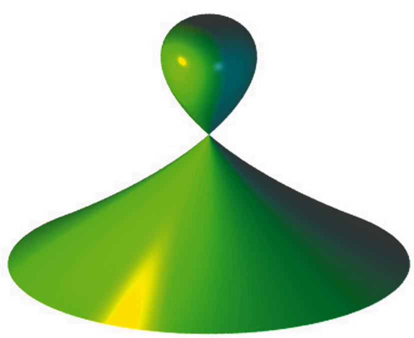

FIGURE 1. *Singular surface Ding-dong*: The zero-set of the equation $x^2 + y^2 = (1 - z)z^2$ in $\mathbb{R}^3$ can be parametrized by $\mathbb{R}^2$ via $(s,t) \to (s(1-s^2)\cdot\cos t, s(1-s^2)\cdot\sin t, 1-s^2)$. The picture shows the intersection of the Ding-dong with a ball of radius 3.

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You will agree that such a parametrization is particularly useful, either to produce pictures of X (at least in small dimensions), or to determine geometric and topological properties of X. The huge number of places where resolutions are applied to prove theorems about all types of objects (algebraic varieties, compactifications, diophantine equations, cohomology groups, foliations, separatrices, differential equations, $\mathcal{D}$-modules, distributions, dynamical systems, etc.) shows that the existence of resolutions is really basic to many questions. But it is by no means an easy matter to construct a resolution for a given X.

Puzzle: Here is an elementary problem in combinatorics — the polyhedral game of Hironaka. Finding a winning startegy for it is instrumental for the way singularities will be resolved. Each solution to the game can yield a different method of resolution. The formulation is simple.

Given are a finite set of points A in $\mathbb{N}^n$, with positive convex hull N in $\mathbb{R}^n$ (see Figure 2),

$$N = \mathrm{conv}(A) + \mathbb{R}^n_{\geq 0}.$$

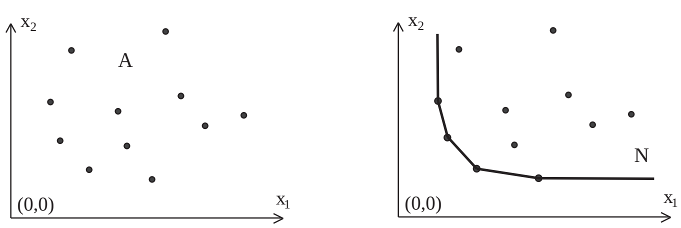

Figure 2. Convex hull of points in $\mathbb{N}^n$.

There are two players, $\mathcal{P}_1$ and $\mathcal{P}_2$. They compete in the following game. Player $\mathcal{P}_1$ starts by choosing a non-empty subset J of $\{1, \ldots, n\}$. Player $\mathcal{P}_2$ then picks a number j in J.

After these "moves", the set A is replaced by the set A' obtained from A by substituting the j-th component of vectors α in A by the sum of the components α_i with index i in J, say $\alpha_j \to \alpha'_j = \sum_{i \in J} \alpha_i$. The other components remain untouched, $\alpha'_k = \alpha_k$ for $k \neq j$ (see Figure 3). Then set $N' = \mathrm{conv}(A') + \mathbb{R}^n_{\geq 0}$.

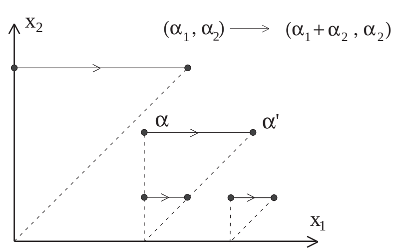

Figure 3. Movement of points.

The next round starts over again, with N replaced by N': $\mathcal{P}_1$ chooses a subset J' of $\{1, \ldots, n\}$, and player $\mathcal{P}_2$ picks j' in J' as before. The polyhedron N' is replaced by the corresponding polyhedron N''. In this way, the game continues.

Player $\mathcal{P}_1$ wins if, after finitely many moves, the polyhedron N has become an orthant, $N = \alpha + \mathbb{R}^n_{\geq 0}$, for some $\alpha \in \mathbb{N}^n$. If this never occurs, player $\mathcal{P}_2$ has won.

Problem: Show that player $\mathcal{P}_1$ always possesses a winning strategy, no matter how $\mathcal{P}_2$ chooses his moves.

To get a feeling for the problem, let us check what happens in two variables, $n = 2$. If $\mathcal{P}_1$ always chooses $J = \{1\}$ or $\{2\}$, the transformation on vectors $\alpha \in \mathbb{N}^2$ is the identity, the polyhedron N' equals N and she loses. So she is forced to eventually choose $J = \{1, 2\}$, and $\mathcal{P}_2$ may hence choose $j = 1$ or $j = 2$ at his taste. Here is the evolution of N in the case $\mathcal{P}_2$ chooses first $j = 1$ (the dotted segments correspond to vertices of N which move under the transformation to the interior of N'; see Figure 4).

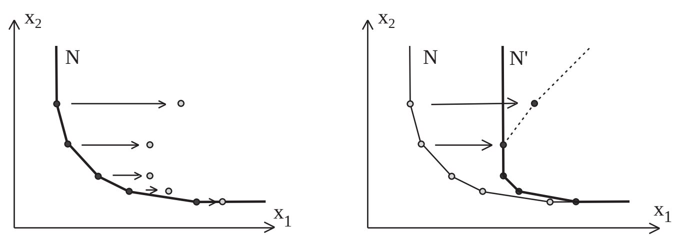

FIGURE 4. Transform of polyhedron N.

If $\mathcal{P}_2$ chooses in the next move again $j = 1$, we get a polyhedron N'' with just one (small) compact edge (see Figure 5).

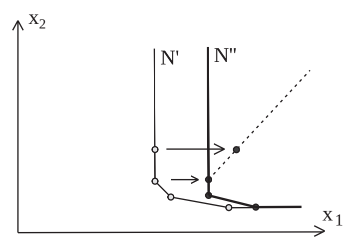

FIGURE 5. Transform of polyhedron N'.

Then let $\mathcal{P}_2$ choose $j = 2$. The vertices move vertically and yield a polyhedron N''' which is already a quadrant (see Figure 6).

From this sequence of pictures it seems clear that N always gets sharper and sharper until all but one vertex have become interior, in which case N is an orthant. So we ask you: *Can you prove this rigorously?*

Should be easy, shouldn't it? So why not try the case $n = 3$. Surprisingly enough, this case is already a real challenge — though maybe not for you. Note that $\mathcal{P}_1$ now has four options for how to choose the set J in $\{1, 2, 3\}$, and not all of them will work in all situations.

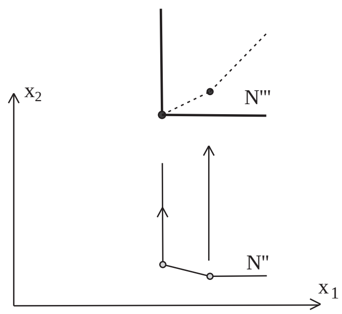

FIGURE 6. Transform of polyhedron N'' has become a quadrant.

What is all this about? Take a polynomial in seven variables of degree twenty-three, for example

$$P=\sqrt{\pi}\cdot x^{23}+y^2z-\cos 2\cdot u^{17}v^4w^2+\frac{1+\sqrt{5}i}{2}\cdot xt^{20}+\Gamma(\frac{3}{2})\cdot yuvt+7e^e.$$

If you wish to locate the zeroes of a polynomial equation, you will have, aside from a few cases, a pretty hard time (and also computers and their graphics programs will have trouble). If at one zero you know already (by chance or by experimentation) a partial derivative of the polynomial does not vanish, you may solve the equation, at least approximately, nearby this zero by the implicit function theorem, getting an expression of the respective variable in terms of the others. Otherwise, if all derivatives vanish, the geometric description of the zeroes will be difficult.

The best you can hope for is to construct a (possibly only local) parametrization of the zero-set of the polynomial by the points of a manifold of the same dimension. The obstruction for finding such a parametrization sits in the points where the zero-set is not smooth, its singular points. There, the geometric situation can be quite mysterious.

Suppose you have a hopeless tangle of wool — in any case you will know that originally this was a smooth and well educated string — and that pulling now at one of its ends will only increase the disaster. You have learned to try to loosen the knot*, pulling gently here and there, until you see some hope coming up.*

Why not do the same here with our zero-set: Let us try to *loosen its singularities.* If we believe that they admit a parametrization (and we will), we may also believe that the singularities arose by squeezing the zero-set into a too narrow ambient space, or by fooling carelessly around with them. So we will try to pull the zero-set apart, and possibly the singularities will become more lucid.

Here is an example (too simple to be a serious candidate, but instructive). Take the curve in $\mathbb{R}^2$ of equation $x^2=y^3$, the famous cusp. Its singularity sits at the origin, and we will have no effort to find a parametrization: $t\to(t^3,t^2)$. But remember that, in general, it won't be possible to guess a parametrization, so we better rely on a systematic method to construct it. What happens if we start to drag the two branches of the cusp? As we need space to move freely, we pull them vertically into three-space rather than staying in the plane. See the comic strip in Figure 7.

At the beginning, the curve will persist to have a singularity at 0, but as the vertical slopes of the branches increase, the singularity may suddenly stretch and

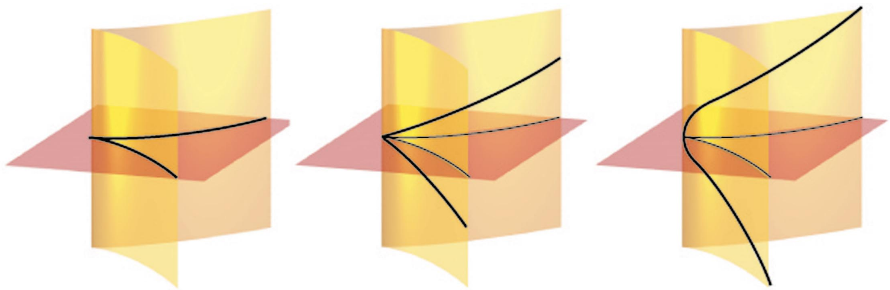

FIGURE 7. The cusp $x^2 = y^3$: It can be resolved by vertical dragging.

become smooth. In this case, we have obtained a space curve, which maps to our original curve in the plane (by vertical projection). There appears an interesting feature: Above the origin, the space curve will be *tangent* to the direction of the projection (and this, in turn, is necessary to produce the singularity below). We end up with an almost philosophical speculation:

Singular curves are the shadow of smooth curves in higher dimensional space.

Of course, the preceding reasoning is purely geometric and thus heuristic (it would certainly have pleased the ancient Greeks). To give it a more solid foundation, let's work algebraically. The problem is to reconstruct the space curve from its shadow in the plane. There is a standard procedure for doing this: Interpret it as a graph. In our case, take the graph (over the plane curve) of the function $(x, y) \to x/y$ (to take precisely this function and not another one will only be justified a posteriori). For simplicity, we discard here the problem which arises at 0. Any point (x, y) of the plane curve is lifted to the point $(x, y, z) = (x, y, x/y)$ in three-space, yielding a space curve with parametrization $(t^3, t^2, t^3/t^2) = (t^3, t^2, t)$. As its derivative is nowhere zero, this is a smooth curve. And indeed, it is the curve appearing at the right end of our comic strip.

You will protest — and you are right do so. Nobody told us why to choose the function $(x, y) \to x/y$ which made the vehicle run. Instead of looking for reasons, let's try it out on another (still trivial) example, the cone $x^2 + y^2 = z^2$. Now we are already in three-space, which limits somewhat our graphics facilities for presenting higher dimensions. So let's get started from the algebraic side. Take the graph over the cone of the map $(x, y, z) \to (x/z, y/z)$. It lives in five-space, and the equations will be $x^2 + y^2 = z^2$, $u = x/z$, $v = y/z$ (you will accept that we refrain from any illustration). Now, reminding our classes of differential geometry, we may visualize this surface by elimination of variables (which corresponds to the projection to the (z, u, v)-space). We get the surface $(uz)^2 + (vz)^2 = z^2$. Factoring z^2, it follows that it has two components, the plane $z = 0$ and the cylinder $u^2 + v^2 = 1$ in three-space, which, of course, is *smooth*. See Figure 8.

Composing the graph with this last projection (and throwing away the plane $z = 0$), we have found a smooth surface which parametrizes the cone, via the map $(z, u, v) \to (uz, vz, z)$. Isn't this convincing?

The two examples were very simple, and the parametrizing manifold was found in one step. In reality, for more serious examples, it takes many steps, but —

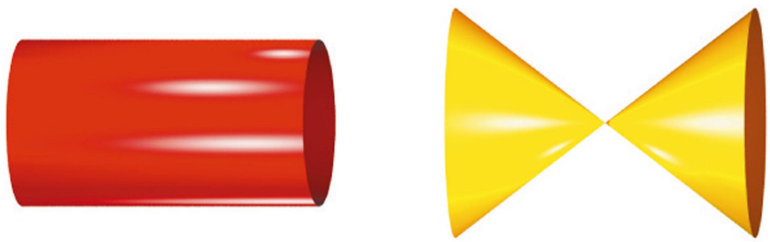

FIGURE 8. The cone parametrized by the cylinder.

surprise! — the technique always works (at least in characteristic zero; in positive characteristic there is still a big question mark).

Showing that one finds by the method of (iterated) graphs a parametrization of the zero-set of polynomials constitutes the theme of the paper.

Here is the semantic upshot of our (virtual) dialogue: Associating to the zero-set of the polynomial the graph of a function of the above type (ratios of the variables) conceptualizes in what is called the *blowup of a manifold in smooth centers*; the resulting parametrization of the zero-set through a manifold is expressed by saying that we *resolve the singularities of X by a finite sequence of blowups*.

We conclude this appetizer with a zero-set for which a parametrization cannot be guessed offhand (see Figure 9).

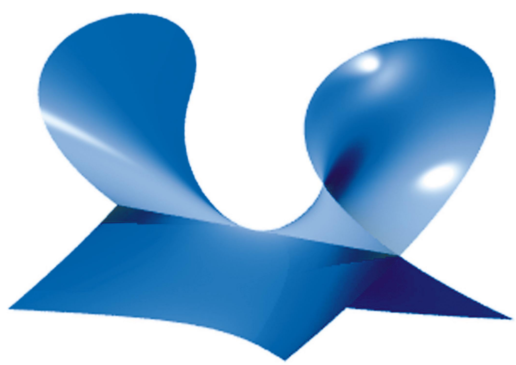

FIGURE 9. The surface $x^2 = z^2\ (z - y^2)$ has its singularities along the y-axis $\{x = z = 0\}$. Outside 0, we perceive, locally along the y-axis, two planes meeting transversally. At the origin, two contracting hoppers meet from opposite sides. This is truly a rather singular point.

IQ: Can you make out which blowups produce a resolution of this surface?

Straight into High-Tech. There is no way to describe fluently resolutions without appropriate language. The most natural context to communicate here are schemes, with the unavoidable drawback that many interested readers get discouraged. We are aware of this common behaviour — and the next to last paragraph will for sure succeed in producing annoyance outside the algebraic geometers' world. But outside, resolution of singularities is also used and recognized, far beyond algebraic geometry. Mathematicians may wish to learn about it.

We propose a compromise: In the section "High-Tech $\rightsquigarrow$ Low-Tech", we will list down-to-earth interpretations of the main objects we are using. This dictionary shall show (as well as the example section towards the end of the paper) that the central problems in this field are not problems on abstract schemes but problems on concrete polynomials, namely their *behaviour under certain well-specified coordinate changes* — a topic which is familiar to all of us. So don't be shocked by the next dozen lines or so; it is just a fancy (though very concise) way to talk about polynomials. Relief is to come afterwards.

Statement to be proven (*High-Tech*): Any reduced singular scheme X of finite type over a field of characteristic zero admits a *strong resolution of its singularities*. This is, for every closed embedding of X into a regular ambient scheme W, a proper birational morphism ε from a regular scheme W' onto W subject to the following conditions.

> *Explicitness.* ε is a composition of blowups of W in regular closed centers Z transversal to the exceptional loci.
>
> *Embeddedness.* The strict transform X' of X is regular and transversal to the exceptional locus in W'.
>
> *Excision.* The morphism $X' \to X$ does not depend on the embedding of X in W.
>
> *Equivariance.* ε commutes with smooth morphisms $W^- \to W$, embeddings $W \to W^+$, and field extensions.
>
> *Effectiveness* (optional). The centers of blowup are given as the top locus of a local upper semicontinuous invariant of X.

The induced morphism $\delta : X' \to X$ is called a *strong desingularization* of X.

Credits: Existence with E_1 and E_2 proven by Hironaka [Hi 1]. Constructive proofs satisfying E_1 to E_5 given by Villamayor [Vi 1], [Vi 2], Bierstone-Milman [BM1], Encinas-Villamayor [EV 1], [EV 2], Encinas-Hauser [EH], and Bravo-Villamayor [BV 2]. Further references are in the first part of the bibliography. Implementation by Bodnár-Schicho [BS1]. Existence of weak resolutions by Abramovich-de Jong, Abramovich-Wang and Bogomolov-Pantev [AJ], [AW], [BP], based on the work of de Jong on alterations [dJ].

Avertissement. We shall follow the opposite démarche of a usual mathematical research paper. Instead of formulating a theorem and then giving its proof in the shortest and clearest possible fashion, the reader will be concerned with finding and developing the proof of the result on her/his own. Thus we will first try to localize and extract the key problem which has to be cracked. Starting from this problem we shall make various attempts and reflections on how to solve it. In the course of this inquiry, we will be able to make many observations on phenomena and properties related to the problem. This in turn will allow us to specify even more the main difficulties and to get a list of possible ways of attack.

In a next step we shall try out these approaches in reality. Some of them may fail; others will look more prospective. The job of the author of this article will be to prevent the reader from falling into traps or pursuing paths which lead to nowhereland.

In this way, the reader will learn peu à peu the rules of the game and get a feeling for the relevant constructions and reasonings. All together, the proof shall be natural and follow from a good understanding of the difficulties, rather than to fall like a shooting star from the sky.

Occasionally, some help will be needed. Accepting (or neglecting) certain technical complications, the reader will see that there is a rather canonical approach for proving resolution of singularities in characteristic zero. Only at a few places one really needs a trick, and the author is willing to provide these gently and tacitly.

> For the convenience of the reader, the precise definitions of all concepts appearing in the text are given in appendix D at the end of the paper; a table of notations appears in appendix E. Nevertheless, a certain familiarity with blowups will be supposed. A very instructive exercise for getting acquainted is to compute the blowup of the Whitney umbrella $x^2 - y^2z = 0$ in $\mathbb{A}^3$ with center once the origin and once the z-axis; cf. the section on blowups in the appendix.

Our exposition follows the proof given in the recent paper [EH] of Santiago Encinas and the author, where many more details and specifications can be found (cf. also the paper [BS1] of Bodnár and Schicho). Here we are more interested in motivating the various constructions. The paper [EH], in turn, relies on the techniques of [Hi 1], [Vi 1], [Vi 2], [BM1], [EV 1]; see the appendix of [EH] for many precise references to these articles. Basic to all of them are the ideas and concepts proposed and developed by Hironaka. It should not be forgotten that Abhyankar also has strongly influenced the research in this field, contributing many substantial ideas and constructions. And, of course, Zariski has to be considered as the grandfather of resolution of singularities.

La démonstration se fait par une récurrence subtile.
(J. Giraud, Math. Reviews 1967)
Le rapporteur avoue n'en avoir pas fait entièrement le tour.
(A. Grothendieck, ICM 1970)

The article has accomplished its goal if the reader starts to suspect — after having gone through the complex and beautiful building Hironaka proposes — that he himself could have proven the result, if only he had known that he was capable of it. This intention is possibly too optimistic; nevertheless, it's time to start understanding the proof of a result which is one of the most widely used in algebraic geometry.

Caveat: The proof we shall go through is by no means simple, and trying to capture its flavour requires patience and stamina.[1]

Thanks: To Orlando Villamayor for introducing the author to the subject and sharing many insights; to Santiago Encinas for providing substantial information on the tricky technical machinery; to Ana Bravo, Gábor Bodnár and Hironobu Maeda for many valuable comments on an earlier draft of the article; to Sebastian Gann from the graphics department (the pictures were created with Adobe Illustrator and POV-Ray); to an anonymous referee for her/his precise suggestions for improving the readability of the paper; to the authors mentioned above for transmitting in their papers and conferences fascinating features of resolution of

[1] *"Der Bergsteiger"*, op. nasc.

singularities; to many other mathematicians for discussing the subject with the author.

The present manuscript is an extended version of lectures delivered at the Research Institute for Mathematical Sciences of the University of Kyoto and at the Mathematics Department of the University of Michigan at Ann Arbor. The author is grateful for the kind hospitality offered by the people at these institutions.

Contents:

CHAPTER 0: OVERVIEW

−2. High-Tech $\rightsquigarrow$ Low-Tech. Some readers may not be so familiar with the language of modern algebraic geometry. The concepts of schemes, sheaves and ideals are useful to handle zero-sets of polynomials which are defined locally in affine space and which are glued together to produce global objects similar to abstract differentiable manifolds. If the reader confines her/himself to affine or local geometry, there is no harm to communicate in more concrete terms. For this translation, we provide below a (very rough) dictionary from technical termini to everyday concepts and examples (the expert reader may excuse the clumsiness). But let us first say two words about blowups.

Blowups are *the* basic device for constructing resolutions of singularities. They constitute a certain type of transformation of a regular scheme (manifold) W yielding a new regular scheme W' above W, the *blowup* of W, together with a projection map $\pi : W' \to W$, the *blowup map*. The role of blowups is to untie the singularities of a given zero-set X in W by looking at its inverse image X' in W'. The scheme W' offers X' more "space" to spread out than W (despite the fact that W and W' are of the same dimension), because W' sits (in a funny way) inside a higher dimensional ambient scheme. The process is repeated until, after finitely many blowups, the final inverse image $\tilde{X}$ of X has been resolved (in a sense that will be laid down with precision).

Each blowup is completely specified by its *center* Z. This is a regular closed subscheme (submanifold) of W which is chosen according to a prescribed rule (depending on the zero-set X we wish to resolve). The center is the locus of points of W above which the map π is not an isomorphism. The construction of $\pi : W' \to W$ from the knowledge of Z as a subscheme of W is best described locally. So let us restrict ourselves to a neighborhood of a point a in Z. There, we may view W as affine space $\mathbb{A}^n$ with $a = 0$ the origin. The center Z can be interpreted, locally at a, as a coordinate subspace of $\mathbb{A}^n$, for example, as the d-dimensional subspace $\mathbb{A}^d \times 0^{n-d} \subset \mathbb{A}^n$ defined by $x_{d+1} = \ldots = x_n = 0$.

Let U be a submanifold of W at a, transversal to Z and of complementary dimension $n - d$. For convenience, we take for U the coordinate subspace $U = 0^d \times \mathbb{A}^{n-d}$ and fix the local product decomposition $W = Z \times U$ of W. Write points $w \in W$ as pairs $w = (w_1, w_2)$ with $w_1 \in Z$ and $w_2 \in U$. The projectivization $\mathbb{P}(U)$ of $U = 0^d \times \mathbb{A}^{n-d}$ is the $(n - d - 1)$-dimensional projective space $\mathbb{P}^{n-d-1}$ of lines through 0 in U. These ingredients already suffice to describe the blowup of W with center Z.

Namely, consider the graph of the map $\lambda : W \setminus Z \to \mathbb{P}^{n-d-1}$ sending a point $w = (w_1, w_2)$ of $W \setminus Z = Z \times (U \setminus 0)$ to the line ℓ_w in $\mathbb{P}^{n-d-1}$ passing through w and its projection w_1 in Z. The graph of λ is a closed regular subscheme (submanifold) Λ of $(W \setminus Z) \times \mathbb{P}^{n-d-1}$, say $\Lambda = \{(w, \ell_w),\, w \in W \setminus Z\}$. With this setting, the blowup W' of W is defined as the *closure* of Λ in $W \times \mathbb{P}^{n-d-1}$,

$$W' = \overline{\{(w, \ell_w),\, w \in W \setminus Z\}} \subset W \times \mathbb{P}^{n-d-1}.$$

It is easy to see that W' is again a regular scheme. The standard affine charts of $\mathbb{P}^{n-d-1}$ induce a covering of W' by affine schemes. The blowup map $\pi : W' \to W$ is given as the restriction to W' of the projection $W \times \mathbb{P}^{n-d-1} \to W$ on the first factor. The *exceptional locus* of the blowup is the inverse image $Y' = \pi^{-1}(Z) = Z \times \mathbb{P}^{n-d-1}$ of the center Z. It is the locus of points of W' where π fails to be an isomorphism. Along the exceptional locus, π contracts the second factor $\mathbb{P}^{n-d-1}$ of Y' to a point. See Figure 10.

This resumes the geometric description of blowups locally at a point of Z. The algebraic formulae and more details can be found in appendix C and in the section "Examples". The papers [BS 2], [BS 3] of Bodnár and Schicho exhibit how to implement the computation of blowups efficiently, illustrated nicely by the blowup of $W = \mathbb{A}^3$ with center Z a circle.

Here now is the promised brief guide on how to translate some notions from algebraic geometry to commonplace mathematical language.

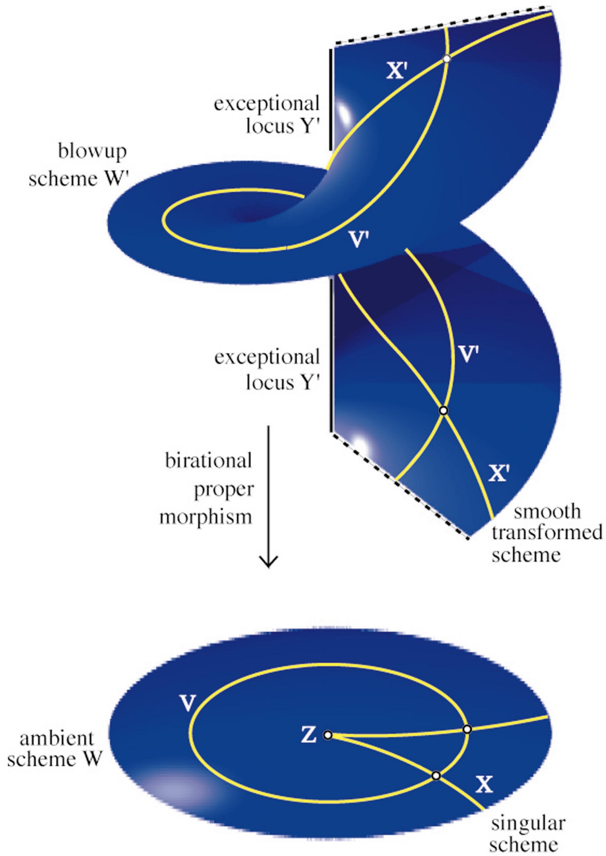

FIGURE 10. *Blowup*: The picture shows (part of) the surface $W' = \{xz - y = 0\}$ in $\mathbb{R}^3$, which can be identified with the affine portion of the graph of the map $\mathbb{R}^2 \to \mathbb{P}^1$ sending (x, y) to $(x : y)$. This surface is the blowup of the real plane $W = \mathbb{R}^2$ with center $Z = \{0\}$ the origin. Inside W' we see the transforms V' and X' of the circle $V = \{x^2 + y^2 = 1\}$ and of the singular plane curve $X = \{x^2 = y^3\}$. The surface W' has to be glued along the dotted lines.

Algebraic geometry	*Intuition / examples*
affine scheme	zero-set given by an ideal of polynomials
(global) scheme	zero-sets glued together suitably
hypersurface / divisor	zero-set of one single polynomial
regular point	point where zero-set is manifold
singular point	the remaining points
reduced scheme	zero-set given by a radical ideal of polynomials
excellent scheme	scheme which behaves well under passage to formal power series
ambient scheme	affine space $\mathbb{R}^n$ or $\mathbb{C}^n$ where zero-set lives

embedded scheme	zero-set considered as a subset of $\mathbb{R}^n$ or $\mathbb{C}^n$
blowup of scheme	modification of scheme by pulling apart certain portions
example of blowup	vertical projection from the corkscrew to the plane
blowup of ambient scheme	higher dimensional analog of the corkscrew
center of blowup	locus above which the blowup is not an isomorphism
exceptional locus of blowup	locus where blowup morphism is not isomorphism
total transform of scheme	inverse image under ambient blowup morphism
strict transform of scheme	(closure of) total transform minus exceptional locus
transversal schemes	manifolds whose tangent spaces meet transversally
normal crossings scheme	transversal manifolds meeting like coordinate subspaces
morphism	polynomial map
birational morphism	map which is almost everywhere an isomorphism
proper morphism	map with compact inverse images of compacta
top locus	points where a given function has its maxima
order of polynomial at point	order of Taylor expansion at point

−1. Explanation of the result. The properties required in the definition of a strong resolution of a singular scheme deserve some further specifications. First some general remarks.

Observe that the singular scheme X may not possess a global embedding $X \subset W$ into a regular ambient scheme W. In this case, one can cover X by affine pieces, embed these, and construct local morphisms ε there. By the excision property, their restrictions to the pullback of X patch and give a birational proper morphism $X' \to X$ with X' regular. Of course, embeddedness can then be asked only locally.

The result holds for reduced excellent schemes whose residue fields are of characteristic zero. The main property needed is that the singular locus of X is a proper closed subscheme of X (which would not hold for non-reduced schemes), and that several constructions extend to the completions of the local rings. The proof gives also a certain resolution for non-reduced schemes (expressed as the monomialization of ideals).

It should be emphasized that strong resolutions of schemes are by no means unique, and many birational morphisms ε will fulfill the required properties. The sometimes misinterpreted notion of *canonical resolution* appearing in the literature does not refer to uniqueness but rather to equivariance.

In the first chapter of Hironaka's paper [Hi 1], several further specifications and variations of the result are described. We are not going to discuss these here. Neither do we plan to list the many differences between Hironaka's proof and the subsequent ones. We restrict ourselves here to as simple a proof as available at the current stage of the field.

Let us now comment more specifically on the five properties of a strong resolution. The first two properties, explicitness and embeddedness, were proven by Hironaka; equivariance appears for the first time in [Vi 2]; excision in [BM1] and [EV 3].

Explicitness. The morphism $\varepsilon : W' \to W$ is given as a composition $W' = W^{(r)} \to W^{(r-1)} \to \ldots \to W^{(1)} \to W^{(0)} = W$ where each $W^{(i+1)} \to W^{(i)}$ is the blowup of $W^{(i)}$ in a closed regular center $Z^{(i)}$ in $W^{(i)}$. Transversality of $Z^{(i)}$ with the exceptional locus $Y^{(i)}$ in $W^{(i)}$ shall mean here that the union $Z^{(i)} \cup Y^{(i)}$ is a normal crossings scheme, where $Y^{(i)}$ denotes the inverse image in $W^{(i)}$ of the first i centers $Z^{(0)}, \dots, Z^{(i-1)}$ under the preceding blowups. By induction, we can assume that $Z^{(j)}$ is transversal to $Y^{(j)}$ for $j < i$. This implies that $Y^{(i)}$ is again a normal crossings divisor; hence it makes sense to demand that $Z^{(i)} \cup Y^{(i)}$ be a normal crossings scheme.

By the general properties of blowups, ε is a proper birational morphism $W' \to W$, and induces an isomorphism $W' \setminus Y' \to W \setminus Y$ outside the final exceptional locus $Y' \subset W'$, where $Y \subset W$ denotes the image of Y' under ε, i.e., the image of all intermediate centers $Z^{(i)}$.

Embeddedness. The strict transform of X is defined as follows. The restriction $\tilde{\varepsilon}$ of ε to $W' \setminus Y'$ being an isomorphism, consider the pullback $\tilde{X}$ of X under $\tilde{\varepsilon}$ in $W' \setminus Y'$. It coincides with the pullback of $X \setminus Y$ in W' under ε. The strict transform X' of X is defined as the Zariski-closure of $\tilde{X}$ in W'. If Z is contained in X, it coincides with the blowup of X with center Z. In terms of ideals, the strict transform is defined locally in the following way: If J defines X in W and has pullback J^* in W', let J' be the ideal generated by all $I(Y')^{-k_f} f^*$, where f^* ranges in J^* and $I(Y')^{k_f}$ is the maximal power of the (monomial) ideal defining Y' in W' which can be factored from f^*.

> The strict transform J' of J can also be calculated in terms of generator systems $f_1, \dots, f_m$ of J, but not any system will do the job. However, if the tangent cones of f_i (i.e., the homogeneous forms of lowest degree) generate the tangent cone of J, the transforms $f_i' = I(Y')^{-k_i} f_i^*$ with $k_i = k_{f_i}$ will generate J'.
>
> Such generator systems were called by Hironaka *standard bases* [Hi 1, chap. III], and are nowadays also known by the name of Macaulay bases. In order to compute and control invariants associated to J, Hironaka was led to introduce reduced standard bases (in the sense of reduced Gröbner bases). Only later [Hi 6] did Hironaka consider monomial orders on $\mathbb{N}^n$ with the corresponding standard bases, initial monomials and division theorems.

Embeddedness requires two properties to hold. First, the strict transform X' has to be regular, and, secondly, it should meet the exceptional locus Y' in W' transversally (in fact, it is even stipulated that $X' \cup Y'$ be a normal crossings scheme). In general, regularity of X' will be achieved earlier in the resolution process, and transversality requires some additional blowups. Example: Blowing up the origin in $\mathbb{A}^2$ desingularizes the curve $X = \{x^2 = y^3\}$, but its strict transform is tangent to the exceptional locus (see Figure 7). One further blowup yields transversality (i.e., transversal tangents), and a third blowup normal crossings (i.e., no three curves meet in a point).

Excision. If $X \subset W$ and $X \subset \tilde{W}$ are two closed embeddings with resolutions $\varepsilon : W' \to W$ and $\tilde{\varepsilon} : \tilde{W}' \to \tilde{W}$, it could happen that the restrictions δ and $\tilde{\delta}$ to X' and $\tilde{X}'$ are different. Excision says that this is not the case: There is an algorithm which constructs for any embedding $X \subset W$ the sequence of blowups $\varepsilon : W' \to W$, and for two different embeddings of X the algorithm produces the same restrictions over X.

Of course, one could choose many other proper birational morphisms $W' \to W$ inducing a desingularization of X. We will see in later sections what is meant by algorithm here. Essentially it is the construction of a center $Z \subset X$ at each stage of the resolution process, and it will be shown that this construction can be realized independently of the choice of the embedding of X.

Equivariance. This is a property with many facets. The commutativity of ε with smooth morphisms $W^- \to W$ means that for any embedding $X \subset W$, the morphism $\varepsilon^- : (W^-)' \to W^-$ induced by $\varepsilon : W' \to W$ (as a fiber product) is a strong resolution of the pullback X^- of X in W^-.

Equivariance contains as special cases the following assertions: ε commutes with the restriction to open subschemes of W, and hence induces also local resolutions. It commutes with automorphisms of W which stabilize X: Any symmetry of X will be preserved by the desingularization; i.e., X' will have the same symmetry. This is often expressed by saying that group actions on X lift to a group action on X' (equivariance of operation of groups). As this holds also for local symmetries, it follows that $X' \to X$ is an isomorphism outside the singular points of X. In particular, all centers will lie over $\mathrm{Sing}(X)$ — and their images in X must fill up $\mathrm{Sing}(X)$ since X' is regular and hence any singular point of X must belong at least once to a center. The property that $X' \to X$ is an isomorphism outside $\mathrm{Sing}(X)$ is called the *economy* of the desingularization (and is not fulfilled for weak resolutions by alterations as proposed by de Jong): regular points of X are not touched by the resolution process.

Another consequence of equivariance is that the desingularization of X commutes with passing to a cartesian product $X \times L$ with L a regular scheme. Said differently, if X is trivial along a regular stratum S in X (locally or globally), then the desingularization δ of X is a (local or global) cartesian product along S (you may consult the literature on equiresolution problems for this topic).

Commutativity with smooth morphisms also implies that the desingularization δ commutes with passage to completions. The proof shows that the result holds for formal schemes as well as for real or complex analytic spaces (with some natural finiteness conditions with respect to coverings to be imposed). In [Hi 1], resolution was proven for schemes of finite type over a field and real analytic spaces. The case of complex analytic spaces was done in [AHV1], [AHV2].

Commutativity with embeddings $W \to W^+$ simply means that the morphism $\varepsilon^+ : (W^+)' \to W^+$ restricts to ε over W; the assertion for field extensions is similar.

All equivariance properties are deduced from the respective properties of the resolution invariant defining the centers of blowup. This is a vector of integers whose components are essentially orders of certain ideals in regular ambient schemes. As long as these ideals behave well with respect to the operations listed in equivariance, their orders will also.

Effectiveness. This is a property which refers more to the actual construction of ε and which need not be imposed on a strong resolution. For implementations it is essential to have the centers given as the top locus of an invariant, i.e., as the locus of points where the invariant takes its maximal value. A prerequisite is that the invariant and the induced stratification of W can actually be computed (cf. the papers of Bodnár and Schicho). There appears a serious complexity problem, since the number of charts tends to explode when iterating blowups, and the chart transformation maps and the local coordinate changes quickly exceed the available capacities. Moreover, the stratifications defined by the invariant have to be computed (up to now) via Gröbner bases, causing the well known troubles when the number of variables increases.

For the future it can be expected (or hoped) that the centers of blowup allow a direct description from the singular scheme. Optimal would be the explicit construction of a non-reduced structure on the singular locus of X which yields, when taken as the center of blowup, the desingularization of X in one blowup. At the moment, it seems that nobody has the slightest idea of how this structure should look (but cf. [Bo2] for a first step in this context).

The five properties discussed here are the most natural and relevant conditions to impose on a desingularization of X. There exist further properties one may ask for, either concerning the monomialization of the defining ideal J of X in W' (see e.g. [BV 1], [BV 2]) or minimality conditions on the resolution (as can be completely established for surfaces and is developed for three-folds in Mori's minimal model program). Other conditions as well as generalizations can be found in the articles of the first list of references. In any case, the most intricate question seems to be the existence of strong resolution for positive characteristics or for arithmetic schemes, both in arbitrary dimension. We shall have a glance at positive characteristics towards the end of the paper.

0. Quick info on proof. For readers in a hurry we quickly describe the rough outline of Hironaka's argument. This, of course, contradicts the strategy of presentation agreed on at the beginning. In addition, several relevant details will have to be skipped. But as not everybody wants to know these, though still wishing to get a general impression on how the proof goes, we annul for a short while our agreement and sketch the main steps.

The patient reader who wishes to explore and develop the proof on her/his own and in chronological order is advised to skip this section and go directly to section 1 for meeting the travel guide.

The question is:

How can I understand in one hour the main aspects of a proof which originally covered two hundred pages?

Here is an attempt at this: You will (have to) believe that resolving a singular scheme is more or less equivalent to monomializing a polynomial ideal by a sequence of blowups. So let us fix an ideal J in a regular scheme W, for simplicity a principal ideal $J = (f)$.

Your personal favorite could be the polynomial $f = x^2 - z^3(z - y^2)$ in $W = \mathbb{R}^3$. The zero-set X of f looks as follows. The two "components" of the surface meet tangentially at the origin of the y-axis. This point is the only singularity of the upper component. The lower component looks like a cylinder (along the y-axis) over the cusp given by its section with the plane $y = \text{const}$ (see Figure 11).

Blow up a (still to be specified) center Z in W and consider the pullback J' of J under the induced blowup morphism $\pi : W' \to W$. Let $Y' = \pi^{-1}(Z) \subset W'$ be the exceptional component.

The singular locus of X in the example is the y-axis $\{x = z = 0\}$. If we take it as the center Z of the blowup, the pullback f' of f under π equals the polynomial $f' = x^2z^2 - z^3(z - y^2)$ (the computation is done in the z-chart of W', which is the

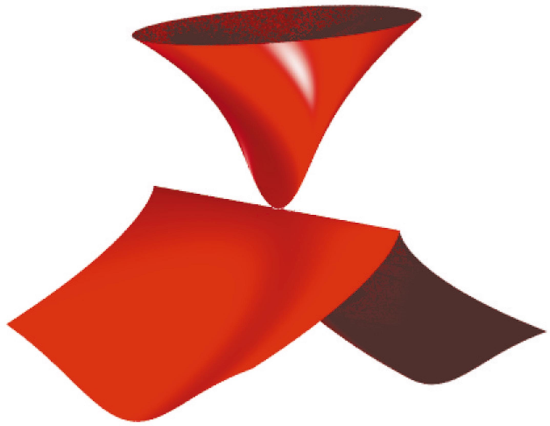

FIGURE 11. Zero-set of $f = x^2 - z^3(z - y^2)$.

chart where the interesting things happen). The pullback in this chart is given by replacing in f the variable x by xz

Define the order of f at a point of W as the order of the Taylor expansion of f at this point. If the order of f at all points of Z was the same, an exercise shows that J' factorizes into $J' = M' \cdot I'$ where M' is a power of the exceptional component Y' (and thus a monomial) and where $I' = (g')$ is an ideal which has at each point of Y' order less than or equal to the order of J along Z (you may either do the exercise or trust that it is easy). The ideal I' is called the weak transform of J.

FIGURE 12. Zero-set of $g' = x^2 - z(z - y^2)$.

In the example, the order of $f = x^2 - z^3(z-y^2)$ at the points of $Z = \{x = z = 0\}$ is everywhere equal to 2. The pullback f' of f factorizes into $f' = z^2 \cdot (x^2 - z(z - y^2)) = z^2 \cdot g'$ with exceptional factor z^2 and weak transform $g' = x^2 - z(z - y^2)$ of f (see Figure 12). Note that g' has at all points order ≤ 2, so the order did not increase. The zero-set of g' looks like this.

The factorization $J' = M' \cdot I'$ is necessary because the order of J' — in contrast to that of I' — will in general increase. As M' is just a monomial, the interesting information lies in I'. For symmetry reasons we shall write $J = M \cdot I$ with $M = 1$ and $I = J$, so that the factorization $J' = M' \cdot I'$ can be interpreted as the transform of the factorization $J = M \cdot I$, with certain transformation rules for M and I (they are not the same for M and for I). With this notation, I' becomes the weak transform of I.

> In the example, we factorize $f = 1 \cdot (x^2 - z^3(z - y^2)) = 1 \cdot g$ and $f' = z^2 \cdot (x^2 - z(z - y^2)) = z^2 \cdot g'$, with g' the weak transform of g.

If we succeed in lowering stepwise the order of I until it reaches 0, the monomial factor M of J will take over more and more of J until it coincides with J. At this final stage, J will have been monomialized, $J = M \cdot 1$.

If the order of I' has dropped at each point of W', we are done, because by induction we may assume to know how to resolve singularities of smaller order. Here we are leaving aside that we cannot specify the center yet (and omitting also some transversality questions which will appear in the further blowups). If the order has remained constant at a point a' of W', we have a problem, because there we have no measure which tells us that the singularity has improved.

> For the transform $g' = x^2 - z(z - y^2)$ of g, there is only one point a' of the same order, namely the origin of the z-chart. There, the order of g' is again 2. In all other points, the order of g' is 1 or 0.

The points where the order has remained constant are rare, since it can be shown that they live in a (regular) hypersurface inside the exceptional component. They will be called *equiconstant* points. We may choose a regular hypersurface V' in W' whose intersection with the exceptional component Y' contains all the points where the order of I' has remained constant. This is always possible (at least locally) and not hard to prove.

It turns out that — by a fluke — appropriate choices of V' stem from regular hypersurfaces below: The image V of V' in W is a regular hypersurface containing Z (again locally). So V' can be considered as the transform of V under the blowup $W' \to W$. In the neighborhood of points a of Z we get the following diagram:

$$\begin{array}{ccccccccc} J' = M' \cdot I' & \text{in} & W' & \supset & V' & \supset & V' \cap Y' & \ni & a' \\ \downarrow & & \downarrow & & \downarrow & & & & \\ J = M \cdot I & \text{in} & W & \supset & V & \supset & Z & \ni & a \end{array}$$

By construction, all equiconstant points a' above a point a of Z lie inside V'.

Let us resume these objects in the situation of the example, where we can pick $V' = \{x = 0\}$ in W' with image $V = \{x = 0\}$ in W.

$$\begin{array}{ccccccccc} f' = z^2 \cdot (x^2 - z(z - y^2)) & \text{in} & (\mathbb{R}^3)' & \supset & \{x = 0\} & \supset & \{x = z = 0\} & \ni & 0 \\ \downarrow & & & & \downarrow & & \downarrow & & \\ f = 1 \cdot (x^2 - z^3(z - y^2)) & \text{in} & \mathbb{R}^3 & \supset & \{x = 0\} & \supset & \{x = z = 0\} & \ni & 0 \end{array}$$

Now comes the decisive idea.

⟦ *In order to measure at equiconstant points of W' the improvement of the singularities when passing from J to J', descend to lower dimension and compare the singularities in V and V'.* ⟧

To this end, associate to the ideal $J = M \cdot I$ in W an ideal J_- in V (this is hence an ideal in fewer variables), and in the same manner associate to $J' = M' \cdot I'$ an ideal $(J')_-$ in V'. These *subordinate* ideals shall then reflect a possible improvement of the singularities between W and W'. Such an improvement can only be detected if it is possible to relate J_- and $(J')_-$ and to find a numerical invariant associated to J_- and $(J')_-$ which drops. This condition restricts our choices of J_-.

How could we compare the two (still unknown) ideals J_- and $(J')_-$? We have already observed that the hypersurface V of W can be chosen to contain (at least locally) the center Z of the blowup $W' \to W$. Therefore, Z defines also a blowup of V, and by the commutativity of blowups with restriction to subschemes, the blowup of V with center Z coincides with our hypersurface V'. We thus obtain a blowup map also in smaller dimension, namely $V' \to V$. To compare the ideals J_- in V and $(J')_-$ in V' it is natural to expect (or postulate) that $(J')_-$ is some *transform* of J_- under this blowup. This is indeed possible, but will require a well-adjusted definition of the ideal J_-. In addition, the type of transformation which occurs when passing from J_- to $(J')_-$ has to be specified.

> Our procedure here is typical for our ongoing search of a resolution proof: We first collect properties which our objects have to satisfy, thus reducing the number of possible candidates. Then we indicate one type of construction for the object — often there are several options — so that all these requirements are met.

We recapitulate the sine-qua-non for the construction of J_-: At any equiconstant point a' of Y' a certain transform $(J_-)'$ of the ideal J_- in V' associated to J shall coincide with the ideal $(J')_-$ associated to the transform J' of J. This is nothing else but saying that two operations on ideals commute, i.e., have the following commutative diagram

$$\begin{array}{ccc} J' & \to & (J')_- = (J_-)' \\ \downarrow & & \downarrow \\ J & \to & J_- \end{array}$$

Expressis verbis: *The descent in dimension before and after blowup commutes at all equiconstant points with the blowup in the actual and in the smaller dimension.*

The descent in dimension has spectacular implications, for it allows us to apply induction on the dimension to the entire resolution problem defined by J_- in V: Once we have defined J_- in V, we may assume that we know how to associate to J_- a suitable center of blowup Z_- and an invariant $i_a(J_-)$ so that blowing up V in Z_- and passing to the transform $(J')_- = (J_-)'$ of J_- makes this invariant drop.

It remains to construct for J the subordinate ideal J_- in V with the appropriate properties. This works only locally. Choose a local equation $x = 0$ for V in W and expand the elements f of I with respect to x, say $f = \sum_k f_k x^k$. Note here that we take elements from I, not from J; this does not matter at the beginning where $J = I$, but after the first blowup it does (the reason for doing so is of rather a technical nature and need not interest us at the moment). The coefficients f_k of this expansion will be polynomials on V. Those with index k smaller than the order of I generate an ideal in V, which will be the correct choice for the ideal J_-. It is called the *coefficient ideal* of $J = M \cdot I$ in V.

> Let us look at what this means for our example $f = x^2 - z^3(z-y^2)$. At the origin, we may choose for V the hypersurface $\{x = 0\}$ (for reasons to be explained later). The expansion of f with respect to x

has only one coefficient with index $k < 2 = \mathrm{ord}_0 f$, viz $-z^3(z-y^2)$, the coefficient of $1 = x^0$. So $J_- = (z^3(z-y^2))$ (it is only accidental here that J_- is the restriction of J to $x = 0$).

The transform $(J_-)'$ of J_- is defined by a suitable transformation rule for J_- under the blowup $V' \to V$ (the usual pullback would be too rude). The clue is that if J_- has constant order along Z, its pullback in V' will factorize similarly as J' factorized in W', and deleting a suitable exceptional factor from this pullback gives the correct transform of J_- (i.e., the *controlled transform* of J_-).

The transform f' of f was $f' = z^2(x^2-z(z-y^2)$. Throwing away the exceptional factor z^2, we get the coefficient ideal $(J')_- = (z(z-y^2))$ in $V' = \{x = 0\}$. It can be checked that it is an appropriate transform of J_- under the blowup $V' \to V$ of V with center Z.

So let us assume that with our definitions, the commutativity relation $(J_-)' = (J')_-$ holds. We may therefore write without ambiguity J'_- for this ideal (only at equiconstant points of W'). As happened for J and J', the ideals J_- and J'_- admit again factorizations $J_- = M_- \cdot I_-$ and $J'_- = M'_- \cdot I'_-$, where M_- and M'_- are exceptional factors in V and V' (at the beginning, M_- will again be trivial and equal to 1). So the correlation between the ideals in W and W' repeats in smaller dimension for their coefficient ideals in V and V'.

Once all this is settled we are almost done.

The center Z is defined by induction on the dimension by setting $Z = Z_-$, with Z_- the center associated to J_-. The improvement from J to J' is measured at the equiconstant points by comparing J_- with J'_-. As before, the relevant invariant is not the order of J_- or J'_- (which may increase), but the order of the factors I_- or I'_-. It turns out that I'_- is the weak transform of I_- whose order never increases. Either the order of I'_- has dropped, and induction applies, or it remained constant, in which case the whole argument of choosing local hypersurfaces can be repeated, producing a second descent in dimension.

In this way, aligning the orders of the various ideals I, I_-, $\ldots$ obtained by successive descent to a vector $i_a(J) = (\mathrm{ord}_a I, \mathrm{ord}_a I_-, \ldots)$ of non-negative integers, we obtain a local invariant, the *resolution invariant* of $J = M \cdot I$ at a. The preceding observations show that it satisfies under blowup the inequality $i_{a'}(J') \leq i_a(J)$ for points a' above a with respect to the lexicographic ordering: The first component of $i_a(J)$, the order of I, does not increase, because I' is the weak transform of I. In the case where it remains constant, the second component of $i_a(J)$, the order of I_-, does not increase, because we are at an equiconstant point in W' where commutativity holds, in particular, where I'_- is the weak transform of I_- and has order $\leq$ the order of I_-. If this order remains constant, the argument repeats in the next smaller dimension. This establishes $i_{a'}(J') \leq i_a(J)$. But in dimension 1, it can be shown that the order of I always drops to 0 (if some earlier component has not dropped), so that the inequality is in fact a strict inequality,

$$i_{a'}(J') < i_a(J),$$

for all points $a \in Z$ and $a' \in Y'$ above a. So our resolution invariant has dropped. This establishes the necessary induction.

Again, we test what was said in the concrete situation of the example, looking at the origins $a = 0 \in W = \mathbb{R}^3$ and $a' = 0$ in the z-chart of $W' = (\mathbb{R}^3)'$. We have $J = (f) = 1 \cdot I$ with

$f = g = x^2 - z^3(z - y^2)$ and $\mathrm{ord}_0 I = \mathrm{ord}_0 g = 2$ in $\mathbb{R}^3$. The coefficient ideal is $J_- = (f_-) = 1 \cdot I_-$ with $f_- = g_- = z^3(z - y^2)$ and $\mathrm{ord}_0 I_- = \mathrm{ord}_0 g_- = 4$ in $\mathbb{R}^2$. Blowing up $Z = \{x = z = 0\}$ in $\mathbb{R}^3$ we get transforms $J' = (f') = M' \cdot I'$ with $f' = z^2(x^2 - z(z - y^2))$, $g' = x^2 - z(z - y^2)$ and $\mathrm{ord}_0 I' = \mathrm{ord}_0 g' = 2$ in $(\mathbb{R}^3)'$. The coefficient ideal of I' is $J'_- = (f'_-) = M'_- \cdot I'_-$ with $f'_- = z(z - y^2)$, $g'_- = z - y^2$ and $\mathrm{ord}_0 I'_- = \mathrm{ord}_0 g'_- = 1$ in $V' = (\mathbb{R}^2)'$. We see that the first component of $i_a(J)$ has remained constant equal to 2 at $a' = 0 \in W'$, and that the second component has dropped from 4 to 1. Therefore $i_{a'}(J') < i_a(J)$ at these points.

Of course, there are many technical complications we have omitted (and which will be discussed in the sequel). But the brief résumé of the proof should at least give you enough information to respond to questions like: *"Do you have any idea how to prove resolution of singularities in characteristic zero?"* Instead of saying, *"Indeed, that's a very good question! It is something I always wanted to know,"* you may start to talk about intricate inductions which are built on each other (actually, you may even mention that a proof as in [EH] requires fourteen such inductions, and that at the moment this is the best one can hope for).

If you wish to clarify the many doubts and questions which may have occurred to you while browsing this quick-info-section, just go on reading.

CHAPTER 1: MAIN PROBLEMS

1. Choice of center of blowup. We now start our journey through the jungle of singularities, blowups and strict, weak and total transforms of ideals. Our first steps will consist of trying to get an overview on the possible paths which could lead us towards a solution to the problem. The centers of blowup constitute our primary object of interest.

Given X, choose a closed embedding of X in a regular ambient scheme W with defining ideal J of the structure sheaf $\mathcal{O}_W$. Resolving X is essentially equivalent to resolving J (in a sense specified later; roughly speaking it means to monomialize J). The problem is to choose the first center Z of blowup. This is a regular closed subscheme of W yielding the blowup $\pi : W' \to W$ and the transforms of X and J in W'. Call $Y' = \pi^{-1}(Z)$ the exceptional component in W'.

Here it has to be decided which kind of transforms of X and J will be considered. Denote by $J^* = \pi^{-1}(J)$ the *total transform* of J, and let J^{st} be the strict transform as defined earlier. It is generated by the elements $\pi^{-1}(f) \cdot I(Y')^{-\mathrm{ord}_Z f}$ for f varying in J, where $I(Y')$ denotes the ideal defining Y' in W' and $\mathrm{ord}_Z f$ is the maximal power of the ideal $I(Z)$ defining Z in W to which f belongs in the localization $\mathcal{O}_{W,Z}$ of $\mathcal{O}_W$ along Z (without passing to the localization it would be the maximal *symbolic* power of $I(Z)$ containing f). It can be shown that $\mathrm{ord}_Z f$ equals the minimal value of the orders $\mathrm{ord}_a f$ of f for a varying in Z.

We shall also consider the *weak transform* $J^\curlyvee = J^* \cdot I(Y')^{-\mathrm{ord}_Z J}$ of J in W'. Here $\mathrm{ord}_Z J$ denotes the minimum of the orders $\mathrm{ord}_Z f$ over all f in J. The fact that $I(Y')$ can be factored from J^* to the power $\mathrm{ord}_Z J$, i.e., the existence of a factorization $J^* = J^\curlyvee \cdot I(Y')^{\mathrm{ord}_Z J}$, is proven by a computation in local coordinates (cf. the appendix). The weak transform $J^\curlyvee$ is an ideal contained in J^{st} whose associated scheme $X^\curlyvee$ in W' may have some components in the exceptional divisor

Y' (and need not be reduced). Algebraically, it is easier to work with weak than with strict transforms (cf. example 6 in the section "Examples"). For hypersurfaces, both notions coincide. We shall always have to do with the weak transform of ideals. A sufficiently explicit and powerful resolution process for weak transforms will allow us to deduce then the desired assertions about strict transforms (or you may restrict to hypersurfaces, which suffices to understand the main argument of the proof).

Before getting stuck at the first steps through the jumble of the jungle, take care to keep apart the three transforms of ideals under blowups $\pi : W' \to W$:

$J^* = \pi^{-1}(J)$	total transform of J,
$J^{st} = (\pi^{-1}(f) \cdot I(Y')^{-\operatorname{ord}_Z f}, f \in J)$	strict transform of J,
$J^\curlyvee = J^* \cdot I(Y')^{-\operatorname{ord}_Z J}$	weak transform of J.

Now, given X in W with ideal J, how shall we decide on the center Z? There is no immediate candidate running around. As we wish to remove the singularities from X, we could take in a first attempt the whole singular locus as center. This works well for curves, because their singularities are isolated points. Once in a while each of them has to be blown up (otherwise the singularity sitting there will never disappear), so we can take them all together as center. Resolution of curves says that these blowups eventually yield a regular curve (but possibly still tangent to some exceptional component), and some further blowups (in the intersection points of the regular curve with the exceptional divisor) allow us to make this curve transversal to the exceptional divisor.

For surfaces, the situation is more complicated, but the idea of taking the singular locus as center still works — provided some cautions are taken. The singular locus of a surface consists of a finite number of isolated points and irreducible curves, which may even be singular.

These curves are not allowed as centers in a strong resolution if they are not regular or if they intersect. But one could try to make them first regular by some auxiliary blowups, separate them from each other by further blowups and then take their union as center. Of course, the singular curves of the singular locus of X can be resolved by point blowups. But it is not clear that their transforms again fill up the whole singular locus of the transform X' of X. And indeed, the singular locus of X' may have new components which lie outside the (strict) transform of the singular locus of X. But, as Zariski observed [Za 4], these new components do not bother us too much since they are regular curves (a fact which fails in higher dimensions, actually already for three-folds; cf. [Ha 2], [Ha 4]). For surfaces it is thus possible to transform the singular locus of X by preliminary blowups into a union of isolated points and regular curves transversal to each other and to the exceptional locus. Further blowups allow us to separate these curves from each other. Taking the resulting union is then a permitted choice of center. By choosing a suitable resolution invariant (for example, as in [Ha 4]) it is shown that blowing up X in this union improves the singularities of X. The invariant drops when passing from X to X'. Now induction applies to show that a resolution of X is achieved in finitely many steps.

For three-folds and higher dimensional schemes, the preceding construction of an admissible center falls short, mostly, because the passage to the singular locus does not commute with blowup: The singular locus of X' may have singular components (inside the exceptional locus) which have nothing to do with the singular locus of

X [Ha 2]. Therefore it seems difficult or even impossible to make the singular locus regular by auxiliary blowups as we did in the case of surfaces.

After this deception, we shall refrain from constructing the center Z directly. Up to now, no ad hoc definition of Z which works in any dimension has been discovered. Instead, we shall proceed in the opposite direction: We shall take any regular subscheme of the singular locus of X as center. We blow it up and then observe what effects on X will follow. Certain centers will have better consequences on X and will be thus preferred. This in turn shall lead us to conditions on Z which may help to determine a suitable class of admissible centers Z.

So assume that we have given a closed regular subscheme Z of W which we take as the center of our first blowup, $\pi : W' \to W$. As we have no exceptional locus yet, no transversality conditions will be imposed on Z. But we will assume that Z sits inside the singular locus of X (since at regular points we won't touch X). For hypersurfaces, this is the locus of points where the ideal J has at least order 2 in W, i.e., of points a with maximal ideal m_a in the local ring $\mathcal{O}_{W,a}$ such that the stalk of J at a is contained in the square of m_a. For non-hypersurfaces, this description of the singular locus only holds if the scheme X is minimally embedded. Let J^* and $J^\curlyvee$ denote the total and weak transform of J in W'. As we have seen above, we may write $J^* = M' \cdot J^\curlyvee$ with $M' = I(Y')^{\mathrm{ord}_Z J}$. As Z is regular, Y' is also regular, and M' is a monomial factor of J^* which should not cause any trouble. The ideal $J^\curlyvee$ is more interesting and contains all the information on the singularities of the scheme defined by J^*. Clearly, at points a' outside Y', its stalks are isomorphic to the stalks of J at the projection point $a = \pi(a')$ (which lies outside Z). So nothing will have changed there. Let us hence look at points a' in Y' above a in Z. There, two interesting things can be observed:

Observation 1. If $\mathrm{ord}_a J = \mathrm{ord}_Z J$, then $\mathrm{ord}_{a'} J^\curlyvee \leq \mathrm{ord}_a J$. Thus the order of the weak transform $J^\curlyvee$ does not increase if the order of J was constant along Z.

> As the order of an ideal at a point is an upper semicontinuous function, the order $\mathrm{ord}_a J$ will be generically constant along Z. Only at points a of a closed subscheme of Z can the order of J be larger than the generic order along Z.

Observation 2. The locus of points a' in Y' with $\mathrm{ord}_{a'} J^\curlyvee = \mathrm{ord}_a J$ lies inside a regular hypersurface of Y'.

We will discuss the second observation in the next section. As for the first we shall ask that the center Z lie inside the locus of points where the order of J is maximal, according to the philosophy that the worst points of X should be attacked first. We thus define $\mathrm{top}(J) = \{a \in W,\ \mathrm{ord}_a J \text{ is maximal}\}$, the *top locus* of J in W. This is a closed subscheme of W, by the semicontinuity of the order, but possibly singular. We agree to postulate:

Requirement. The center Z shall be contained in $\mathrm{top}(J)$.

This condition — which will be imposed in the sequel of the paper on all centers which appear — still leaves a lot of freedom how to choose Z. As a general rule, large centers tend to improve the singularities faster than small centers. This would suggest taking for Z a regular closed subscheme of $\mathrm{top}(J)$ of maximal possible dimension. Such a center will not be unique, e.g., if $\mathrm{top}(J)$ consists of two transversal lines, and one of them has to be chosen as the center. In this case it may happen that the scheme X has a symmetry, obtained e.g. by interchanging two variables, and yielding a permutation of the two lines (i.e., the permutation group S_2 acts on X). You may not want to destroy this symmetry by an asymmetric choice of the center, so none of the lines is a good candidate. Instead you may prefer to take the only subscheme which is S_2-invariant, namely the intersection point, which in turn may be a too small center. So there is some ambiguity about how to choose a regular subscheme of maximal dimension of $\mathrm{top}(J)$.

Also, Z has to be chosen globally: If $\mathrm{top}(J)$ is the node $x^2 - y^2 - y^3 = 0$ in the plane, locally at the origin 0 one could choose one of its branches as center. Globally, this branch will return to the origin yielding a normal crossings singularity in the intersection point. Therefore the origin 0 is the only possible choice for Z here.

> It is a good moment to prove now that the inclusion $Z \subset \mathrm{top}(J)$ indeed implies the stated inequality $\mathrm{ord}_{a'} J^\curlyvee \leq \mathrm{ord}_a J$. This is a local statement at a and a', which allows us to restrict to the local blowup $\pi : (W', a') \to (W, a)$ given by the inclusion of local rings $\mathcal{O}_{W,a} \subset \mathcal{O}_{W',a'}$. By the upper semicontinuity of the order of ideals, it suffices to check the inequality at closed points. For simplicity, we assume that the ground field is algebraically closed.
>
> We may then choose local coordinates $x_1, \ldots, x_n$ at a (i.e., a regular system of parameters of $\mathcal{O}_{W,a}$) such that $\mathcal{O}_{W,a} \subset \mathcal{O}_{W',a'}$ is given by a monomial substitution of the coordinates, say $x_i \to x_i \cdot x_k$ for $1 \leq i \leq k-1$, and $x_i \to x_i$ for $k \leq i \leq n$, where $k \leq n$ is such that $x_1, \ldots, x_k$ define Z in W locally at a. Passing to the completions $\hat{\mathcal{O}}_{W,a} \subset \hat{\mathcal{O}}_{W',a'}$ does not alter the order of J. Expand elements f of $\mathcal{O}_{W,a}$ as a power series in $x_1, \ldots, x_k$, say $f = \sum_{\alpha \in \mathbb{N}^n} c_\alpha x^\alpha$. Then $\mathrm{ord}_a f = \min\{|\alpha|,\ c_\alpha \neq 0\}$. Set $o = \mathrm{ord}_a J = \min_{f \in J} \mathrm{ord}_a f$.
>
> By assumption, $Z \subset \mathrm{top}(J)$; hence $\mathrm{ord}_Z f \geq o$ for all $f \in J$. As $I(Z) = (x_1, \ldots, x_k)$ we get, setting $|\alpha|^k = \alpha_1 + \cdots + \alpha_k$, that $|\alpha|^k \geq o$ for all α with $c_\alpha \neq 0$. The total transform f^* has expansion $f^* = \sum c_\alpha x^{\alpha^*}$ with $\alpha^* \in \mathbb{N}^n$ given by $\alpha_i^* = \alpha_i$ for $i \neq k$ and $\alpha_k^* = \alpha_k + \alpha_1 + \cdots + \alpha_{k-1} = |\alpha|^k$. Factoring $I(Y')^{\mathrm{ord}_Z J}$ from J^* yields $f^* \cdot x_k^{-o} = \sum c_\alpha x^{\alpha^* - o e_k}$ where e_k is the k-th standard basis vector of $\mathbb{N}^n$.
>
> But $o = \min\{|\alpha|,\ c_\alpha \neq 0\} = \min\{|\alpha|^k,\ c_\alpha \neq 0\}$; hence for any α realizing this minimum, we will have $|\alpha^* - o e_k| = |\alpha| + \alpha_1 + \cdots + \alpha_{k-1} - o \leq |\alpha|$, with equality if and only if $\alpha_1 + \cdots + \alpha_{k-1} = o$, say $a_k = 0$. This shows that $\mathrm{ord}_{a'}(f^* \cdot x_k^{-o}) \leq o$, from which $\mathrm{ord}_{a'} J^\curlyvee \leq o = \mathrm{ord}_a J$ is immediate. The claim is proven.

What we have learned in this section will form our point of attack for proving resolution. We therefore repeat: The inequality $\mathrm{ord}_{a'} J^\curlyvee \leq \mathrm{ord}_a J$ between the orders of an ideal and its weak transform will hold for $a' \in Y'$ above $a \in Z$ for any regular center $Z \subset \mathrm{top}(J)$, independently of its dimension or location inside $\mathrm{top}(J)$. This suggests considering the maximal value of $\mathrm{ord}_a J$ on X (or W) as the invariant which should improve under blowup. We have seen at least that it cannot increase. However, at certain points a' it may remain constant. If this happens, either the choice of our center was not a good one or we need some extra measure besides the maximal value of the order of J to show that the situation has also improved at these points, though in a less evident way. This leads us to take a closer look at these points.

2. Equiconstant points. Given an ideal J in W, let $\pi : W' \to W$ be the blowup with center $Z \subset \mathrm{top}(J)$. Let a be a point of Z, and set $o = \mathrm{ord}_a J = \mathrm{ord}_Z J$. Points a' in the exceptional locus Y' will be called *equiconstant points* if the order of the weak transform $J^\curlyvee$ of J has remained constant at a':

$$\mathrm{ord}_{a'} J^\curlyvee = \mathrm{ord}_a J.$$

Classically, these points were called very infinitely near points. Let us first check that such points actually may occur. In example 1, they could be avoided by a different choice of the center; in example 2, only one center is possible, and for this center, an equiconstant point appears in the blowup.

Example 1. Blowing up the origin of the Whitney umbrella X defined in $\mathbb{A}^3$ by $x^2 - y^2 z = 0$ produces at the origin of the z-chart the same singularity, $x^2 - y^2 z = 0$. Indeed, J^* is generated by $x^2 z^2 - y^2 z^3 = z^2(x^2 - y^2 z)$, yielding weak transform $J^\curlyvee$ generated by $x^2 - y^2 z$. (As we have a hypersurface, the strict and the weak transforms coincide.) Hence the origin of the z-chart is an equiconstant point. But as the singularity of X' is exactly the same there, no other invariant besides the order can have improved. Therefore the choice of our center was wrong. Taking for Z the origin yields a center which is too small. Observe that the top locus of X is the z-axis and could have equally been taken as center (this is the only other option). Doing so, the computation of $J^\curlyvee$ in the two charts shows that this larger center does improve the singularities. Actually, the singularities of X are resolved when blowing up the z-axis.

> For the reader's convenience, we include the computation: As $I(Z) = (x, y)$, we have two charts to consider, the x- and the y-chart. The total transform J^* is generated there by $x^2 - x^2 y^2 z = x^2(1 - y^2 z)$ and $x^2 y^2 - y^2 z = y^2(x^2 - z)$ respectively. The polynomials in parentheses define the weak transform of J. They are both regular, say of order ≤ 1 at any point of Y'.

This suggests again that the center should better be chosen large (and then, in this example, no equiconstant point will appear). In any case, we conclude that we cannot choose for the center just any closed regular subscheme of $\mathrm{top}(J)$. Some (still unknown) precautions will have to be taken.

Let us now look at an example where also equiconstant points occur in the exceptional locus, but where only one choice of center is possible.

Example 2. Consider $x^3 - y^2 z^2 = 0$ in $\mathbb{A}^3$. It can be checked that the top locus consists of one point, the origin, and that J has order 3 there (for more information on the order and how to compute top loci, see the appendix). We must therefore choose $Z = \{0\}$ as our center. The total transforms in the three charts are as follows: $x^3 - x^4 y^2 z^2 = x^3(1 - xy^2 z^2)$ in the x-chart, $x^3 y^3 - y^4 z^2 = y^3(x^3 - yz^2)$ in the y-chart, and symmetrically in the z-chart (using that J is invariant under exchanging y and z). The polynomials in parentheses define the weak transform of J. In the x-chart, $J^\curlyvee$ has order 0 along the exceptional divisor $x = 0$ (i.e., the scheme $X^\curlyvee$ does not intersect Y' there). In the y-chart, the origin is the only equiconstant point, and the singularity there is $x^3 - yz^2 = 0$. An analogous statement applies in the z-chart. As the center was prescribed by $\mathrm{top}(J)$, these two equiconstant points cannot be avoided by choosing another center.

At the origins of the y- and z-chart, the order of $J^\curlyvee$ remains constantly equal to 3. However, there seems to be some finer improvement at this point with respect to the initial singularity $x^3 - y^2z^2 = 0$: The exponent of y in the second monomial has dropped from 2 to 1. But this decrease is coordinate dependent and therefore not an intrinsic measure of improvement. We will investigate in the section "Independence" how to measure the improvement here in a coordinate free and intrinsic way.

Example 3. Consider $x^3-y^3z^3 = 0$ in $\mathbb{A}^3$. The top locus consists of two transversal lines, the y- and the z-axis, and J is S_2-invariant by interchanging y and z. The symmetric choice of center would be the origin $Z = \{0\}$, because none of the lines can have a preference, and we wish to choose Z as symmetrically as possible. Blowing up 0 in $\mathbb{A}^3$ yields total transforms $x^3 - x^6y^3z^3 = x^3(1 - x^3y^3z^3)$ regular in the x-chart, and $x^3y^3 - y^6z^3 = y^3(x^3 - y^3z^3)$ in the y-chart (again we may omit the z-chart by symmetry). In the y-chart we definitely have a problem: Our choice of center was the only natural one, but the singularity has remained the same, namely $x^3 - y^3z^3$. We will discuss at the end of the section "Setups" how to overcome this impasse.

> Taking instead of $x^3 - y^3z^3$ the polynomial $x^3 - y^4z^4$, we get in the y-chart the weak transform $x^3 - y^5z^4$, and the singularity seems to have become worse.

The next natural question, based on the preceding observations, is to locate the equiconstant points inside the exceptional divisor and to describe criteria when they appear. The proof of $\mathrm{ord}_{a'} J^\curlyvee \leq \mathrm{ord}_a J$ shows that the homogeneous part of the lowest degree of the elements of J plays a decisive role. The ideal generated by these homogeneous polynomials is the *tangent cone* $\mathrm{tc}(J)$ of J at a, say $\mathrm{tc}(J) = (\sum_{|\alpha|=o_f} f_a x^\alpha,\ f \in J)$ where o_f denotes the order of f at a, and $x_1, \ldots, x_n$ are some local coordinates. Let x^α be a monomial of degree o with non-zero coefficient f_α in the expansion of $f \in J$ with $\mathrm{ord}_a f = o = \mathrm{ord}_a J$. We saw above that if $\alpha_k > 0$, then $\alpha_1 + \cdots + \alpha_{k-1} < o$ and hence $\mathrm{ord}_{a'} f' < \mathrm{ord}_a f$, setting $f' = f^* \cdot x_k^{-o}$. Here, a' is the origin of the x_k-chart in W' induced by the choice of coordinates $x_1, \ldots, x_n$ in W at a.

We can now specify observation 2 from the preceding section.

Observation 2. If $x_1, \ldots, x_n$ are coordinates of W at a with $I(Z) = (x_1, \ldots, x_k)$ and such that the coordinate x_i appears in the tangent cone $\mathrm{tc}(J)$ of J, for some $i \leq k$, then the equiconstant points a' above a lie in the hyperplane $x_i = 0$ of W'. More precisely, let $V \in W$ be the regular hypersurface $x_i = 0$, and let $V' \subset W'$ be its strict transform. Then all equiconstant points of J lie in V'.

> We need here that the coordinates are chosen so that the number of coordinates appearing in the tangent cone of J is minimal.

Here, by slight abuse of notation, $x_1, \ldots, x_n$ denote also the induced coordinates of W' at a' (see the appendix for more details on how to choose local coordinates for blowups). This notation is justified by the fact that the local blowup $(W', a') \to (W, a)$ has an expression in affine charts going from $\mathbb{A}^n$ to $\mathbb{A}^n$. Said differently, the observation can be expressed as follows: The equiconstant points are contained in the strict transform V' of any local hypersurface V in W at a which contains Z locally at a and whose equation appears as a variable in the tangent cone of J when it is written in the minimal number of variables.

Let us compute such hypersurfaces V in some examples.

Example 4. Let us start with a plane curve of equation $x^o + y^q = 0$ in $\mathbb{A}^2$ with $q \geq o$. The order at 0 is o, and the origin is the only point of this order, thus $\mathrm{top}(J)$ is the origin. The blowup W' of $W = \mathbb{A}^2$ is covered by two affine charts, the x-chart and the y-chart, with respective total transforms $x^o + x^q y^q = x^o(1 + x^{q-o}y^q)$ and $x^o y^o + y^q = y^o(x^o + y^{q-o})$. The polynomials in parentheses are the weak transforms. We are only interested in points $a' \in Y'$ (where Y' is isomorphic to projective space $\mathbb{P}^1$). It is useful here to partition Y' into two sets, the entire x-chart and the origin of the y-chart. In the first set, the order of $J^\curlyvee$ is everywhere 0 as is checked by inspection, so we need not consider these points. We are left with the origin of the y-chart with weak transform defined by $x^o + y^{q-o} = 0$. Note that the origin lies in the hypersurface $x = 0$ in the y-chart of W', which agrees with our second observation, since x appears in the tangent cone of J (which is x^o). Whether the order has dropped or not depends on the value of $q - o$. If $q \geq 2o$, the order has remained constant, else it has decreased. For later reference we note that the order of the restriction $J_{|V}$ comes into play, where V is the hypersurface $\{x = 0\}$. The strict transform V' of V contains the only possible equiconstant point. Moreover, if the origin of the y-chart is an equiconstant point, the improvement of $J^\curlyvee$ seems to be captured by $J^\curlyvee{}_{|V'} = (y^{q-o})$, whose order is strictly smaller than $J_{|V} = (y^q)$. Of course, these observations are coordinate dependent and as such not very meaningful or intrinsic. But they already give a feeling for the phenomena we are going to study later on.

Example 5. We consider now a surface, e.g. $x^o + y^r z^s$ with $r + s \geq o$. Here, the order o at 0 is the maximal value of the local orders at points of $\mathbb{A}^3$. The Whitney umbrella from above is a special case, with $o = r = 2$ and $s = 1$. Let us first determine $\mathrm{top}(J)$. Apparently, it depends on the values of o, r and s. If $r, s < o$, it is reduced to the origin. If $r < o \leq s$, it is the y-axis, and symmetrically for $s < o \leq r$. If $r, s \geq o$, then $\mathrm{top}(J)$ is the union of the y- and the z-axis, and three choices for Z are possible: the origin or one of these axes (we do not study the interesting question of normal crossings center here; see [Ha 4]).

In all cases, the tangent cone of J consists of the monomial x^o, except if $r+s = o$, in which case it is $x^o + y^r z^s$. The hypersurface $V = \{x = 0\}$ is always a good candidate for finding equiconstant points, because x appears in the tangent cone (we could also take $x + a(y, z)$ with a any polynomial in y and z of order at least 2). From what we have seen earlier, we know that its strict transform V' in W' contains all equiconstant points of Y' (outside Y', all points will be equiconstant, but have no relevant interest). Let us assume that we have chosen $Z = \{0\}$ the origin in $\mathbb{A}^3$, because $r, s < o$. As $V' \cap Y'$ lies entirely in the complement of the x-chart, we may omit the x-chart from our considerations and look only at the y- and the z-chart. Up to interchanging y and z we may place ourselves in the y-chart, where the total transform J^* equals $x^o y^o + y^{r+s} z^s = y^o(x^o + y^{r+s-o} z^s)$. The origin of this chart is an equiconstant point if and only if $r + 2s - o \geq o$, say $r + 2s \geq 2o$. As $s < o$, the y-exponent has decreased from r to $r - (o - s)$. If $s \geq 0$, we could have chosen a larger center, namely the y-axis (in which case we would have to look at the z-chart, observing that there the z-exponent drops at the equiconstant point).

The various computations in local coordinates ask for giving a more geometric description of regular hypersurfaces of W' containing the set $O' \subset Y'$ of equiconstant points a' of $J^\curlyvee$, i.e., points where $o' = o$, denoting by o' the order of $J^\curlyvee$ at a' in W'. Recall that, by definition, equiconstant points lie inside the exceptional locus Y', so that there is a lot of flexibility in choosing regular hypersurfaces V' of W' containing O'. In any case we may choose them transversal to Y'. This does not imply yet that their image V in W under π is regular. But if V in W is also a regular hypersurface, then V' appears as the strict transform of V. The computation in the proof of $o' \leq o$ has shown that such V exist in W, at least locally at any point $a \in Z$. They are characterized as follows: Choose local coordinates $x_1, \ldots, x_n$ in W at a so that $x_1, \ldots, x_m$ appear in the tangent cone of J and such that $m \leq n$ is minimal with this property. Then any V defined by an equation in which no $x_{m+1}, \ldots, x_n$ appears in the linear term will do the job. If some $x_{m+1}, \ldots, x_n$ appears linearly, the strict transform V' of V may not catch all equiconstant points. Of course, any higher order terms are permitted in the expansion of the defining equation of V, because they do not alter the intersection of V' with Y' in which we are interested.

This property can be expressed by saying that V has reasonably good contact with the subscheme $\mathrm{tc}(X)$ of W defined by the tangent cone $\mathrm{tc}(J)$ of J. In particular, any such regular V must be tangent to the coordinate plane $x_1 = \ldots = x_m = 0$. This condition can be made coordinate independent. Let U be (locally at a, or, say, in the completion of the local ring) the regular subscheme of W of maximal dimension contained in $\mathrm{tc}(X)$. Then any regular V containing U will work.

Our attempts to nail down possible hypersurfaces V is not efficient as long as the approach remains computational and the improvement of the singularities at equiconstant points is measured by ad-hoc and coordinate dependent objects. Nevertheless we emphasize:

Conclusion. For any ideal J in W, not necessarily reduced, and any $a \in W$ there exists locally at a a regular hypersurface V in W whose strict transform V' contains all equiconstant points a' in W' above a. We call such hypersurfaces *adjacent for* J.

At this point, an apparently unmotivated but natural question will be highly instructive and significant:

If V is adjacent to J, is its strict transform V^{st} again adjacent to $J^\curlyvee$?

This question is surprisingly subtle, and its investigation (which we will give in the section "Obstructions") will split the world into a characteristic zero and characteristic p hemisphere. A positive answer to it would allow us to choose locally in W regular hypersurfaces whose successive strict transforms contain, for any sequence of blowups with regular centers inside the top loci of J and its weak transforms, all equiconstant points above a. If such hypersurfaces exist — and we will show that they do in characteristic zero, whereas they need not exist in positive characteristic — they would accompany the whole resolution process until eventually the maximal value of the order of J in W drops. Such hypersurfaces will be said to have *permanent contact* with J.

3. Improvement of singularities under blowup. This section contains the gist of Hironaka's argument, so take your time. It is inspired by Jung's method of projecting surfaces to the plane. If you understand this section, you will be in good shape to be considered an 'insider'.

In the last section we realized that at equiconstant points a' the ideal $J^\curlyvee$ need not be obviously simpler than J at a. The order is the same, and exponents of the Taylor expansions of elements of J may have decreased, increased or remained constant. As a consequence, we suspect that the order is not sufficiently fine to detect an improvement, and we should look for other invariants. On the other hand, the order is the first and simplest invariant associated to an ideal.

In order to confront this irritating quandary, let us return to the curve singularity $x^o + y^q = 0$ in $\mathbb{A}^2$ with $q \geq o$. If $q = o$ and the characteristic is different from o, the order of $J^\curlyvee$ has decreased everywhere and there are no equiconstant points in Y'. If $q = o$ and the characteristic is equal to o, replacing x by $x + y$ transforms $x^o + y^o$ into x^o, which is a monomial and cannot be improved under blowup. So we may assume that $q > o$. There then appears at most one equiconstant point, the origin of the y-chart, and $J^\curlyvee$ is defined by $x^o + y^{q-o}$. If $q \geq 2o$, this is an equiconstant point; otherwise the order has dropped. So let us assume for the sequel that $q \geq 2o$. In the given coordinates, the improvement of the singularity at a' appears in the change from y^q to y^{q-o}. This is a coordinate dependent description of the improvement. But working with fixed local coordinates, although useful for computations, is not appropriate if we wish to argue globally and with intrinsic objects defined for singularities in any dimensions. Therefore we would like to give the monomial y^q in $x^o + y^q$ a coordinate-free meaning, hopefully independent of the dimension.

We proceed as follows. The line $V = \{x = 0\}$ will be adjacent to J at the origin $a = 0$. The first thing to think of is the restriction of $x^o + y^q$ to the hypersurface V. Accordingly, at a', we would take the restriction of $J^\curlyvee$ to $V' = V^{st}$. The resulting polynomials depend on the choice of V, and there is no immediate candidate for V which would make them intrinsic. Actually, this might be asking too much, but at least the orders of the resulting polynomials should be intrinsic. In our example, if o does not divide q or the characteristic is different from o, the orders q and $q - o$ of $J_{|V}$ and $J^\curlyvee{}_{|V'}$ are not intrinsic. Replacing V by $\tilde{V} = \{x + y^2 = 0\}$ gives another adjacent hypersurface with $\mathrm{ord}_a(J_{|V}) = 2o$ (since we assumed $q \geq 2o$) and $\mathrm{ord}_{a'}(J^\curlyvee{}_{|V'}) = o$.

Among the many possible values of $\mathrm{ord}_a(J_{|V})$ for varying hypersurfaces V, only two play a special role: the minimal and maximal values. It is easy to see that the minimal value is not a good choice, since it is always equal to o (though the corresponding V need no longer be adjacent, as is seen in the example above, where $V = \{x + y = 0\}$ realizes the minimum but is not adjacent). The maximal value is more interesting. It is certainly intrinsic, and — as we will show later — can be realized by an adjacent hypersurface (mainly this holds because changing higher order terms in the equation of V preserves adjacency). So let us look more closely at this maximal order (see [Ab 8] for a detailed discussion of it in the curve case).

The maximal value of $\mathrm{ord}_a(J_{|V})$ provides an intrinsic measure of the complexity of the singularity of curves, subordinate to the order itself. We denote it by

$$o_- = \max_V \{\mathrm{ord}_a(J_{|V})\},$$

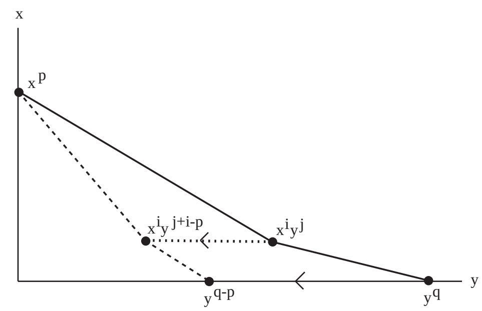

FIGURE 13. *Transformation of Newton polygon under blowup*: In the picture, the x-axis is drawn vertically.

the maximum being taken over all regular local hypersurfaces V of W at a, and call it the *secondary order* of J. The minus sign in the index refers to the decrease in the embedding dimension. What we get is a local invariant of J at a.

Our definition of o_- has an immediate and unpleasant defect which will force us to modify the definition a little bit. In many cases, the restriction $J_{|V}$ of J to V will be the zero ideal and then carries no information at all. The simplest example is $x^o + xy^q$. Of course, the defining equation of V could be factored from the polynomial, but this is not of much use, because the factor will again depend on the choice of V. In this situation it helps to look at a slightly more general example.

Example 6. Let J be generated by $f = x^o + \sum_{i<o} f_i(y)x^i$ with polynomials $f_i(y)$ in one variable y of order $\geq o - i$ at 0. By this assumption on the f_i, the order of J at 0 is o. For simplicity, let us assume that $f_i(y) = c_i y^{(o-i)d}$ for some $d \in \mathbb{N}$ and constants c_i. The preceding example, $x^o + y^q = 0$, had $f_i = 0$ for $i > 0$ and $f_0 = y^q$. Let us observe how the coefficients of f transform at an equiconstant point. The coordinates are chosen so that the only equiconstant point can appear at the origin of the y-chart. The corresponding substitution of the variables for the blowup is $x \to xy$ and $y \to y$. It gives the total and weak transform (in parentheses) of f as

$$\begin{aligned} f^* &= x^o y^o + \sum_{i<o} c_i y^{(o-i)d} x^i y^i \\ &= y^o(x^o + \sum_{i<o} c_i y^{(o-i)d-(o-i)} x^i) \\ &= y^o(x^o + \sum_{i<o} c_i y^{(o-i)(d-1)} x^i). \end{aligned}$$

We see that the change of the exponents of y inside the parentheses is given by replacing the factor d by $d-1$. This is nicely illustrated by the change of the Newton polygon, which, by definition, is the convex hull in $\mathbb{R}^2$ of the exponents of the monomials appearing in a polynomial (see Figure 13.)

It is now clear which number will reflect the improvement in this case. It is precisely the slope d of the segment between the points $(o, 0)$ and $(0, od)$ of the Newton polygon. Instead, we may as well take the projection of the Newton polygon from the point $(o, 0)$ to the y-axis given for $(i, j) \in \mathbb{N}^2$ with $j < o$ by $(i, j) \to (0, \frac{oj}{o-j})$. This projection of integral points of $\mathbb{N}^2$ can also be reflected by associating to f the ideal generated by all equilibrated powers of its coefficients $f_i(y)$, defined by

$$\mathrm{coeff}_x(f) = (f_i(y)^{\frac{o}{o-i}}).$$

There appears the inconvenience of having rational exponents, so we should better take

$$\mathrm{coeff}_x(f) = (f_i(y)^{\frac{o!}{o-i}}).$$

But we prefer to stick to the geometric picture of the projection in $\mathbb{N}^2$ and keep the first definition. We call this "ideal" the coefficient ideal of f with respect to V. To avoid confusion with rational exponents of ideals, we can define $\mathrm{ord}_a(f_i(y)^{\frac{o}{o-i}}) = \frac{o}{o-i} \cdot \mathrm{ord}_a f_i$ to make things well defined. This order may still depend on the choice of coordinates x and y, but the maximal value over all choices of coordinates is per definition intrinsic. It is thus a good measure of complexity subordinate to the order of f.

> The passage to the coefficient ideal explains why many articles on resolution of singularities start with singularities of type $f = x^o + g(y)$ without any monomials of form x^i with $i > 0$. As for the coefficient ideal, such polynomials yield the same amount of difficulty as arbitrary ones. So their study is representative for the general situation.

Conclusion. We have associated to f and given coordinates x, y an ideal $\mathrm{coeff}_x(f)$ in $V = \{x = 0\}$ whose order measures the improvement at equiconstant points of f in W'. This order is independent of any choices if V is chosen so that $\mathrm{ord}_a(\mathrm{coeff}_x(f))$ is maximal.

We shall say that such a hypersurface V has *weak maximal contact* with f. Obviously, $\mathrm{coeff}_x(f)$ makes sense also when y is a vector of coordinates, though it is not clear (and actually not true) that its order drops at equiconstant points if y has more than one component.

This approach to finding finer complexity invariants looks too simple to be of any value, and indeed it is, as we can already see in the case of surfaces.

Example 7. Consider $f = x^o + y^r z^s$ with $r, s > o$, and blow up the origin of $\mathbb{A}^3$. The secondary order of f at the origin is $\mathrm{ord}_0\, y^r z^s = r + s$. We look at the origin a' of the y-chart. The weak transform equals $f^\curlyvee = x^o + y^{r+s-o} z^s$; hence a' is an equiconstant point and the secondary order has increased from $r + s$ to $r + s + (s - o)$. So the singularity has gotten worse.

This is bad news. But it is easy for the attentive reader to protest, since the assumption $r, s > o$ implies that the top locus, $\mathrm{top}(f)$, consists of the y- and z-axis, which could also have been taken as the center (except, possibly, if $r = s$, in which case we wish to preserve the symmetry of the singularity). So you could claim that we have taken the wrong center. The protest is immediately rejected by the following, slightly more complicated, singularity.

Example 8. Consider $f = x^o + y^r z^s + y^{3(r+s)} + z^{3(r+s)}$ with $r, s > o$. Here the top locus is reduced to the origin, which has to be chosen as our center of blowup. At the origin of $\mathbb{A}^3$, the secondary order of f is again $r+s$. We look at the origin a' of the y-chart. The weak transform equals $f^\curlyvee = x^o + y^{r+s-o} z^s + y^{3(r+s)-o} + y^{3(r+s)-o} z^{3(r+s)}$; hence a' is an equiconstant point. The secondary order of f has again increased from $r + s$ to $r + s + (s - o)$.

This is really embarassing. The argument which worked so nicely for plane curves completely breaks down for surfaces: the secondary order may increase at equiconstant points. So instead of an improvement, the singularities seem to get worse.

As mentioned in the introduction, we will proceed in this article by entering without fear into dead-end streets, convinced that we will be able to struggle our way out by carefully investigating the congestion.

In this spirit, let us look more closely at the transformation law the coefficient ideal undergoes at an equiconstant point and why its order may increase. For simplicity we assume that J is generated by a polynomial $f = x^o + g(y)$ with $y = (y_1, \dots, y_m)$ and $\mathrm{ord}_a\, g \geq o$. We may assume that the center is the origin $a = 0$ and that a' is the origin of the y_1-chart. The coefficient ideal of f in $V = \{x = 0\}$ is generated by $g(y)$. We get total transform $f^* = x^o y_1^o + g^*(y)$, where g^* denotes the total transform of $g \in \mathcal{O}_{V,a}$ under the induced blowup $V' \to V$ of center $Z = \{a\}$, say $g^*(y) = g(y_1, y_2 y_1, \dots, y_m y_1)$. Let q be the order of g at 0, $q \geq o$. We now compute the weak transform $f^\curlyvee$ of f in W' and get

$$\begin{aligned} f^\curlyvee &= y_1^{-o} f^* = x^o + y_1^{-o} g^*(y) \\ &= x^o + y_1^{-o} y_1^q g^\curlyvee(y) \\ &= x^o + y_1^{q-o} g^\curlyvee(y) \end{aligned}$$

with $g^\curlyvee$ the weak transform of g under $V' \to V$. But the coefficient ideal of $f^\curlyvee$ in V' is $y_1^{-o} g^*(y) = y_1^{q-o} g^\curlyvee(y)$, which is different from $g^\curlyvee(y)$ whenever $q > o$. Hence $(g) = \mathrm{coeff}_V(f)$ has a law of transformation *different* from that of f, because its transform equals the weak transform *multiplied* by a power of the exceptional monomial y_1. This is a transform $(g)^! = (\mathrm{coeff}_V(f))^!$ in between the weak and the total transform which depends on o; it is defined by $(g)^! = y_1^{-o}(g)^*$ (hence depends on the order o of f at a) and is called the *controlled transform* of (g) with respect to the *control* o (the number o does not appear in the notation $(g)^!$, but whenever you see a ! in the exponent you should be reminded to watch out for the respective control).

As in general the order of the weak transform of an ideal may remain constant, the order of the coefficient ideal of $f^\curlyvee$ may really increase, as we have seen in example 8 above. In contrast, the order of the weak transform of the coefficient ideal $\mathrm{coeff}_V\, f$ of f won't increase at a', because the center was included in its top locus. Put together, the problem is that the coefficient ideal of the weak transform $f^\curlyvee$ of f need not coincide with the weak transform of the coefficient ideal of f. This is a failure of commutativity between our two basic operations: the passage to the coefficient ideal and the passage to the weak transform.

We get at equiconstant points the following commutative diagrams of local blowups:

$$\begin{array}{ccc} (W', a') & \rightsquigarrow & (V', a') \\ \downarrow & & \downarrow \\ (W, a) & \rightsquigarrow & (V, a) \end{array}$$

and ideals:

$$\begin{array}{ccc} J' = J^\curlyvee & \rightsquigarrow & C' = C^! \\ \downarrow & & \downarrow \\ J & \rightsquigarrow & C \end{array}$$

where $a' \in W'$ is an equiconstant point of J (i.e., $\mathrm{ord}_{a'}(J^\curlyvee) = \mathrm{ord}_a J$) under the blowup $W' \to W$ with center $Z \subset V$, the hypersurface V is adjacent to J (i.e., its strict transform $V' = V^{st}$ contains all equiconstant a' above a), C and C' denote the coefficient ideals of J and J' with respect to V and V' and $C^!$ is the controlled transform of C with respect to $o = \mathrm{ord}_a J$.

This illustrates quite explicitly that the transformation laws for J and its coefficient ideal are different when passing to an equiconstant point of J. So there is no hope of having the order of C decrease (or at least not increase).

The decrease in the curve case was just a lucky circumstance which is not representative for the more general situation of arbitrary dimension. The natural conclusion is that our choice of the coefficient ideal of an ideal as a secondary measure of the complexity of a singularity (after the order of the ideal) is not appropriate. We have to look for a better candidate. That's a good point at which to get stuck and to give up trying to prove resolution of singularities (or reading this paper).

Before doing so we should at least look back and see what we have done so far.

So let us contemplate again the difference between examples 7 and 8. In the latter, f equals $x^o + y^r z^s + y^{3(r+s)} + z^{3(r+s)}$ and the monomials $y^{3(r+s)}$ and $z^{3(r+s)}$ appearing in the expansion of f prohibited us choosing as center a coordinate axis. In contrast, in example 7, we could choose a line as center. We had $f = x^o + y^r z^s$ there so that its coefficient ideal in $V = \{x = 0\}$ is a monomial, namely $y^r z^s$. No matter how large its exponents r and s are, blowing up one of the coordinate axes will make them decrease (putting aside the symmetry problem for the moment). It is reasonable to expect that this works also in more variables — and it does: For any $f = x^o + y_1^{r_1} \cdots y_m^{r_m}$ there exists a choice of a coordinate plane inside $\mathrm{top}(f)$ so that blowing it up makes some exponent r_i decrease at a chosen equiconstant point of f. It suffices to take a minimal subset $i_1, \ldots, i_k$ of $\{1, \ldots, m\}$ so that $r_{i_1} + \ldots + r_{i_k} \geq o$ and to set $Z = \{x = y_{i_1} = \ldots y_{i_k} = 0\}$. It is clear that $Z \subset \mathrm{top}(f)$. We leave it as an exercise to show that some r_i drops under blowup (this is obligatory homework).

What have we learnt from this? If the coefficient ideal is a principal monomial ideal, we will have no (say, almost no) problems choosing a suitable center and measuring the improvement of f. On the other hand, if it is not a monomial, it seems impossible to lower the order of the coefficient ideal by blowup.

This suggests changing our strategy drastically:

Instead of trying to lower the order of the coefficient ideal of f, try to transform the coefficient ideal into a principal monomial ideal.

We have already seen that the coefficient ideal of the weak transform $f^\curlyvee$ of f factors into an exceptional monomial part and a remaining ideal, which is the weak transform of the coefficient ideal of f, a simple observation which will allow us to overcome the recent depression and to attack the problem of resolution with new impetus.

This change of strategy perfectly fits into the classical idea of principalization of ideals: to transform *ab initio* the ideal J into a principal monomial ideal, taking after each blowup the total transform (or, if you prefer, the controlled transform, but not the weak or strict transform). Let us formulate our new objective:

Monomialization of ideals. Given an ideal J in a regular ambient scheme W, construct a sequence of blowups $W' \to \ldots \to W$ in regular centers transversal to the exceptional loci so that the pullback of J in W' (i.e., the total transform) is a principal monomial ideal. In addition, we may require that the normal crossings divisor defined by the pullback of J is supported by the exceptional locus, i.e., consists of an exceptional component raised to a certain power.

This clarification and precision of what we are aiming at will be of crucial importance. It will allow a quite systematic approach to the before mentioned descent in dimension via coefficient ideals.

As a first consequence of our new orientation, we will have to adapt the notation. We shall write the ideal J as a product $J = M \cdot I$ where M is a principal monomial ideal supported by exceptional components and where I is the ideal whose order we wish to drop successively until $J = M \cdot 1$. At the beginning, M will be 1 and $J = I$. After each blowup, we will collect in M as much of the new exceptional component as possible. Hence, denoting by J', M' and I' the corresponding objects after blowup, we will have:

$$\begin{array}{ll} J' = M' \cdot I' \quad \text{with} & \\ J' = J^* \text{ or } J' = J^! J^* \cdot I(Y')^{-c} & \text{the total or controlled transform of } J, \\ I' = I^\curlyvee = I^* \cdot I(Y')^{-\operatorname{ord}_Z I} & \text{the weak transform of } I, \\ M' = M^* \cdot I(Y')^{\operatorname{ord}_Z I} & \text{for } J' = J^* \text{ the total transform}, \\ M' = M^* \cdot I(Y')^{\operatorname{ord}_Z I - c} & \text{for } J' = J^! \text{ the controlled transform.} \end{array}$$

Here, M' is defined precisely so that J' admits again a product decomposition $J' = M' \cdot I'$ analogous to $J = M \cdot I$.

The next considerations can be seen, in one version or other, as the core of most proofs on resolution of singularities in characteristic zero.

Let $V \subset W$ be a local hypersurface at a which is adjacent for I. Hence $V' \subset W'$ will contain all points a' where $\operatorname{ord}_{a'} I' = \operatorname{ord}_a I$. Observe that we will now assume that $Z \subset \operatorname{top}(I)$ instead of $Z \subset \operatorname{top}(J)$ so that $\operatorname{ord}_a I = \operatorname{ord}_Z I$ for all $a \in Z$. To observe the improvement of I at equiconstant points we shall consider the coefficient ideal $\operatorname{coeff}_V(I)$ of I in V (and not that of J). As it will play in V the same role as J in W, we shall call it J_- (the minus sign referring to the decreasing dimension of $V = W_-$ compared with W). Assume that $J_- = M_- \cdot I_-$ is a factorization with M_- a principal monomial ideal in V (again, before any blowup, M_- will be 1 and $J_- = I_-$). Let us now look at our commutative diagrams from above. The upper right hand corner admits two candidates as ideals to appear there: The coefficient ideal $(J')_-$ of I' in V' and the transform $(J_-)'$ of J_-. It is not clear in general which transform of J_- has to be taken, and if it is possible to choose it so that the equality $(J')_- = (J_-)'$ holds in V' (the controlled ideal will be a good candidate),

$$\begin{array}{ccc} (W', a') & \rightsquigarrow & (V', a') \\ \downarrow & & \downarrow \\ (W, a) & \rightsquigarrow & (V, a) \end{array}$$

As for the ideals we get

$$\begin{array}{cccc} J' = M' \cdot I' & \rightsquigarrow & (J')_- \leftrightsquigarrow & (J_-)' \\ \downarrow & & & \downarrow \\ J \;= M \cdot I & \rightsquigarrow & & J_- \end{array}$$

To establish the commutativity of the diagram with ideals, we shall choose, guided by the curve case, for the transform $(J_-)'$ of J_- the controlled transform $(J_-)^!$ of J_- with respect to $o = \operatorname{ord}_a I$, say $(J_-)' = (J_-)^! = (J_-)^* \cdot I(Y')^{-o}$. With this choice, the equality $(J')_- = (J_-)'$ holds, and the proof is the same as for curves:

We choose local coordinates $x, y_1, \dots, y_n$ in W at a so that $V = \{x = 0\}$, the ideal $I(Z)$ of Z is given as $(x, y_1, \dots, y_k)$ and a' is the origin of the y_1-chart (such a choice exists; cf. appendix C). Expand $f \in I$ into $f = \sum_{i<o} a_{f,i}(y)x^i$ modulo x^o locally at a with coefficients $a_{f,i} \in \mathcal{O}_{V,a}$. For arbitrary ideals (i.e., not necessarily principal) the coefficient ideal J_- of I in V is defined as

$$J_- = \mathrm{coeff}_V\, I = \sum_{i<o} (a_{f,i}(y), f \in J)^{\frac{o}{o-i}}.$$

Observe here that the ideals generated by the i-th coefficients are raised to a certain power, and not the coefficients themselves. This has technical reasons we are not going to explain any further. Using $a_{f^\curlyvee,i} = (a_{f,i})^* \cdot I(Y' \cap V')^{i-o}$ and $o' = \mathrm{ord}_{a'} I' = \mathrm{ord}_a I = o$, we compute as follows:

$$\begin{aligned}
&\mathrm{coeff}_{V'}(I') = \mathrm{coeff}_{V'}(I^\curlyvee) \\
&= \mathrm{coeff}_{V'}(\sum_{i<o'} a_{f^\curlyvee,i} \cdot x^i,\ f^\curlyvee \in I^\curlyvee) \\
&= \mathrm{coeff}_{V'}(\sum_{i<o'} a_{I(Y'\cap V')^{-o}\cdot f^*,i} \cdot x^i,\ f^\curlyvee \in I^\curlyvee) \\
&= \mathrm{coeff}_{V'}(\sum_{i<o} (a_{f,i} \cdot x^i)^* \cdot I(Y' \cap V')^{-o},\ f \in I) \\
&= \sum_{i<o} (a_{f,i}^*,\ f \in I)^{o/(o-i)} \cdot I(Y' \cap V')^{-o} \\
&= (\sum_{i<o} (a_{f,i},\ f \in I)^{o/(o-i)})^* \cdot I(Y' \cap V')^{-o} \\
&= I(Y' \cap V')^{-o} \cdot (\mathrm{coeff}_V\, I)^* = (\mathrm{coeff}_V\, I)^!.
\end{aligned}$$

This proves the desired equality.

Let us write down explicitly the main ingredient of the local descent in dimension.

Commutativity for coefficient ideals. Given an ideal $J = M \cdot I$ in W, we may associate to the local blowup $(W', a') \to (W, a)$ of W in $Z \subset \mathrm{top}(I)$ an adjacent hypersurface $V = W_-$ of W at a and an ideal $J_- = M_- \cdot I_-$ in V, the coefficient ideal of I in V, so that if a' is an equiconstant point for I and hence $a' \in V'$, the total transform $J' = M' \cdot I'$ admits a coefficient ideal $(J')_- = (M')_- \cdot (I')_-$ in $V' = (W')_- = (W_-)'$ at a' which is the controlled transform $(J_-)'$ of J_- with respect to the control $\mathrm{ord}_a I$.

As the descent in dimension commutes with the local blowup at equiconstant points, we may write

$$(J')_- = (J_-)' = J_-' = M_-' \cdot I_-'.$$

This is a significamt advance in reaching our objective. The statement of commutativity implies that the monomialization problem for $J = M \cdot I$ can be transferred to lower dimension and expressed there by the analogous problem for $J_- = M_- \cdot I_-$. In particular, both the search for a suitable center of blowup Z in W and the measure of improvement of the singularities in W' at a' can be expressed as problems in lower dimension. In V we may apply *induction on the local embedding dimension* to solve both problems:

For the center Z in W we can choose the center Z_- in V associated to $J_- = M_- \cdot I_-$, yielding a commutative diagram of local blowups:

$$\begin{array}{ccccccc} Y'_- & \subset & (V', a') & \subset & (W', a') & \supset & Y' \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ Z_- & \subset & (V, a) & \subset & (W, a) & \supset & Z \end{array}$$

For the resolution invariant $i_a(J)$ of J we can take the vector

$$i_a(J) = (\mathrm{ord}_a(I), i_a(J_-)),$$

where $i_a(J_-)$ is the invariant associated to J_- in $V = W_-$; it exists by induction on the dimension. This vector is considered with respect to the lexicographic ordering. If its first component $\mathrm{ord}_a(I)$ drops under blowup, we have $i_{a'}(J') < i_a(J)$ and we are done. If it remains constant, we look at the remaining components of $i_a(J)$. As we are in this case at an equiconstant point a' of I, commutativity holds, say $(J')_- = (J_-)'$. Hence

$$i_{a'}(J') = (\mathrm{ord}_{a'}(I'), i_{a'}((J')_-)) = (\mathrm{ord}_a(I), i_{a'}((J_-)')).$$

By induction on the dimension we know that $i_{a'}((J_-)') < i_a(J_-)$ holds lexicographically. Hence $i_{a'}(J') < i_a(J)$. In both cases, the resolution invariant has dropped.

Unfortunately, this elegant reasoning does not go through as smoothly as we might have dared to hope. Before seeing this, the reader may wish to recapitulate the large amount of information packed into the last section.

4. Obstructions. The preceding constructions and the ensuing descent in dimension produce a natural and efficient setting for proving resolution of singularities. There are, however, several obstructions which prevent us from applying the method directly without further modifications.

(1) It has to be verified that the induction basis where W has dimension 1 is valid.

(2) The controlled transform $(J_-)^!$ of J_- in W' at a' is defined with respect to the order of J at the image a of a' in W. It hence depends on the stratum of $\mathrm{ord}\ J$ in which a lies. In this way, $(J_-)^!$ is only coherent in W' along the inverse image of each stratum where $\mathrm{ord}\ J$ is constant, but not on the whole W'.

(3) The coefficient ideal depends on the choice of the local hypersurface V, and different choices of V may yield different invariants and a center which is only locally defined. We have to specify a class of hypersurfaces V so that the invariant and the center do not depend on the special choice of V within this class and so that the local definitions coincide on overlaps of charts.

(4) The center Z_- associated to $J_- = M_- \cdot I_-$ will be contained in $\mathrm{top}(I_-)$ but maybe not in $\mathrm{top}(I)$. As we plan to set $Z = Z_-$, the inclusion of Z_- in $\mathrm{top}(I)$ will become mandatory so that the order of the weak transform $I' = I^\curlyvee$ of I does not increase. Therefore we have to adjust the construction of J_- so that $Z_- \subset \mathrm{top}(I)$. Actually, we could also choose Z differently from Z_-, but this would yield new complications.

(5) The center Z_- may not be transversal to already existing exceptional components in W. This would destroy the monomiality of the exceptional factors M and M_- of J and J_- and produce an exceptional locus which is not a normal crossings divisor.

(6) In some cases, namely when $I = (x^o)$, the coefficient ideal of I is zero and the descent in dimension breaks down. Some substitute for coefficient ideals has to be found in this case.

(7) When J_- is already resolved and thus a principal monomial ideal, viz $J_- = M_- \cdot 1$, the inequality $i_{a'}((J_-)') < i_a(J_-)$ used above will no longer hold, since the invariant of a resolved ideal should be the minimal possible value which cannot drop again. Some other argument has to be found to show that $i_a(J)$ drops. In particular, a suitable center for J has to be defined directly.

(8) It has to be shown that the invariant is upper semicontinuous and that its top locus is regular.

(9) The hypersurface V of W is chosen adjacent to I at a. Its transform V' in W' need no longer be adjacent to I' at a'. Thus commutativity as stated in the last section may fail for the next blowup if V' is not replaced by a hypersurface which is adjacent for I'.

In the next sections we shall show how to overcome all these difficulties. Only the last problem (9) will require us to assume that the characteristic of the ground field is zero. The other problems are solved without reference to the ground field.

Let us first comment on the above list. Problems (1) and (2) are easy: In dimension 1, the ideal J is locally at a just the power of a coordinate (x^o) supported on the point. Blowing it up yields $J' = (x^o) \cdot 1$ with $M' = (x^o)$ and $I' = 1$, so that J' is resolved. As I was not 1 (else, already J would have been resolved), its order $\mathrm{ord}_a\, I$ drops from $o > 0$ to 0.

Problem (2) is handled by allowing stratified ideals. Equipping W with a suitable stratification (given by the constancy of the order of the relevant ideals), we require only that the stalks of our ideals are the stalks of a coherent ideal along the respective strata.

The other difficulties are much more serious. (3) is overcome by allowing only hypersurfaces V which have weak maximal contact with I, i.e., which maximize the order of $\mathrm{coeff}_V(I)$. Then the order of $\mathrm{coeff}_V(I)$ is by definition independent of the choice of V. It has to be shown that when a varies, this order defines an upper semicontinuous function along $\mathrm{top}(I)$. This is done by showing that locally along $\mathrm{top}(I)$, the same V can be chosen for all stalks of I.

As for (4), note that when passing to the coefficient ideal of I at a, we have the local inclusion $\mathrm{top}(\mathrm{coeff}_V(I), o) \subset \mathrm{top}(I)$ where $o = \mathrm{ord}_a\, I$ and $\mathrm{top}(\mathrm{coeff}_V(I), o)$ denotes the locus of points in V where $J_- = \mathrm{coeff}_V(I)$ has order $\geq o$. This is immediately verified from the definition of coefficient ideals and is left as an exercise. If J_- factorizes into $J_- = M_- \cdot I_-$, then $\mathrm{top}(I_-)$ need not be contained in $\mathrm{top}(J_-, o)$ if the order of I_- is small (in particular, this happens if $I_- = 1$). But the inclusion $\mathrm{top}(I_-) \subset \mathrm{top}(J_-, o)$ is the only way to get the inclusion $\mathrm{top}(I_-) \subset \mathrm{top}(I)$, which in turn is necessary to know that the center Z_- associated to $J_- = M_- \cdot I_-$ satisfies not only $Z_- \subset \mathrm{top}(I_-)$ but also $Z_- \subset \mathrm{top}(I)$. Recall that these inclusions are used when showing that the order of the weak transforms of I and I_- do not increase.

This trouble is overcome by replacing I by a modified ideal P before passing to the coefficient ideal J_-. This *companion ideal* P of I equals I if the order of I is sufficiently large and is the sum of I with a convenient power of M if this order is small (the analogous construction will be applied for I_-; see "Setups" and the appendix). The companion ideal is modelled so that $\mathrm{top}(P_-) \subset \mathrm{top}(J_-, o)$ if $J_- = M_- \cdot I_-$. Moreover, the commutativity relation shall be preserved: At equiconstant points of I the weak transform of P equals the companion ideal of the weak transform $I^\curlyvee$ of I. The passage to the companion ideal before taking the coefficient ideal is a technical complication which has no deeper reason other than guaranteeing the inclusion $\mathrm{top}(I_-) \subset \mathrm{top}(I)$ without losing commutativity.

Let us now discuss problem (5), the transversality of Z with the exceptional locus. This is a delicate point which has caused many troubles in the past. The idea for how to attack it nowadays appears for the first time in [Vi 1]. Let F denote the exceptional locus at the current stage of the resolution process. Let Z be the subscheme Z_- of W associated to J_- neglecting in its construction the transversality problem. If $Z = Z_-$ is not transversal to F, we could formulate a new resolution problem by considering the ideal $I_V(Z_- \cap F)$ defining the intersection of Z_- with F in V. Resolving it would yield a total transform which is a principal monomial ideal (supported by the new exceptional components which arise during its resolution process); hence the weak transform would be equal to 1. This signifies that the corresponding blowups separate Z_- from F, i.e., achieve $Z_- \cap F = \emptyset$.

We see here that the separation of schemes can be easier formulated as a resolution problem (the order of the ideal defining the intersection should become 0) than the transversality of schemes (there is no intrinsic invariant known which is able to measure in a reasonable way the distance of a scheme from being a normal crossings scheme). Therefore we prefer to separate Z_- from F in order to solve the transversality problem.

There is one notational inconvenience with this. While separating Z_- from F, new exceptional components will appear, and these should not be separated from Z_-, since separating them will create new components meeting Z_-, and so on. Therefore it is necessary to put the new exceptional exponents in a bag and to distinguish them from the old ones, which constitute F. The new components will a priori be transversal to Z_- and its transforms, since the centers chosen to separate Z_- from F lie inside Z_-.

Again we encounter a technical complication: We must record the exceptional components which may fail to be transversal to the foreseen center. Moreover, we should solve permanently the transversality problem for these dangerous components. Once separation is established, the anticipated virtual center can really be chosen as actual center, and blowing it up should improve the original singularities.

Instead of treating the resolution of J and the separation of the virtual center from the exceptional locus alternately, both problems are taken care of simultaneously. The trick is to multiply the companion ideal P of I by a suitable transversality ideal Q before passing to the coefficient ideal J_-. So we set $K = P \cdot Q$ where $Q = I_W(E)$ is the ideal defining the dangerous exceptional components of F in W. This ideal K is called the *"composition ideal"*. In this way, the virtual center will automatically be contained in $\mathrm{top}(K) = \mathrm{top}(P) \cap \mathrm{top}(Q)$, hence in $\mathrm{top}(Q)$ (here, the equality $\mathrm{top}(K) = \mathrm{top}(P) \cap \mathrm{top}(Q)$ is understood for the local top loci at points a of W). It thus lies by construction inside all dangerous exceptional components of F (which will then no longer be dangerous).

We thus get the following sequence of ideals:

$$\begin{aligned} J &\rightsquigarrow J = M \cdot I \rightsquigarrow P \rightsquigarrow K = P \cdot Q \rightsquigarrow J_- \\ &\qquad = \mathrm{coeff}_V(K) \rightsquigarrow J_- = M_- \cdot I_- \rightsquigarrow P_- \rightsquigarrow K_- = P_- \cdot Q_- \rightsquigarrow \dots \end{aligned}$$

We have to ensure here for all ideals the correct commutativity relations with respect to blowup, the appropriate inclusions of the various top loci and the transversality of the resulting center with the exceptional locus. The burdening of the notation caused by the introduction of the ideals P, Q and K is considerable. There is no apparent way to avoid these. In addition, we have to know in each dimension the multiplicities (= exponents) of the monomial factors M, M_-, ... and the collection of dangerous exceptional components E, E_-, ... of F. A crucial point is to distinguish carefully between objects which are globally defined and intrinsic, i.e., do not depend on the choice of the local hypersurfaces, and those which depend on these choices, are only defined locally, and just play an auxiliary role.

The global and intrinsic objects are collected in the resolution datum, which will be called *"singular mobile"*. A mobile consists of the ideal J; a number c, the control, which prescribes the law of transformation for J; and two sets D and E of exceptional components. Both are strings $(D_n, \dots, D_1)$ and $(E_n, \dots, E_1)$ of normal crossings divisors supported by the exceptional locus, where the index i refers to the embedding dimension where the divisor will be used. The first divisors D_i define in each dimension i the monomial factor M_i of the ideal J_i (here, $J_n = J$ and $J_{n-1} = J_-$). We have to know this factor in order to be able to split from J_i the singular factor I_i we are interested in. Only a posteriori will I_i appear as the weak transform of some ideal from the previous stage of the resolution process. The second divisors E_i collect precisely the dangerous exceptional components with regard to the virtual center. Again, these appear in each dimension.

The local and non-intrinsic objects associated to a singular mobile are collected in its *"punctual setup"*. A setup is given by the choice of local flags $W_n \supset W_{n-1} \supset \cdots \supset W_1$ of regular i-dimensional subschemes W_i of W at a given point a (here, $W_n = W$ and $W_{n-1} = W_- = V$). These local hypersurfaces W_{i-1} of W_i are chosen adjacent to the respective ideals I_i (more precisely, to their companion ideals P_i). They allow us to define and construct all the ideals mentioned above. These depend on the choice of the flag, but their orders, which will constitute the components of our local resolution invariant, will be shown to be independent of this choice. In this way the invariant is again intrinsic and an honest measure of the singular complexity of the mobile.

Let us return to our list of obstructions. The next one is number (6) and relates to the problem that the coefficient ideal of an ideal of form $K = (x^o)$ is zero. This problem is easily settled. If $K = (x^o)$, then its support is regular and defined by $\{x = 0\}$. It may therefore be taken as the center (provided it is transversal to the exceptional locus, as we shall assume in this discussion temporarily). Blowing it up transforms K into 1 (since the composition ideal K passes as I under blowup to its weak transform). Thus the ideal can easily be resolved. For notational reasons it is however convenient to pursue the descent in dimension also in this case, so as to have always the same length of the setup and the invariant. As the zero ideal would produce infinite orders and unpleasant terminology, we prefer to define J_- in this case as the ideal 1, with trivial factorization $J_- = M_- \cdot I_-$. In particular, the companion ideal P_- will also be 1. As for the transversality ideal Q_-, it can be shown that once K equals (x^o) the transversality problem has already disappeared in the respective dimension, so that Q_- can also be taken equal to 1 (say $E_- = \emptyset$). Hence $K_- = 1$. Now the descent to the next smaller dimension continues in precisely the same way.

A similar reasoning applies to problem (7), which concerns the case where $J = M \cdot 1$ is already resolved. All subsequent ideals in lower dimension will be set equal to 1. However, one has to show that a center can be chosen which decreases the order of the controlled transform of J (in order to lower eventually the order of the ideals in one dimension higher). The choice of the center in this case (again a priori transversal to the exceptional locus) is of a combinatorial nature and was sketched in the hypersurface case $x^o + y_1^{r_1} \cdots y_k^{r_k}$ above. The procedure works in general. However, the improvement is captured through the transformation of M instead of I and requires us to insert a combinatorial component in between the components of the resolution invariant in order to reflect the improvement of an already resolved ideal $J_i = M_i \cdot 1$ in some dimension $i < \dim W$.

We come to problem (8), the upper semicontinuity of the invariant and the regularity of the center. The first has already been discussed above and relies on the fact that the constructions of punctual setups can be done locally in an entire neighborhood of a point. Here it is important to know that the construction of the various ideals commutes with specialization at points. For this, the notion of *"tunedness"* is useful: Stalks of ideals associated to other stalks of ideals are tuned if both admit coherent representatives such that the correspondence is valid for the stalks of these representatives at all points of suitably small neighborhoods. A minor technical complication due to the consideration of stratified ideals consists in defining tunedness with respect to the strata of an underlying stratification of the ambient scheme. We refer to [EH] for more details.

The last problem (9) is the persistency of adjacent hypersurfaces under blowup. Historically, this was the main obstacle to overcome in the fifties, when Hironaka and Abhyankar studied resolution of singularities under the guidance of Zariski. Nowadays its solution is known by the name of hypersurfaces of maximal contact (in the terminology of Hironaka) or of Tschirnhaus coordinate transformation (in the terminology of Abhyankar). Both only work in characteristic zero (or a characteristic prime to the order of the ideal considered). In positive characteristic, the failure of maximal contact could not yet be replaced by some other concept.

Recall that we call a local regular hypersurface V in W at a adjacent to an ideal I if under blowing up a regular center Z inside V, the locus of equiconstant points a' in W' above a is contained in the weak transform V' of V. This notion refers to one blowup. In order to make the induction on the local embedding dimension work along several blowups, we have to ensure that a hypersurface V can be chosen in W at a whose successive transforms contain all equiconstant points above a for any sequence of blowups of W in regular centers contained in V and its transforms. This property of V will be called *permanent maximal contact* with I at a. In the literature, such hypersurfaces are simply called hypersurfaces of maximal contact. As we will have to treat several concepts of contact simultaneously, it is better to emphasize in the naming the key property (in this case permanence). Permanent maximal contact guarantees that after each blowup, if the order of the ideal has remained constant, the transforms of the *same* hypersurface of W can be chosen to perform the descent in dimension. As a consequence, commutativity holds along the whole sequence of local blowups until the order of I eventually drops (and not just for one blowup).

> This is a crucial and delicate point in the whole story, so the reader may want to contemplate this last paragraph.

Two things have to be distinguished here. The existence of hypersurfaces of permanent contact is a property of the series of equiconstant points under a sequence of blowups. It may hold or fail, depending on how the world decides to be. Once the abstract existence is confirmed it remains to realize such hypersurfaces through a special choice or construction. This is human labour. In characteristic zero, a miraculously simple trick produces an explicit selection of hypersurfaces of permanent maximal contact (not for all of them). This construction cannot be carried out in positive characteristic, and one may ask for alternative constructions.

But, hélas, there are none: It can be shown by examples that in positive characteristic the series of equiconstant points above a given point a may leave at some instant the transforms of any regular (or even singular) hypersurface through a. The characteristic p world decided not to admit hypersurfaces of permanent contact. In fact, the departure of equiconstant points from accompanying hypersurfaces happens only after several blowups (see the section "Problems in positive characteristic"). So characteristic p is not harder because our techniques are not appropriate or too limited; it is per se more difficult.

Let us now explain how one can find hypersurfaces of permanent maximal contact in characteristic zero. Suppose that given an ideal I in W at a and assuming for simplicity that I is principal, $I=(f)$ for some polynomial f. Choose local coordinates $x, y_1, \ldots, y_n$ so that $V=\{x=0\}$ is adjacent to f. Let $o=\operatorname{ord}_a f$. As x appears in the tangent cone of f by adjacency, a generic linear coordinate change (or simply replacing y_i by $y_i+c_i x$ with generic constants c_i) will allow us to assume that x^o appears in the expansion of f with non-zero coefficient, say equal to 1. We may write $f=x^o+\sum_{i<o} f_i(y)x^i$ modulo x^{o+1} with coefficients $f_i \in \mathcal{O}_{V,a}$.

As seen earlier, the coordinates may in addition be chosen so that $Z=\{x=y_1=\ldots=y_k=0\}$ and that the equiconstant point a' is the origin of the y_1-chart. Thus the substitution of variables for the blowup is given by $x \to xy_1$, $y_1 \to y_1$ and $y_i \to y_i y_1$ for $2 \le i \le k$, respectively $y_i \to y_i y_1$ for $k<i$. It yields at a' the weak transform

$$f' = x^o + y_1^{-o} \sum_{i<o} f_i(y_1, y_2y_1, \ldots, y_ky_1, y_{k+1}, \ldots, y_n) y_1^i x^i$$

modulo x^{o+1}. We wish to find criteria on $V=\{x=0\}$ so that its transform $V'=\{x=0\}$ at a' is again adjacent to f'. This signifies that we have to ensure that x appears in the tangent cone of f' provided that the order of f' has remained constant at a' (i.e., equal to the order o of f at a). Now the key idea — and this is not obvious (but has been explained many times in the literature) — is to look at the coefficient $f_{o-1}(y)$ of the monomial x^{o-1} of f. If it is identically zero, also the coefficient $f'_{o-1}(y)$ of the monomial x^{o-1} of f' will be identically zero, by the special choice of our coordinates and the induced coordinate substitution. This is immediate. So this property, though coordinate dependent, is persistent under blowup at equiconstant points. In addition, $f_{o-1}(y)=0$ implies that $V=\{x=0\}$ is adjacent to f in W at a. Accordingly, also $V'=\{x=0\}$ is adjacent to f' in W' at a'. We have found a stronger condition which automatically persists under blowup.

It remains to show that the coordinates $x, y_1, \ldots, y_n$ can always be chosen so that $f_{o-1}(y)$ is identically zero. Here, characteristic zero appears on the scene, and the proof is then very short. Assume that $f_{o-1}(y)$ is non-zero. Replace x by $x-\frac{1}{o}f_{o-1}(y)$ in f (this is not possible in characteristic p dividing o). It is immediately checked that, in the new coordinates, the coefficient of x^{o-1} has become zero. This change of coordinates may a priori alter the generators of $I(Z)$. But as f has order o along Z, the polynomial $f_{o-1}(y)$ belongs to $I(Z)$, so that replacing x by $x - f_{o-1}(y)$ stabilizes $I(Z)$.

We say that the hypersurface $V = \{x = 0\}$ is *osculating* for $I = (f)$ at a if the coefficient of x^{o-1} in the expansion of f is identically zero. For ideals I which are not principal, it is required that there is at least one element f in I of order $o = \mathrm{ord}_a I$ with this property.

We have seen before that osculating to I implies adjacent to I. As we have just shown that osculating persists at equiconstant points under blowup, we conclude that osculating also implies permanent maximal contact, and that's what we were looking for.

This is one of the rare instances in mathematical research where reality produces a truly favorable coincidence. The existence of hypersurfaces of permanent maximal contact in characteristic zero is a *coup de chance*, which, presumably, has no deeper reason than a simple computation on Taylor expansions and blowups. Once you have made the right guess by looking at the x^{o-1}-coefficient, the proofs are really an exercise, as we saw above. All this breaks down in positive characteristic, and no substitute for permanent maximal contact has been found up to now.

This concludes our discussion of the various obstructions which have to be handled before setting up the strategy for the proof of resolution of singularities in characteristic zero. In the next sections, we shall describe the respective solutions to the problems in more technical detail. After this we will compute a few explicit examples from scratch.

The busy or moderately interested reader is advised to conclude here the lecture of the article.

CHAPTER 2: CONSTRUCTIONS AND PROOFS

5. Mobiles. With this section we start to formalize the ideas and concepts described heuristically up to now. This will require us to read the next paragraphs more carefully and to carry out occasionally private computations. At first sight, the concept and use of mobiles is not easy to grasp. A concrete example running with the definitions will help to capture their flavour. Nevertheless, mobiles can only be understood together with their setups, which will be explained in the section thereafter. The reader is advised to go through this and the next section concurrently with the section “Examples”.

The resolution of singular schemes and the monomialization of ideals will be deduced from the resolution of a more complicated object called singular mobile. A mobile consists of data which allow us to describe at each stage of the resolution process the portion of a given ideal and of its successive coefficient ideals which has already been monomialized and which keeps track of the dangerous exceptional components which may fail to be transversal to the local flags. Such objects appear in various disguises in the literature, e.g. by the name of resolution datum or idealistic exponent (Hironaka), trio, quartet and quintet (Abhyankar), basic object (Encinas-Villamayor) and infinitesimal presentation (Bierstone-Milman). They are all differently defined—sometimes local, sometimes global, intrinsic or not intrinsic—and do not necessarily gather the key information, so they have either to be completed by additional information (viz the knowledge of the prior sequence of blowups) or to be considered modulo equivalence relations.

The advantage of mobiles in comparison to these other resolution objects is that they collect precisely the intrinsic global information which we wish to know at each stage of the resolution process and that they allow a local surgery (through the construction of setups) which produces the local resolution invariant defining the center and yielding the required induction. As such, we can define the transform of a mobile under blowup without needing any additional ingredient, which in turn allows us to speak of the resolution of a mobile. So let us define them.

A *singular mobile* in a regular ambient scheme W is a quadruple $\mathcal{M} = (\mathcal{J}, c, D, E)$ where $\mathcal{J}$ is a coherent ideal sheaf in a regular locally closed subscheme V of W, c is a positive number, the *control*, and $D = (D_n, \dots, D_1)$ and $E = (E_n, \dots, E_1)$ are strings of (stratified) normal crossings divisors D_i and E_i in W (here, the index n denotes the dimension of V).

In most cases, $\mathcal{J}$ is the ideal we wish to resolve or to monomialize and is an ideal of W, so that $V = W$ in this case. Ideals living in locally closed regular subschemes of W appear during the descent in dimension; thus in the definition, the ambient scheme of the ideal $\mathcal{J}$ is V instead of W. The control c prescribes the transformation rule for J: Under blowup, J will pass to the controlled transform $J^! = J^* \cdot I(Y')^{-c}$ of J with respect to c. For $\mathcal{J}$ itself one could take $c = 0$ so that $\mathcal{J}$ passes to the total transform, but when descending in dimension the coefficient ideals will have to pass under blowup — as we have seen earlier — to the controlled transform in order to ensure commutativity. Therefore it is convenient to consider from the beginning controlled transforms. In addition, the control fixes the objective of the resolution process: When the order of J drops below c, we declare our goal to be achieved.

> The control c should actually carry an index $+$, say $c = c_+$, to indicate that it comes from a dimension one higher than the dimension of V. When constructing the setups of a mobile, the ideals J_i will have controls c_{i+1} given as the orders of the ideals K_{i+1}.
>
> As a general policy, we will try to treat each dimension in precisely the same fashion. This makes things more systematic and helps to produce automatized proofs. In particular, the restriction of a mobile to some regular smaller dimensional subscheme should again be a mobile.

The divisors D_i and E_i of a mobile will be supported by the exceptional components produced so far by the resolution process. So, for instance, at the beginning of the resolution process, all D_i and E_i are empty. Under blowup they obey a precise law of transformation, prescribed by the resolution invariant. The divisors D_i will carry multiplicities; their restriction to the member W_i of the local flag of a setup indicates the monomial factor M_i which will be taken off the ideal J_i of a punctual setup of the mobile, say $J_i = M_i \cdot I_i$ with $M_i = I_{W_i}(D_i \cap W_i)$ a principal monomial ideal. (See the next section for the precise definition of a punctual setup of a mobile and of local flags; here and in the sequel, $I_V(X)$ denotes the ideal defining a subscheme X of V.) The string $D = (D_n, \dots, D_1)$ is called the *combinatorial handicap* of $\mathcal{M}$.

Observe that the D_i are divisors in W, whereas the monomials M_i live in W_i (as said earlier, the index i of D_i refers to the dimension where D_i will operate).

The reason is that, as such, the D_i can be defined intrinsically, i.e., independently of the chosen flag, whereas the monomials M_i will depend on W_i. In order to know that M_i is really a principal monomial ideal it has to be shown that W_i meets D_i transversally. This is one of the transversality assertions which has to be established in the course of the proof.

In practice, the D_i carry some small additional combinatorial information, their label (cf. [EH] for the precise definition). The label of D_i is a pair of natural numbers. It serves only in case the ideal $J_i = M_i \cdot I_i$ is resolved, say $J_i = M_i$ and $I_i = 1$, in which case the various components of D_i have to be ordered in some way to enable one to choose systematically one of their intersections as center. The label induces such an ordering.

The divisors $E = (E_n, \ldots, E_1)$ are reduced and will be called the *transversal handicap* of the mobile. Once a local flag $W_n \supset \ldots \supset W_1$ is chosen (there are certain rules for doing this), the divisor E_i collects those exceptional components which may fail to be transversal to W_{i-1}.

The shift by 1 in the indices is due to the fact that the transversality ideal $Q_i = I_{W_i}(E_i \cap W_i)$ associated to E_i lives in dimension i but operates in dimension $i-1$. As W_i will be transversal to E_i, Q_i is a principal monomial ideal in W_i. By the choice of E_i, its top locus $\mathrm{top}(Q_i)$ lies inside the components of the exceptional locus F to which the next hypersurface W_{i-1} may not be transversal. This is a clever device to achieve the transversality of the center Z (which will lie inside $\mathrm{top}(Q_i)$) with F without introducing a singular ideal of possibly large order (as $I_{W_{i-1}}(E_i \cap W_{i-1})$ would be).

A thorough investigation of the transversality problem shows that it is more practicable to treat in each dimension the transversality problem of W_i with the exceptional locus rather than to formulate only one transversality problem for the center Z at the end of the descent in dimension. Thus transversality ideals appear in each dimension.

To summarize, the ingredients of a mobile $\mathcal{M} = (\mathcal{J}, c, D, E)$ prescribe for the ideal $\mathcal{J}$ a transformation law under blowup, given by the control c; a factorization law for the successive coefficient ideals J_i of a setup of $\mathcal{M}$ with monomial factor M_i, given by the combinatorial handicaps D_i; and a partition of the exceptional components of F for transversality records, given by the transversal handicaps E_i.

Let us look at a concrete example: Consider the ideal J generated by the polynomial $f = x^5 + y^7 + z^d$ with $d \geq 7$. The ambient scheme is $V = W = \mathbb{A}^3$. Assume we wish to resolve the scheme X defined by J in W. It has only one singularity, at the origin 0 of $\mathbb{A}^3$, and there the order of J is 5. By induction, it is certainly sufficient to make the order of f drop below 5. So we will set the control c of our mobile equal to 5. In W there are no exceptional components yet; hence the combinatorial and transversal handicaps are trivial, $D_3 = D_2 = D_1 = \emptyset$ and $E_3 = E_2 = E_1 = \emptyset$. The mobile $\mathcal{M}$ we associate to J is

$$\mathcal{M} = (J, 5, (\emptyset, \emptyset, \emptyset), (\emptyset, \emptyset, \emptyset)).$$

The next steps will be to define the transform of a mobile under blowup and to formulate what we mean by a resolution of a mobile. The transform of $\mathcal{M}$ is only defined for a specific blowup of the ambient scheme W, the one given by the center Z constructed from the mobile as the top locus of the local invariant associated to it. To prove that mobiles admit a resolution we will have to show that the transform $\mathcal{M}'$ of the mobile $\mathcal{M}$ under the blowup of W with this center has improved, which will be reflected by the decrease of the local invariant when passing from $\mathcal{M}$ to $\mathcal{M}'$.

Everything relies on constructing an appropriate local upper semicontinuous invariant of mobiles which stratifies W and measures the local complexity of $\mathcal{M}$. It should not depend on any choices but just on the mobile. This invariant has to satisfy two conditions. The locus of points where it attains its maximal value on W, its top locus, shall define a permitted center Z of blowup. And when blowing up this center, the invariant of the transformed mobile shall decrease at all points of the new exceptional component Y'.

This conceptually simple program requires us to develop an adequate definition of the local invariant — which is a truly subtle task. It is much harder to find a suitable invariant than to prove afterwards that it actually works. It's the same as with differential equations: Once you guess a solution correctly, it is almost trivial to verify that it is indeed a solution. For the invariant the verification is not as easy, but rather because of the technical complexity than for other reasons. Most proofs are straightforward. However, it will be instrumental to set up all objects and the relations between them very systematically (although this may create some objects which only play the role of a stowaway). Otherwise the proofer would be quickly lost in the thicket.

The resolution invariant we shall associate to a mobile is a local numerical object for each point a of W. It should live in a well ordered set so as to allow induction. In our case it will be a vector of non-negative numbers in $\mathbb{N}^{4n}$ (equipped with the lexicographic ordering), each component being the order of an ideal (with the exception of the components m_i, which are pairs of numbers always equal to $(0,0)$, except once, when o_i is for the first time zero, in which case m_i is a label),

$$i_a(\mathcal{M}) = (o_n, k_n, m_n, o_{n-1}, k_{n-1}, m_{n-1}, \dots, o_1, k_1, m_1).$$

The definition of $i_a(\mathcal{M})$ requires the concept of a punctual setup of a mobile at a point (described in the next section). For the moment it suffices to know that each mobile we shall meet admits punctual setups and that a setup at a is given by a flag $W_n \supset \ldots \supset W_1$ of locally closed regular i-dimensional subschemes W_i of W at a. To such a flag we shall associate strings of ideals $(J_n, \dots, J_1)$, $(I_n, \dots, I_1)$ and $(K_n, \dots, K_1)$, the indices referring to the scheme W_i where the respective ideals live. They are related to each other as follows: J_n is the stalk of $\mathcal{J}$ at a. Each J_i factors into $J_i = M_i \cdot I_i$ with $M_i = I_{W_i}(D_i \cap W_i)$ the exceptional monomial prescribed by the combinatorial handicap D_i. The ideals K_i are products $P_i \cdot Q_i$ where P_i is the companion ideal of J_i (if you have forgotten the definition, you may as well think of P_i as being I_i), and where $Q_i = I_{W_i}(E_i \cap W_i)$ is the transversality ideal prescribed by the transversal handicap E_i. Finally, J_{i-1} is the coefficient ideal of K_i in W_{i-1}.

We accept and perfectly understand that the reader may consider this bulk of ideals hopelessly confusing. To comfort him, let us see to what they boil down at the beginning of the resolution process where all D_i and E_i are empty. Then the ideals $J_i = I_i = K_i$ are equal for each i and $J_{i-1} = \mathrm{coeff}_{W_{i-1}}(J_i)$ since $P_i = I_i$

and $Q_i = 1$. So there is just one string of ideals $(J_n, \ldots, J_1)$ appearing in the flag $W_n \supset \ldots \supset W_1$ at a, and J_{i-1} is the coefficient ideal of J_i in W_{i-1}. In the general case, we shall denote a setup also by $(J_n, \ldots, J_1)$, though the flag and the other ideals are part of the structure. It can be checked that the flag determines all the remaining ingredients of a setup. Not all mobiles admit punctual setups (because the factorization $J_i = M_i \cdot I_i$ need not hold if D_i is not appropriate), but those arising under blowup as the transforms of a mobile with initially empty combinatorial handicap will do. These are the only ones we shall consider.

> We illustrate these objects in our current example. At $a = 0$, the punctual setup and the local flag $W_3 \supset W_2 \supset W_1$ for $\mathcal{M}$ will be defined as follows (see the next section for the justification): We set $W_3 = W = \mathbb{A}^3$ and $J_3 = J = (x^5 + y^7 + z^d)$. There are no factorizations yet; hence $J_3 = I_3 = M_3 \cdot I_3$ with $M_3 = I_{W_3}(D_3) = 1$. For the same reason, all companion ideals P_i equal I_i, and, since $E_i = \emptyset$, the transversality ideals are trivial $Q_i = 1$, so that $K_i = I_i$. The first descent in dimension goes via the coefficient ideal J_2 of $K_3 = I_3$ in $W_2 = \{x = 0\}$, say $J_2 = (y^7 + z^d)$. Again, $J_2 = I_2 = M_2 \cdot I_2$ with $M_2 = I_{W_2}(D_2) = 1$. The second descent is similar, $W_1 = \{x = y = 0\}$ and $J_1 = (z^d) = I_1 = M_1 \cdot I_1$ with $M_1 = I_{W_1}(D_1) = 1$.

The components o_i and k_i of the invariant are defined as the orders $o_i = \operatorname{ord}_a I_i$ and $k_i = \operatorname{ord}_a K_i$ of the ideals I_i and K_i. At the beginning of the resolution process we have $o_i = k_i = \operatorname{ord}_a J_i$ since $J_i = I_i = K_i$. The combinatorial components m_i will be explained later. All of the m_i except one play no role at all. We just write them in order to keep the components of the invariant systematic in each dimension. The only relevant m_i occurs for the maximal index i so that $o_i = 0$. This signifies that $I_i = 1$ locally at a; hence $J_i = M_i$ is already monomialized (say resolved). As we have seen in the last section, it is necessary in this case to choose the center combinatorially and to build up a separate induction. The nontrivial component m_i is precisely the combinatorial resolution invariant which takes care of a monomial principal ideal J_i supported by exceptional components.

> In the example, we have $o_3 = k_3 = 5$, $o_2 = k_2 = 7$, $o_1 = k_1 = d$ and all $m_i = (0, 0)$.

The critical point in the definition of the invariant is to specify a class of local flags $W_n \supset \ldots \supset W_1$ so that the resulting invariant does not depend on the choice of the flag within this class. This will be achieved by requiring that the hypersurface W_{i-1} of W_i is osculating for the companion ideal P_i of J_i. Then all components of the invariant become automatically intrinsic, and one can show that $i_a(\mathcal{M})$ defines an upper semicontinuous function on W.

Once we dispose of the local invariant, we set

$$Z = \{a \in W,\ i_a(\mathcal{M}) \text{ is maximal lexicographically}\} = \operatorname{top}(i_a(\mathcal{M})).$$

This is a closed subscheme of W. By induction on the dimension it is shown that it is regular and transversal to the current exceptional locus. Let $W' \to W$ denote the induced blowup with new exceptional component Y'. We are going to define the transform $\mathcal{M}'$ of the mobile $\mathcal{M} = (\mathcal{J}, c, D, E)$ in W'. After this is done it has to be shown that $\mathcal{M}'$ admits again punctual setups at each point. So its invariant $i_{a'}(\mathcal{M}')$ is well defined and can be compared with $i_a(\mathcal{M})$. Using the various commutativity relations (they carry on the ideals of the setups; see the section "Commutativity"), one can then simply show that $i_{a'}(\mathcal{M}') \leq i_a(\mathcal{M})$ holds with respect to the lexicographic order. Indeed, the components of the invariant are orders of ideals which pass under blowup by "Commutativity" to their weak transform (so their order does not increase, since the center lies in their top locus), provided that the earlier components of the invariant have remained constant. This yields $i_{a'}(\mathcal{M}') \leq i_a(\mathcal{M})$.

In order to show that the invariant actually drops, say

$$i_{a'}(\mathcal{M}') < i_a(\mathcal{M}),$$

we are confronted with two scenarios. Assume that $i_{a'}(\mathcal{M}') = i_a(\mathcal{M})$ has remained constant. If all components o_i of $i_a(\mathcal{M})$ are positive, we get a contradiction, because the weak transform of I_1 will be 1 (this always happens for ideals in one dimensional ambient spaces), so $o_1' = 0 < o_1$. Otherwise, let d be the maximal index so that $o_d = 0$. Hence $I_d = 1$ and $J_d = M_d$ is a monomial. A computational argument shows that the combinatorial component m_d of $i_a(\mathcal{M})$ drops, $m_d' < m_d$; see the section "Shortcuts" for details. We conclude that for any point a' above $a \in Z$ the invariant drops, $i_{a'}(\mathcal{M}') < i_a(\mathcal{M})$. This establishes the required induction step for the resolution of mobiles.

So let us indicate briefly how the transform $\mathcal{M}'$ is defined. The rules are prescribed by the desired transformation laws for the respective ideals of a punctual setup of $\mathcal{M}$. The ideals I_i — which yield the components o_i of the invariant — shall pass to their weak transforms at a' as long as the higher indexed components of the invariant have remained constant at a' (otherwise $o_i = \operatorname{ord}_a I_i$ and the lower indexed components are irrelevant, by definition of the lexicographic order). As $J_i = M_i \cdot I_i$, this gives the transformation formula for the transversal handicap D_i, depending on the value of the truncated invariant $(o_n, \ldots, o_{i+1}, k_{i+1}, m_{i+1})$ at a'. Namely, if $(o_n', \ldots, o_{i+1}', k_{i+1}', m_{i+1}') = (o_n, \ldots, o_{i+1}, k_{i+1}, m_{i+1})$, we shall set $D_i' = D_i^* + (o_i - c_{i+1}) \cdot Y'$ where c_{i+1} is the control defining $J_i' = J_i^! = J_i^* \cdot I(Y')^{-c_{i+1}}$. As J_i is the coefficient ideal of K_{i+1} at a, c_{i+1} will equal $\operatorname{ord}_a K_{i+1}$ in order to have J_i' the coefficient ideal of $K_{i+1}' = K_{i+1}^\curlyvee$ in W_i'. If $(o_n', \ldots, o_{i+1}', k_{i+1}', m_{i+1}') < (o_n, \ldots, o_{i+1}, k_{i+1}, m_{i+1})$ at a', then D_i' is set equal to 1 (because otherwise the factorization $J_i' = M_i' \cdot I_i'$ may fail, and since the components of the invariant in dimension i are irrelevant in this case).

This is a bit technical. The computation of transforms of mobiles will be extensively practiced in the example section. In the example from above the transform $\mathcal{M}'$ of $\mathcal{M}$ is defined as follows.

The center of the blowup is the origin $Z = \{0\}$ of $W = \mathbb{A}^3$. As we chose the control $c = 5 = \operatorname{ord}_a J$, we have as controlled transform $J' = J^* \cdot I(Y')^{-5}$ of J the weak transform $J' = J^\curlyvee$ of J. The transversal and the combinatorial handicaps are stratified divisors, so their definition depends on the points of W' we are looking at. We shall only pick one point a', the origin of the z-chart, which is the most interesting point.

In the induced coordinates in the z-chart, the exceptional component Y' is defined by $\{z = 0\}$, and the total transform J^* of J equals $J^* = (x^5z^5 + y^7z^7 + z^d)$ (replace x and y by xz and yz). Therefore, $J' = J^* \cdot z^{-5} = (x^5 + y^7z^2 + z^{d-5})$, which we may write as the first ideal $J_3' = J'$ of our punctual setup of $\mathcal{M}'$. As this ideal equals the weak transform of $I_3 = J_3$, we have the factorization $J_3' = I_3' = M_3' \cdot I_3'$

with $I_3' = (x^5 + y^7z^2 + z^{d-5})$ and $M_3' = 1$. It follows that the first component D_3' of the combinatorial handicap D' of $\mathcal{M}'$ equals $\emptyset$.

If $d < 10$, the order of I_3' has dropped at a' from 5 to $d-5$. In this case, our objective to lower the order of J is already achieved. If $d \geq 10$, the order has remained constant and we have to look at further components of the setup and the invariant. As the order of I_3' has remained constant at a', we may take for the local hypersurface W_2' the transform $W_2^\curlyvee$ of $W_2 = \{x = 0\}$ (see again the next section for justification). In the z-chart, W_2' is given as $\{x = 0\}$. As Z is transversal to W_2 (because it is contained in it), Y' and W_2' will be transversal. This means that Y' will pose no transversality problems with respect to W_2'; hence we set the transversal handicap E_3' equal to $E_3' = \emptyset$ (recall that the transversal handicap E_{i+1} takes care of the member W_i of the flag in one dimension less, which may cause transversality problems).

As $D_3' = E_3' = \emptyset$, the companion ideal P_3' equals I_3' and the transversality ideal Q_3' equals 1, so that $K_3' = I_3' = (x^5 + y^7z^2 + z^{d-5})$. Its coefficient ideal in $W_2' = \{x = 0\}$ is $J_2' = (y^7z^2 + z^{d-5})$. It factorizes into $J_2' = (z^2)\cdot(y^7 + z^{d-7}) = M_2' \cdot I_2'$ with $M_2' = (z^2)$ and $I_2' = I_2^\curlyvee = (y^7 + z^{d-7})$ the weak transform of $I_2 = (y^7 + z^d)$. Therefore the second component D_2' of the combinatorial handicap D' of $\mathcal{M}'$ equals at a' the divisor $D_2' = 2 \cdot Y'$.

At this stage, we have to distinguish two cases. If $d < 14$, the order of I_2' has dropped at a' from 7 to $d - 7$, and a new hypersurface W_1' has to be chosen to complete the construction of the setup (which, in any case, is irrelevant, since we know already that the invariant $i_{a'}(\mathcal{M}')$ has dropped lexicographically at a').

We shall concentrate on the more delicate situation where $d \geq 14$, so that $\mathrm{ord}_{a'} I_2' = \mathrm{ord}_a I_2 = 7$. As before, we may conclude that $E_2' = \emptyset$, so that $Q_2' = 1$ and $K_2' = P_2' = I_2'$. Accordingly, we choose for the next member W_1' of the local flag the transform $W_1' = W_1^\curlyvee$ of $W_1 = \{x = y = 0\}$, which, in the z-chart we are considering, has the same equation $W_1' = \{x = y = 0\}$. The resulting coefficient ideal of $K_2' = I_2'$ equals $J_1' = (z^d) = (z^d) \cdot 1$ with $M_1' = (z^d)$ and $I_1' = I_1^\curlyvee = 1$. We see that the order o_1' of I_1' has dropped at a' from $o_1 = d$ to $o_1' = 0$, and that $D_1' = d \cdot Y'$.

The transformed mobile $\mathcal{M}'$ will be, on the stratum of the local invariant through a' and in case $d \geq 14$, of form

$$\mathcal{M}' = (J', 5, (\emptyset, 2 \cdot Y', d \cdot Y'), (\emptyset, \emptyset, Y')).$$

Recall that the various divisors of the handicaps are stratified, so that they may look differently at other points of W'. The components of the invariant of $\mathcal{M}'$ at a' are

$$\begin{aligned} o_3' &= k_3' = 5, \\ o_2' &= k_2' = 7, \\ o_1' &= k_1' = 0, \\ m_3' &= m_2' = (0, 0), \\ m_1' &= (d, *). \end{aligned}$$

The combinatorial invariant m_1' in dimension 1 has as its first component the order of M_1'; the second $*$ is part of the label of M_1' which we need not specify here. You may believe that the next center Z' of blowup is the origin a' of the chosen

z-chart in W'. We leave it as a lengthy but worthwhile exercise to compute the next transform $\mathcal{M}''$ of the mobile and its punctual setups at all points of W''.

We return to the general framework of mobiles and setups. Similar to the ideals I_i, the ideals K_i — which yield the components k_i of the invariant — shall pass under blowup to their weak transforms at a' as long as the earlier components $(o_n, \ldots, o_{i+1}, k_{i+1}, m_{i+1}, o_i)$ of the invariant have remained constant at a', by the same reasoning as before. As $K_i = P_i \cdot Q_i$ and $P_i' = P_i^\curlyvee$ at a', we conclude that $Q_i' = I_{W_i'}(E_i' \cap W_i')$ should be the weak transform of Q_i. This is fine, because it turns out that we can choose for W_i' the weak transform $W_i^\curlyvee$ of W_i at a'. Setting $E_i' = E_i^\curlyvee$ is the correct choice for the transversal handicap, because the new exceptional component Y' will a priori be transversal to $W_{i-1}' = W_{i-1}^\curlyvee$.

A detailed explanation may be helpful here to make things more explicit: The hypersurface W_{i-1} of W_i is chosen osculating for P_i, and P_i' will equal $P_i^\curlyvee$ at points a' where the truncated invariant $(o_n', \ldots, o_{i+1}', k_{i+1}', m_{i+1}', o_i')$ of $\mathcal{M}'$ equals the truncation $(o_n, \ldots, o_{i+1}, k_{i+1}, m_{i+1}, o_i)$ of the invariant of $\mathcal{M}'$. By the persistence of osculating hypersurfaces at equiconstant points, $W_{i-1}' = W_{i-1}^\curlyvee$ will again be osculating for P_i' (since $o_i' = \operatorname{ord}_{a'} P_i^\curlyvee = \operatorname{ord}_a P_i = o_i$ at a').

As the center Z was contained locally in W_{i-1}, the new exceptional component Y' is automatically transversal to $W_{i-1}^\curlyvee$. Hence it need not be added to the transversal handicap E_i'. Only if a new osculating hypersurface W_{i-1}' has to be chosen (because the order of $P_i' = P_i^\curlyvee$ has dropped or because $P_i' \neq P_i^\curlyvee$), E_i' must contain in addition to the transform of E_i also the component Y'.

At points a' where $(o_n, \ldots, o_{i+1}, k_{i+1}, m_{i+1}, o_i)$ has dropped we set E_i' equal to the whole exceptional locus minus the components belonging to $E_n', \ldots, E_{i+1}'$ which have already been taken care of by the transversal handicaps in higher dimensions. We see here clearly why both D_i' and E_i' will be *stratified* normal crossings divisors in W'.

We are left to define the control c' of $\mathcal{M}'$. It is simply c except if the maximal order of $\mathcal{J}'$ on W' has dropped below c, in which case this maximum is chosen. This gives a rough idea how a mobile transforms under blowup. The important thing is that $\mathcal{M}'$ does not depend on any choices (i.e., not on the flags chosen for $\mathcal{M}$), because the components of the invariant are independent. In conclusion, mobiles are a somewhat heavy and slowly moving vehicle, but they run and run and run … until they are resolved.

6. Setups. To motivate the definition of setups of mobiles, let us assume that we have already blown up several times and have thus arrived at an ideal J in W. It will be the controlled transform of the ideal we started with — with respect to a given control c (you may just as well think of J as being the total transform). The exceptional components in W produced by the prior blowups will be denoted by F. The ideal J factorizes into $J = M \cdot I$ where I denotes the weak transform of the ideal we started with, and where M is a locally principal monomial ideal supported by F.

The mobile $\mathcal{M}$ for which we wish to construct a punctual setup will be of the form $\mathcal{M} = (J, c_+, D, E)$ where we assume for simplicity that J is an ideal in W (and not in a regular subscheme). The control c_+ carries now an index $+$. This is for notational reasons, because the ideals J_i will be governed by controls c_{i+1} associated to ideals K_{i+1} in one dimension higher. So c_+ shall suggest c_{n+1} with

$n = \dim W$. We shall assume that the handicaps D and E have been constructed as certain transforms of the handicaps of the initial mobile.

Recall that we wish to transform the entire ideal J into a principal monomial ideal. We measure the distance of J from being a principal monomial ideal by factorizing J into $J = M \cdot I$ and by considering the order o of I. The smaller o is, the closer we are to the final goal. Observe here that this kind of measurement is relatively dull. For instance, if J is from the beginning a principal monomial ideal, our invariant is not able to capture this. Instead it is necessary to blow up several times until all factors of J appear as exceptional components (with multiplicities), i.e., $J = M \cdot 1$. It's only then that the invariant will tell us that the ideal is resolved.

There have been several attempts to measure directly the distance of an ideal from being a principal monomial ideal or, said geometrically, of a scheme from being a normal crossings divisor. None of them succeeded in building up a general induction argument for resolution. The problem is that the natural invariants one could think of are related to the Newton polyhedron of the ideal and are therefore very much coordinate dependent. This would not matter too much if the coordinate choices were compatible with the coordinate substitutions occurring in blowups. There are two types.

The monomial substitutions of the coordinates do not pose problems, but the translations which are necessary in the exceptional divisor to compute Taylor expansions do. These translations can also be applied before blowing up and then correspond to Borel linear coordinate changes, i.e., changes given in suitable coordinates by upper or lower triangular matrices. Such changes affect considerably the shape of the Newton polyhedron.

However, some characteristics of the Newton polyhedron remain unchanged, e.g. its (integral) distance from the origin (which is just the order of the ideal at the origin) or certain projections of the Newton polyhedron to coordinate planes (which correspond to passing to the coefficient ideal of the ideal). It is certainly worth searching for further measures of monomiality.

The ideal I is the part of J we are mainly interested in and whose order shall decrease. Our center Z for the next blowup will be chosen in its top locus $\mathrm{top}(I)$. As we already explained (and this is particularly relevant in the next smaller dimensions), the center should also be contained in $\mathrm{top}(J, c_+)$ of points where j has order at least c_+. Namely, if J is the coefficient ideal of some ideal K_+ of order c_+ at a, as will occur in lower dimensions, we have $\mathrm{top}(J, c_+) \subset \mathrm{top}(K_+)$ and wish to ensure that the center lies inside $\mathrm{top}(K_+)$. The simplest way to achieve this is to require that the center lies in $\mathrm{top}(J, c_+)$.

As $\mathrm{top}(I)$ may fall outside $\mathrm{top}(J, c_+)$, we replace I by an ideal which is sufficiently close to it and which ensures the inclusion of top loci. This substitute for I is the companion ideal P of $J = M \cdot I$ and the control c_+. It shall satisfy $\mathrm{top}(P) \subset \mathrm{top}(I) \cap \mathrm{top}(J, c_+)$, have the same order as I and behave under blowup similarly as I. A suitable definition of P is as follows:

$$
\begin{aligned}
&P = I + M^{\frac{o}{c_+ - o}} && \text{if } 0 < o = \mathrm{ord}_a I < c_+, \\
&P = I && \text{otherwise.}
\end{aligned}
$$

The rational exponent could be avoided by placing $P = I^{c_+ - o} + M^o$ if $0 < o < c_+$. But then I and P would have different orders, thus burdening the notation and complicating the transformation laws. We leave it to the reader to define ideals with rational exponents as equivalence classes of pairs consisting of an ideal and a number. The companion ideal satisfies $\mathrm{top}(P) \subset \mathrm{top}(I) \cap \mathrm{top}(J, c_+)$ as desired. Moreover, it behaves well with respect to taking weak transforms. Indeed, if $c'_+ = c_+$ and $J' = J^! = M' \cdot I'$ with $I' = I^\curlyvee$, and if $o' = o$ holds for the orders of I and I', then the companion ideal P' of J' with respect to $J' = M' \cdot I'$ and c'_+ equals the weak transform $P^\curlyvee$ of P.

We give the proof of this commutativity relation: If $o' \geq c'_+$, the assertion is clear since $P = I$ and $P' = I'$ in this case. So assume that $o' < c'_+$. From $M' = M^* \cdot I(Y')^{o-c_+}$ we get $P' = I' + (M')^{\overline{\frac{o'}{c'_+ - o'}}} = I^\curlyvee + (M^*)^{\overline{\frac{o}{c_+ - o}}} \cdot I(Y')^{-o} = (I^* + (M^*)^{\overline{\frac{o}{c_+ - o}}}) \cdot I(Y')^{-o} = P^\curlyvee$.

We now choose, locally at a, a regular hypersurface $V = W_-$ in W which is osculating for P. Such hypersurfaces exist locally in characteristic zero, though need not patch globally. We shall use V to define the coefficient ideal which performs the descent in dimension. Before doing so, notice that V need not be transversal to the exceptional locus F. As the center will be locally included in V, it may neither be transversal. However, if we knew that it is contained in the intersection of V with F, transversality would be guaranteed. This inclusion can be achieved by multiplying P with the ideal Q defining F in W. The *composition ideal* $K = P \cdot Q$ will then satisfy $\mathrm{top}(K) \subset \mathrm{top}(P) \cap \mathrm{top}(Q)$, which in turn lies in all components of F it meets.

> Other choices of K are possible, for instance the equilibrated sum of powers of P and Q; cf. [EV 1]. The exponents have to be chosen so that $\mathrm{top}(K) \subset \mathrm{top}(P) \cap \mathrm{top}(Q)$ and the definition is compatible with passage to weak transforms under blowup. The definition $K = P \cdot Q$ used here and in [EH] is simpler than in [EV 1], but less efficient for implementations of the algorithm.

The center of blowup will lie inside $\mathrm{top}(K)$ and hence in all components of F it meets. Therefore it would be appropriate to set $Q = I(F)$. It turns out that with this choice of Q the corresponding centers would be too small, and actually a resolution would never be achieved (essentially, because V can never be separated by blowups from the whole exceptional locus F, since the new components which appear during the separation process will again meet V). So we take for Q instead of $I(F)$ the ideal $I(E)$ where E collects only the "dangerous" components of F with respect to V, i.e., those which may fail to be transversal to V. These can be determined easily and have been described in "Mobiles". The ideal Q is called the *transversality ideal* of $J = M \cdot I$. It depends on J because E depends on J.

Let us pause to summarize what we are up to: First we decompose J into $J = M \cdot I$ and agree to take $o = \mathrm{ord}_a I$ as the first component of the resolution invariant $i_a(\mathcal{M})$. As the center will be defined as $\mathrm{top}(i_a(\mathcal{M}))$, it will lie in $\mathrm{top}(o) = \mathrm{top}(I)$. Next we associate to I the companion ideal P and an osculating hypersurface V for P. We specify a (reduced) divisor E of exceptional components and set $Q = I(E)$ and $K = P \cdot Q$. We do this for two reasons: first to force the center inside all components of F it intersects (with the drawback of getting possibly a relatively small center); second, because resolving K instead of I will separate F from V, and then the center will and can be chosen sufficiently large. To include the resolution (= monomialization) of K to our program, we define the second component of the invariant $i_a(\mathcal{M})$ as the order k of K at a,

$$i_a(\mathcal{M}) = (o, k, \ldots) = (\mathrm{ord}_a I, \mathrm{ord}_a K, \ldots).$$

In this way, the center $Z = \mathrm{top}(i_a(\mathcal{M}))$ will lie inside $\mathrm{top}(I)$ and $\mathrm{top}(K)$, the latter being included in $\mathrm{top}(P)$ and $\mathrm{top}(Q)$, hence in the intersection of the components of E. As $\mathrm{ord}_a K = \mathrm{ord}_a P + \mathrm{ord}_a Q = \mathrm{ord}_a I + \mathrm{ord}_a Q$, the second component k measures the order of Q, i.e., how far Q is from being resolved. Under blowup, all four ideals—I, P, Q and K—will pass at equiconstant points (here equiconstant is meant with respect to the prior components of the invariant) to their respective weak transforms.

> It could be suspected that it is easier to postpone the transversality problem until the moment a virtual center C is found via I, P and their successive coefficient ideals. If C is already transversal to the entire exceptional locus F, it is taken as the actual center Z; if not, the ideal $I(C \cap F)$ is added to the resolution problem.
>
> This quite natural approach leads to an unpleasant situation which is due to the combinatorial invariant associated to a resolved ideal $J_i = M_i \cdot 1$ in a certain dimension i. It appears as a component m_i in the resolution invariant and is the only component which may increase when the center is too small. As resolving the ideal $I(C \cap F)$ by auxiliary blowups requires small centers, the component m_i will indeed increase, and the induction breaks down. The invariant m_i and the combinatorial resolution problem will be described later on.

Let us continue on our way through the setup of a mobile. The next step is the descent in dimension. Let J_- be the coefficient ideal of K in V at a (recall that V is a local hypersurface of W at a which is osculating for P). If K is *bold regular*, i.e., generated by a power of a coordinate, $K = P \cdot Q = (x)^k$, and consequently $V = \{x = 0\}$, then the coefficient ideal of K in V would be zero, so we set instead $J_- = 1$ in this case. Otherwise we may really take for J_- the coefficient ideal of K as defined earlier, since it will be non-zero.

When passing to the coefficient ideal, a certain coherence property is required. The local hypersurface V of W is chosen for each $a \in W$, and different a may yield quite different hypersurfaces and coefficient ideals. This in turn may destroy the upper semicontinuity of the order of the coefficient ideal (which will form a component of the resolution invariant), and also the (local) coherence of the setup in lower dimensions.

To confront this problem, the easiest and most efficient solution is to choose the same hypersurface V for all points a of the top locus $\mathrm{top}(P)$ of P, at least locally along $\mathrm{top}(P)$. This is possible in characteristic zero, since the construction of V via derivatives extends along $\mathrm{top}(P)$ to a neighborhood of the given point a (in positive characteristic, some detour has to be taken by stratifying $\mathrm{top}(P)$ further, but the construction still works). It is checked that V contains $\mathrm{top}(P)$ locally at a and that V is osculating for P at all points of $\mathrm{top}(P)$ sufficiently close to a. Thus we could really work in the open subschemes of a sufficiently fine open covering of W instead of working in local rings. As all constructions extend to neighborhoods, we shall stick for simplicity to the punctual setting via local rings. That's why setups are called punctual. It is shown that punctual setups can be defined as the specializations of setups at points in entire neigborhoods and are therefore in a suitable sense "coherent" (with respect to the underlying stratification).

> In positive characteristic, the top locus of P has to be stratified by its local embedding dimension to find hypersurfaces which work for whole neighborhoods of points on a stratum. Osculation with P cannot be realized and has to be replaced by weak maximal contact, with the drawback that this property is not persistent under blowup.

We shall not discuss the aspects of coherence any further, and content ourselves to state that all constructions involved in a punctual setup of a mobile (factorizations, companion, transversality and composition ideals as well as coefficient ideals) are sufficiently coherent. The coherence always refers to the strata of the stratification given by the earlier components of the invariant and is encapsulated in [EH] by the concept of tunedness.

So let J_- be the coefficient ideal of K in V at a, locally along $\mathrm{top}(K)$. This is by definition the local top locus of K, i.e., the top locus of K restricted to a sufficiently small neighborhood of a in W. Thus along $\mathrm{top}(K)$ the order of K is constant and equal to $\mathrm{ord}_a K$. We denote by c this value and associate it to J_- as its *control* on $V = W_-$. This signifies that under blowup, J_- will pass to the controlled transform $J'_- = J^!_- = J^* \cdot I(Y')^{-c}$ with respect to c. This ensures that the coefficient ideal of $K' = K^\curlyvee$ in $V' = V^\curlyvee$ equals J'_- at points a' where the order of K has remained constant (K-equiconstant points).

> We should briefly indicate the difference between the order $k = \mathrm{ord}_a K$ of K at a point a of W and the control c of J_- in V. The first will vary along W and form the second component of the invariant $i_a(\mathcal{M})$. The second is given for each a in W as the value of k along the local stratum $\mathrm{top}(K)$ and will be considered as a constant associated to the local hypersurface V of W at a. As the same V can be chosen locally for all points a of $\mathrm{top}(K)$ and as k is constant along $\mathrm{top}(K)$, this choice for the control c of J_- is justified.

The definition of c and the construction of J_- conclude the descent in dimension, which started with J. It is carried out locally along the stratum of the first two components o and k of the invariant. The local hypersurface $V = W_-$ will contain an open set of this stratum, and there all stalks of J pass to the respective stalks of J_- at points of W_-.

It remains to indicate how J_- factorizes. This is prescribed by the second component $D_- = D_{n-1}$ of the transversal handicap D of the mobile. Before any blowup, $D_- = \emptyset$ and $J_- = I_-$. After a sequence of blowups, D_- will be the normal crossings divisor which yields $J_- = M_- \cdot I_-$ with $M_- = I_{W_-}(D_- \cap W_-)$ and I_- the weak transform of the ideal preceding it before the last blowup. This description of D_- only holds if the first two components (o, k) of the invariant have remained constant during this blowup. If they have dropped lexicographically, we set $D_- = \emptyset$ and $J_- = I_-$. This is justified by the fact that it does not matter lexicographically that the component $o_- = \mathrm{ord}_a I_-$ of the invariant may have increased, if one of the earlier components has dropped. Recall here that commutativity for I_- is only needed at equiconstant points, say, if (o, k) has remained constant, and that otherwise I_- need not be the weak transform of its predecessor.

Once we know that J_- factorizes (and this will be proven in more detail in "Commutativity"), we can start over again, associating to J_- the ideals P_-, Q_- and K_-, a local hypersurface $(W_-)_-$ in $W_- = V$ with control c_- and the coefficient ideal $(J_-)_-$ of K_- in $(W_-)_-$. We can continue like this down to ambient dimension equal to 1, getting a local flag $W = W_n \supset W_- = W_{n-1} \supset \ldots \supset W_1$ and a string of ideals $(J = J_n, J_- = J_{n-1}, \ldots, J_1)$. It may happen that from some index d on, all J_i are equal to 1 (this occurs if K_{d+1} is bold regular or 1). If d denotes the maximal index with $I_d = 1$, then $o_i = 0$ for $i \leq d$ and also $k_i = 0$ (since it turns out that $E_d = \emptyset$ and therefore $Q_d = K_d = 1$ in this case). The remaining members $W_{d-1}, \ldots, W_1$ of the flag are irrelevant. The top locus of $(o_n, k_n, \ldots, o_1, k_1)$ equals the support W_d of K_{d+1}, which would then be the center — leaving aside the combinatorial components m_i of the resolution invariant which may reduce the center further. Let us investigate this combinatorial situation more closely.

7. Shortcuts. What happens under blowup if $I_d = 1$ for some $d \geq 1$ (if $I_i \neq 1$ for all i, we set $d = 0$)? What should be the correct choice of center?

We then have $J_d = M_d$ a principal monomial ideal, and the component $o_d = 0$ of the invariant cannot improve further. But it may happen that the vector $(o_n, k_n, , \ldots, o_{d+1}, k_{d+1})$ remains constant at some point a' of Y', and then no improvement will be observed. At this stage the size of the center comes into play. Up to now it only mattered that the center be included in the various top loci, independently of how large it was chosen. In all cases the orders o_i and k_i did not increase, and, by induction, at least one of them would decrease if all $o_i > 0$. If $J_d = M_d$ in W_d, the companion ideal P_d of I_d is 1 and its top locus equals the whole ambient scheme W_d. Choosing as center W_d will not be permitted, since it is not contained in $\operatorname{top}(J_d, c_{d+1})$, and hence not in $\operatorname{top}(K_{d+1})$, except if K_{d+1} was bold regular of support W_d and $J_d = 1$.

Let us look at example 7 of the section "Improvement of singularities" and its generalization. It was given by $x^p + y_1^{r_1} \cdots y_d^{r_d}$ in $W_{d+1} = \mathbb{A}^{d+1}$ at $a = 0$. Here the polynomial generates K_{d+1}, W_d is given by $x = 0$, and $J_d = (y_1^{r_1} \cdots y_d^{r_d})$ consists of exceptional components. Assume that $J_d = M_d$ and $I_d = 1$. We have seen that any center $Z = \{x = y_{i_1} = \ldots y_{i_m} = 0\}$ with indices $i_1, \ldots, i_m$ ranging in a minimal subset of $\{1, \ldots, d\}$ so that $r_{i_1} + \ldots + r_{i_m} \geq p$ will make the order $r_1 + \ldots + r_m$ of $M_d = (y_1^{r_1} \cdots y_d^{r_d})$ drop when passing to the controlled transform with respect to $c_{d+1} = \operatorname{ord}_a K_{d+1}$.

This example is representative for our problem and anticipates what has to be done. The center Z should be an intersection of some components of M_d along which M_d has order $\geq \operatorname{ord}_a K_{d+1}$ and should have maximal possible dimension. Then the order of the controlled transform of $J_d = M_d$ with respect to c_{d+1} will drop. At a', either the earlier components of the invariant have dropped or, if they have remained constant, the ideal M'_d of the setup of J' will be precisely the controlled transform of M_d with respect to the control c_{d+1} (since $J_d = M_d$ and $J'_d = M'_d$). This suggests adding the order of M_d as the last nontrivial component of our invariant (this will be the first component of the combinatorial invariant m_d). A computation shows that it only drops when the center is really maximal with the above property; smaller centers will make the order increase.

There is another problem with the choice of the center. There can appear several maximal candidates among the respective intersections of components of M_d.

We cannot just choose one of them ad hoc, since our construction is local and the choice of the center at some other point may not be compatible and would hence prohibit getting a globally defined center. Therefore we have to choose the center everywhere subject to the same rule. In example 3 we have seen that we might run into symmetry problems, because several candidates are permuted by a symmetry of K_{d+1}. There is a nice escape button in our case, using that M_d consists of exceptional components. These components have appeared in the preceding resolution process one after the other, so they are naturally ordered, for instance by the moment of their respective appearance. The rule to select a center among the possible candidates could then be to take the intersection of components whose total age (= sum of individual ages of the components) is the largest (or the smallest).

The precise choice of the center is a little bit involved. Let us call a *shortcut* of a divisor F any divisor obtained from F by deleting some of its components. As just explained, the center will be the intersection of the components of a certain shortcut of M_d. This intersection is the top locus of the shortcut. We will put different labels on all shortcuts of the combinatorial handicaps D_i in order to distinguish them and to make the choice of the center systematic. Labels are simply positive integers, and weak transforms of shortcuts will get the same label as their preimage. The center is then chosen as the top locus of a shortcut N_d of M_d whose order is $\geq \mathrm{ord}_a K_{d+1}$ and which has no proper shortcuts $\tilde{N}_d$ of order $\geq \mathrm{ord}_a K_{d+1}$ (this guarantees that the center is maximal). If there are several such shortcuts available, take the one with maximal order. There may still be several of them. In this case take the one with maximal label. This one is then unique, since no two shortcuts will have the same label. Even though defined locally, the local pieces of the center will patch on overlaps (since defined through the order of the shortcuts of M_d and the labels of the shortcuts of the global divisors D_d inducing labels on the shortcuts of M_d) and give a globally defined subscheme of W.

All this is incorporated in the combinatorial component m_d of the invariant; see [EH] for more details. It is a pair whose first number is the order of the unique shortcut N_d specified before and whose second number is the label of N_d. Thus

$$m_d = (\mathrm{ord}_a N_d, \mathrm{lab}_a N_d).$$

The combinatorial component m_d is only needed in dimension d. To smoothen the invariant we set $m_i = (0,0)$ for $i \neq d$ (i.e., if $I_i \neq 1$ or $i < d$) and set

$$i_a(\mathcal{M}) = (o_n, k_n, m_n, \ldots, o_1, k_1, m_1) \in \mathbb{N}^{4n}.$$

This is the final definition of the invariant through a punctual setup of the mobile $\mathcal{M}$ at a. In each dimension it consists of a quadruple (o_i, k_i, m_i) associated to the ideals I_i, K_i and M_i of a setup of $\mathcal{M}$ at a. Once we know that the invariant does not depend on the choice of the local hypersurfaces W_i and that the required commutativity and transversality relations hold, the preceding discussion applies and shows that $i_a(\mathcal{M})$ decreases lexicographically under blowup of $\mathcal{M}$ in $Z = \mathrm{top}(i_a(\mathcal{M}))$.

> There is a small delicacy in the decrease of m_d at points where the earlier components of the invariant have remained constant. By definition, m_d' equals $(\mathrm{ord}_{a'} N_d', \mathrm{lab}_{a'} N_d')$ where N_d' is the shortcut of $M_d' = M_d^!$ defined analogously to N_d for M_d. If N_d' is the weak transform of N_d, its order will have decreased, so that $m_d' < m_d$.

If not, N'_d will be a newly chosen shortcut of M'_d, hence the weak transform of a shortcut $\tilde{N}_d$ of M_d. As $\operatorname{ord}_a N_d$ was maximal among all considered shortcuts of M_d, we get $\operatorname{ord}_{a'} N'_d \leq \operatorname{ord}_a \tilde{N}_d \leq \operatorname{ord}_a N$. Thus the first component of m_d does not increase. If it remains constant, which can only occur if N'_d is a newly chosen shortcut of M'_d, the label of $N'_d = \tilde{N}_d^{\curlyvee}$ will be the label of $\tilde{N}_d$ and hence strictly smaller than the label of N_d. This shows that $m'_d < m_d$ holds lexicographically in all cases.

8. Commutativity. The idea of commutativity has appeared several times up to now, and often with different meanings. We shall now give a precise description of what is meant by commutativity in each context.

Let $\mathcal{M} = (\mathcal{J}, c, D, E)$ be a mobile in W, and let $\mathcal{M}'$ be the transform of $\mathcal{M}$ under the blowup $W' \to W$ with center $Z = \operatorname{top}(i_a(\mathcal{M}))$. Let $(J_n, \ldots, J_1)$ be a punctual setup of $\mathcal{M}$ at a, and let a' be a point above a. Roughly speaking, commutativity shall express the property that the ideals of the setup of $\mathcal{M}$ have transforms in W' (weak or controlled transform, according to the ideal) which define the ideals of a truncated setup $(J'_n, \ldots, J'_j)$ of $\mathcal{M}'$ at a' down to the index j until which the truncated invariant has remained constant at a.

This is used in two ways. First, to show that $\mathcal{M}'$ admits again punctual setups at all points a' of W' (the remaining components $J'_{j-1}, \ldots, J'_1$ are easily determined since all $M'_{j-1}, \ldots, M'_1$ and $Q'_{j-1}, \ldots, Q'_1$ will be trivial equal to 1, so that $J'_i = I'_i = P'_i = K'_i$ for $i < j$; hence $J'_{i-1} = \operatorname{coeff}_{W'_{i-1}} J'_i$). Second, to be able to compare the invariant $i_{a'}(\mathcal{M}')$ of $\mathcal{M}'$ with $i_a(\mathcal{M})$: The components of $i_{a'}(\mathcal{M}')$ will be the orders of the weak transforms of the ideals defining the components of $i_a(\mathcal{M})$ as long as the earlier components have remained constant.

More explicitly, commutativity means that, for each $n \geq j \geq 1$, we have $W'_i = W_i^{\curlyvee}$ for $n \geq i \geq j$ if $(o_n, \ldots, o_{j+1})$ has remained constant at a'; $J'_i = J_i^!$ and $I'_i = I_i^{\curlyvee}$ for $n \geq i \geq j$ if $(o_n, \ldots, m_{j+1})$ has remained constant at a'; $P'_i = P_i^{\curlyvee}$, $Q'_i = Q_i^{\curlyvee}$ and $K'_i = K_i^{\curlyvee}$ for $n \geq i \geq j$ if $(o_n, \ldots, o_j)$ has remained constant at a'. Observe the inductive nature of these conditions, with j decreasing from n to 1.

Quite generally, we can introduce the following concept of commutativity: Let $\sharp : R \to R^\sharp$ be a map sending ideals R in W at a to ideals $R^\sharp$ at a in a regular locally closed subscheme V of W. Let Z be a regular closed subscheme W contained locally at a in V. Denote by $W' \to W$ and $V' \to V$ the induced blowups with center Z (observe that the blowup V' of V in Z equals the weak = strict transform of V under the blowup $W' \to W$). Assume given prescribed transformation rules $R \to R'$ and $S \to S''$ for ideals R of W and ideals S of V with respect to the blowups of W and V with center Z. We say that $\sharp : R \to R^\sharp$ commutes with blowup if for any choice of Z the following diagram commutes (exceptionally we write here the arrow of blowups in the opposite direction, which, in any case, is a matter of taste)

$$\begin{array}{ccccc} R' & \to & (R')^\sharp & \leftrightsquigarrow & (R^\sharp)'' \\ \uparrow & & & \nearrow & \\ R & \to & R^\sharp & & \end{array}$$

This is equivalent to saying that

$$(R')^\sharp = (R^\sharp)''.$$

Obviously we may require this commutativity relation to hold only at specific points a' above a, usually those where the prior components of the invariant have remained constant (where the adjective "prior" has to be interpreted correctly).

Let us list what kind of maps $\sharp : R \to R^\sharp$ we have met in the definition and construction of setups: Associating to J the factors M and I of J in the product $J = M \cdot I$, the companion ideal P of I associated to $J = M \cdot I$ and the control c_+, the transversality ideal Q given by an exceptional divisor E associated to P, the composition ideal $K = P \cdot Q$, and the coefficient ideal J_- of K. Analogously to ideals, we may also say that the passage to the osculating hypersurface $V = W_-$ for P commutes with blowup, i.e., $(W')_- = (W_-)^\curlyvee$ if $P' = P^\curlyvee$ and the order of P has remained constant.

Almost all commutativity relations for these ideals have already been established. The factorization $J' = M' \cdot I'$ for $J' = J^!$ the controlled transform with respect to c_+ with $I' = I^\curlyvee$ and $M' = M^* \cdot I(Y')^{o-c_+}$ for $o = \mathrm{ord}_Z\, I$ holds because the passage to the total transform is multiplicative. Commutativity of the companion ideal P is due to the special choice of the exponents in P (and requires $J' = J^!$ and that the control c_+ and the order of I have remained constant at a'), and that of Q holds by definition of E and E'. Commutativity for $K = P \cdot Q$ is then trivial, and that of coefficient ideals has been proven by an explicit calculation on the Taylor expansions, provided that $(W')_- = (W_-)^\curlyvee$. All these local ideals are actually determined by the choice of the local flags in W and W'.

As for W_-, the situation is more delicate, because various W_- and $(W')_-$ could be chosen. So the commutativity assertion says in this case that if $\mathrm{ord}_{a'}\, P' = \mathrm{ord}_a\, P$, then the weak transform of an osculating hypersurface W_- for P is again osculating for $P' = P^\curlyvee$ (which was also expressed as the persistence of osculating hypersurfaces under blowup). We have proven this by a computation in local coordinates. Note that commutativity need not hold for the weaker condition that W_- has weak maximal contact with P (see the section "Problems in positive characteristic" below).

We conclude that commutativity relations are almost automatic to verify, provided all ideals are correctly defined (which is not obvious to do when constructing them).

9. Independence. We shall describe now why the invariant $i_a(\mathcal{M})$ of a mobile $\mathcal{M}$ does not depend on the choice of the local flags. This is needed in three regards. First, it implies that our local definition of the center will yield a globally defined center. Second, the transform of a mobile under a blowup $W' \to W$, being defined through the truncated invariants in W and W', will not depend on the local flags. And finally, if the invariant would depend on some choices its decrease under blowup may not be significant for the actual resolution of the mobile, because another choice may produce an increase.

Recall that given a mobile $\mathcal{M}$ in W, its invariant $i_a(\mathcal{M})$ at $a \in W$ was defined and constructed through the choice of a local flag $W_n \supset \ldots \supset W_1$ and the resulting punctual setup $(J_n, \ldots, J_1)$ of $\mathcal{M}$ at a. Here, the hypersurfaces W_{i-1} of W_i were subject to be osculating for the companion ideals P_i of $J_i = M_i \cdot I_i$. We have seen earlier that if V is osculating for P, then it has weak maximal contact with P, i.e., maximizes the order of $\mathrm{coeff}_V(P)$. Therefore this order does not depend on the choice of V.

In a punctual setup, we take the coefficient ideal not of P but of the product $K = P \cdot Q$ with Q the transversality ideal. So we need an extra argument to show that the order of $\mathrm{coeff}_V(K)$ is independent of V. One possible way to see this is to observe that if V has weak maximal contact with P, then it has also weak maximal contact with any product $P \cdot Q$ (provided $P \neq 0, 1$). We will sketch the proof of this for principal ideals; for the general case and more details we refer to the paper [EH].

So let P and Q be generated by $f = x^p + \sum_{i<p} f_i x^i$ and $g = x^q + \sum_{i<q} g_i x^i$ modulo x^{p+1} respectively x^{q+1}, with coefficients f_i and g_i in the local ring of the hypersurface $V = \{x = 0\}$. Assume that V has weak maximal contact with P, i.e., maximizes $\min_i(\mathrm{ord}_a f_i^{p/(p-i)})$. Let $N(f)$ and $N(g)$ denote the Newton polyhedra of f and g in $\mathbb{N}^n$ with respect to local coordinates $x, y_1, \ldots, y_{n-1}$. The order of the coefficient ideals of f and g in V is given by the projections $\pi : (i, \alpha) \to \frac{p}{p-i} \cdot \alpha$ and $\tau : (i, \alpha) \to \frac{q}{q-i} \cdot \alpha$ from $\mathbb{N}^n$ to $\mathbb{Q}^{n-1}$ as $\min_{\alpha \in \mathrm{supp}\, f} \pi(|\alpha|)$ and $\min_{\alpha \in \mathrm{supp}\, g} \tau(|\alpha|)$.

Now let $h = f \cdot g = x^{p+q} + \sum_{i<p+q} h_i x^i$ and consider the corresponding projection $\sigma : (i, \alpha) \to \frac{p+q}{p+q-i} \cdot \alpha$. Denote by e_f, e_g and e_h the respective orders of the coefficient ideals of f, g and h in V. Using the three projections and the expression of the coefficients h_i of h through fi and g_i, it follows from a computation that $\frac{e_h}{p+q} = \min\{\frac{e_f}{p}, \frac{e_g}{q}\}$.

Assume now that V maximizes e_f. If $e_f/p \leq e_g/q$, then $\frac{e_h}{p+q} = e_f/p$ is already maximal. Otherwise, assume that $\frac{e_h}{p+q} = e_g/q$ is not maximal. Then a coordinate change $(x, y) \to \varphi(x, y) = (x + b(y), y)$ with $\mathrm{ord}\, b = e_g/q$ would allow us to increase $\frac{e_h}{p+q}$. From $e_f/p > e_g/q$ it follows that the order of the coefficient ideal of f with respect to $\varphi(V)$ would be e_g/q; hence $\frac{e_h}{p+q} = e_g/q$ would remain constant, i.e., was already maximal.

This gives an idea of why the order of the coefficient ideal J_{i-1} of $K_i = P_i \cdot Q_i$ does not depend on the choice of W_{i-1}. On the other hand, the handicaps D_{i-1} and E_{i-1} do not depend on any choices (when constructing the transform of a mobile, they were defined by the values of the earlier components of the invariant, which, by decreasing induction on the dimension, can already be assumed to be independent of any choices). It will be shown in "Transversality" that W_{i-1} is transversal to D_{i-1} and E_{i-1}. Hence the orders of $M_{i-1} = I_{W_{i-1}}(D_{i-1} \cap W_{i-1})$ and $Q_{i-1} = I_{W_{i-1}}(E_{i-1} \cap W_{i-1})$ equal the orders of D_{i-1} and E_{i-1}. This in turn shows that the orders of I_{i-1}, P_{i-1}, Q_{i-1} and K_{i-1} do not depend on the choice of the local flag. Actually, W_{i-1} maximizes all these orders.

> The difference $k_i - o_i = \mathrm{ord}_a Q_i$ could be taken in the invariant instead of the component k_i, since it carries the same amount of information, and since it is preceded by o_i in the invariant.

We have seen that $K_i = P_i \cdot Q_i$ and an osculating W_{i-1} for P_i induce intrinsic values of $o_{i-1} = \mathrm{ord}_a I_{i-1}$ and $k_{i-1} = \mathrm{ord}_a K_{i-1}$. This is not sufficient yet, since the next components of the invariants $o_{i-2}, \ldots$ and $k_{i-2}, \ldots$ could in principle depend on the choice of W_{i-1}.

In order to show that they do not depend, there are three options: either to show as above via Newton polyhedra and iterated projections to lower dimensions that the respective orders are indeed maximal or to allow only flags $W_n \supset \ldots \supset W_1$ which maximize the whole vector $(o_n, k_n, m_n, \ldots, o_1, k_1, m_1)$ lexicographically (not just component by component), with the drawback of having to show that the weak transform of such a flag maximizes the vector $(o'_n, k'_n, m'_n, \ldots, o'_1, k'_1, m'_1)$ of the transformed mobile $\mathcal{M}'$. Finally, one can also use a trick of Hironaka which shows that the truncated invariant depends only on the top loci and their behaviour under certain auxiliary blowups; cf. [EH]. All three possibilities are mostly computational, and we do not intend to explain these details here.

10. Transversality. There are several transversality conditions we have met up to now. The handicaps D and E of a mobile shall be normal crossings divisors. The hypersurfaces W_i shall be transversal to D_i and E_i (meaning that the unions $W_i \cup D_i$ and $W_i \cup E_i$ are normal crossings schemes). The center Z shall be transversal to the exceptional locus F.

The clue to establish all these conditions is the easy fact that when blowing up a regular center Z in W, any normal crossings divisor to which Z is transversal will remain a normal crossings divisor in W' and will be transversal to the new exceptional component Y'. This is best proven in local coordinates for which the blowup is given by a monomial substitution of the coordinates.

Let us briefly indicate how this applies in our situation. Assume that W_i is transversal to D_i and E_i. We shall use that the center Z is locally contained in W_i (for $i \geq d$, where d is maximal with $o_d = 0$) and transversal to the exceptional locus F. As D_i and E_i are supported by exceptional components, Z is also transversal to them. This implies that $W_i^\curlyvee$ is transversal to $D_i^\curlyvee$ and $E_i^\curlyvee$ and to Y'. By definition of D'_i and E'_i and the choice of W'_i as in "Commutativity", either W'_i is a newly chosen hypersurface, in which case D'_i and E'_i are chosen to be empty, or $W'_i = W_i^\curlyvee$, in which case D'_i and E'_i are composed by $D_i^\curlyvee$ and Y', respectively $E_i^\curlyvee$ and Y'. In all cases, the same transversality property holds again in W'. The argument uses part of "Commutativity", which in turn uses the transversality conditions in W; cf. the next section, "Cartesian induction", for the respective implications between the various arguments.

There is one more property needed in the proofs here, namely that the support of $E_n \cup \ldots \cup E_1$ fills up the the whole exceptional locus F. This again follows from the transformation rule for the transversal handicap. The rule is defined through the loci where the truncated invariants remain constant. It uses that the entire invariant always drops, so the locus where all truncated invariants remain constant is empty.

It should be emphasized that W_{i-1} need not be transversal to E_i. Indeed, by multiplying the ideal Q_i of E_i to P_i when defining the composition ideal K_i, we ensure that the center is contained in all components of F which could possibly be non-transversal to W_{i-1}. These are just the components of E_i. This is a subtle point of the construction of setups, which, in particular, allows us to interpret the center as the top locus of the invariant. Otherwise we would have to intersect this top locus with the intersection of the dangerous components of F (getting possibly a singular intersection), making the setting much less systematic.

11. Cartesian induction. We come to the end of our search for a proof of resolution of singularities. It is a good moment to resume the overall outset and to pin down the internal logical structure of the argument. Its basis is a cartesian induction: The horizontal induction on the local embedding dimension is amalgamated with the vertical induction on the resolution invariant. The interested reader may compare this with Hironaka's original induction argument [Hi 1, chap. I.2, p. 170], where four (relatively complicated) inductive statements are interwoven.

We start from a mobile $\mathcal{M}$, associate to it local setups $(J_n, \dots, J_1)$ which in turn define the local invariant $i_a(\mathcal{M})$. Its top locus Z defines the first center and the blowup $W' \to W$. Looking at the truncated setup and the truncated invariant, we define the transform of the mobile $\mathcal{M}'$ in W' by descending induction on the dimension (i.e., D'_i and E'_i are defined for descending index i). For this construction we need the transversality properties of D, E and Z in W and the commutativity relations for all diagrams appearing for the ideals of the truncated setups with respect to the blowup $W' \to W$. Again, these diagrams are run through by descending dimension.

Once we have defined the transform of the mobile, we have simultaneously shown that it admits punctual setups, hence an invariant. The commutativity relations show that the invariant does not increase under the blowup. By induction on the dimension and/or by the combinatorial component of the invariant and the blowup of the top locus of shortcuts, it is shown that the invariant actually decreases. This in turn is used together with the commutativity relations to establish the transversality properties for D', E' and Z' in W'.

We see here that the implications spiral up along the resolution process. This can be schematized as follows:

$$\begin{array}{lccccc}
\downarrow & & \text{commutativity'} & \longrightarrow & \dots & \\
 & & & \nwarrow & & \\
W' & & & \text{transversality'} & & \\
 & & & & \nwarrow & \\
\downarrow & & \text{commutativity} & \longrightarrow & \text{decrease of invariant} & \\
 & & & \nwarrow & & \\
W & & & \text{transversality} & & \\
 & & & & \nwarrow & \\
\downarrow & & & & \dots &
\end{array}$$

From this it is clear why this type of reasoning is a cartesian induction: The properties *commutativity* and *decrease* are proven by descending horizontal induction on the embedding dimension and refer to the vertical map given by the blowup; the property *transversality* is proven by vertical induction on the sequence of blowups and refers to the horizontal structure in W.

12. Examples. To see whether we have really understood the preceding constructions, let us carry them out in concrete examples. You are invited to sharpen your pencil. More examples can be found in [EV 2], [BM3], [BM5], [BS1], [BS 3]. As the number of charts increases quickly after each blowup, we shall restrict ourselves sometimes to the most interesting points of the exceptional divisor and thus compute only local resolutions (more precisely, the local data of the resolution along a certain valuation). The first two examples are rather trivial but a good testing ground to become familiar with the whole story of mobiles and their setups.

Example 1. Plane curve. Let us return to example 1 from the beginning and see what the invariant comes up to. The mobile consists of the ideal $J = (x^p + y^q)$ with $p \le q$, control $c_+ = 1$ and empty handicaps D and E. The setup at $a = 0$ is given by the flag $W_2 = W = \mathbb{A}^2$ and $W_1 = \{x = 0\}$ with ideals $J_2 = I_2 = P_2 = K_2 = J$

and $J_1 = I_1 = P_1 = K_1 = (y^q)$ (since it was assumed that $q \geq p$). The invariant $i_a(\mathcal{M}) = (o_2, k_2, m_2, o_1, k_1, m_1)$ at the origin is

$$i_a(\mathcal{M}) = (p, p, (0,0), q, q, (0,0))$$

with $(0,0)$ the two trivial combinatorial invariants. Its top locus is the origin, so that Z will be set equal to $\{0\}$ yielding the point blowup $W' \to W$ of the plane. At the origin of the x-chart the order of I_2 drops to 0 and the ideal J' is monomialized there. The complement of the origin of the x-chart lies entirely in the y-chart, to which we may therefore restrict. There, the total transform is given by $x^p y^p + y^q = y^p(x^p + y^{q-p})$ with $M_2' = (y^p)$ and $I_2' = (x^p + y^{q-p})$. The equiconstant points for I_2 lie in $\{x = 0\}$. Hence the origin a' of the y-chart is the only possible candidate for an equiconstant point. If $q < 2p$, the order of I_2' will have dropped at a', and we put, similarly to points outside the origin, $D_1' = \emptyset$, $E_2' = Y'$ and $E_1' = \emptyset$. The hypersurface W_2' at a' has been newly chosen (and will be $\{y = 0\}$). If $q \geq 2p$, the origin is an equiconstant point for I_2 and we can take $W_1' = W_1^\curlyvee$. We get $D_1' = \{y^{q-p} = 0\}$ and $E_2' = E_1' = \emptyset$. Therefore $J_1' = M_1' \cdot 1 = (y^{q-p})$ is resolved, and the invariant will be

$$i_a(\mathcal{M}) = (p, p, (0,0), 0, 0, (q-p, *))$$

with $(q-p, *)$ the combinatorial invariant of $M_1' = (y^{q-p})$ (we do not specify the label here). The next center is the origin a' of the y-chart, and this continues in the same fashion until the order of I_2 at the origin of the respective y-charts has dropped below p.

Example 2. Cylinder over plane curve in $\mathbb{A}^3$. As a variant of the preceding example, consider the surface in $W = \mathbb{A}^3$ defined by $x^p + y^q$. The same considerations as before apply, except that the invariant has four more zero components and that the center is always the z-axis. The resolution is the cartesian product of the resolution of example 1 with the z-axis.

Example 3. Plane curve embedded in $\mathbb{A}^3$. The mobile consists of $J = (x^p + y^q, z)$, $c_+ = 1$ and D and E empty. The invariant $i_a(\mathcal{M})$ at the origin is

$$i_a(\mathcal{M}) = (1, 1, (0,0), p, p, (0,0), q, q, (0,0))$$

with $(0,0)$ the two trivial combinatorial invariants. Its restriction to $\{z = 0\}$ equals the invariant of the mobile of example 1. The resolution is induced by the resolution of example 1, taking the same centers, but embedded in three-space.

Example 4. Whitney umbrella. Let us now consider the surface $x^2 + yz^2$ in $W = \mathbb{A}^3$ (for notational reasons, we have replaced here $y^2 z$ by yz^2). It is immediately checked that blowing up its singular locus, which is the y-axis, removes all singularities and yields a regular scheme. This is, however, not the way our invariant will proceed, because it is confined to a completely systematic treatment of the singularities and cannot recognize advantageous ad hoc centers.

The mobile is defined similarly as before, with $J = (x^2 + yz^2)$, control $c_+ = 1$ and empty handicaps D and E. At the origin $a = 0$ of W the hypersurfaces $W_2 = \{x = 0\}$ of $W_3 = W$ and $W_1 = \{y = 0\}$ of W_2 will define osculating hypersurfaces for $P_3 = I_3 = J_3 = K_3 = J = (x^2 + yz^2)$ and $P_2 = I_2 = J_2 = K_2 = (yz^2)$. We have $J_1 = I_1 = P_1 = K_1 = (z^3)$. The invariant $i_a(\mathcal{M}) = (o_3, k_3, m_3, \ldots, o_1, k_1, m_1)$ equals at $a = 0$ the vector

$$i_a(\mathcal{M}) = (2, 2, (0,0), 3, 3, (0,0), 3, 3, (0,0))$$

with trivial combinatorial components $(0,0)$. For $a \neq 0$ but inside the y-axis (which is the top locus of $J_3 = I_3$), we have

$$i_a(\mathcal{M}) = (2, 2, (0, 0), 2, 2, (0, 0), 0, 0, (0, 0)).$$

The top locus of the invariant and hence the first center of blowup is the origin of $\mathbb{A}^3$. Let $W' \to W$ be the corresponding blowup with exceptional component Y' isomorphic to $\mathbb{P}^2$. The scheme W' is covered by three affine charts. The equiconstant points of I_3 lie in the weak transform $W_2^{\curlyvee}$ of W_2.

In points where the order of I_3 has decreased we set $E_3' = Y'$ because we will have to choose there a new osculating hypersurface W_2' for $P_3' = I_3'$, and it may fail to be transversal to Y' (thus, Y' is a 'dangerous' component). Then $E_2' = E_1' = \emptyset$ since no further transversality problem occurs. The combinatorial handicaps D_i' defining M_i' are given by the transformation rules for J_i and I_i.

So let us determine the equiconstant points for I_3. If there are some, the top locus of $\mathcal{M}'$ will be contained in the locus of equiconstant points for I_3. So it is reasonable to look for these first. As any equiconstant point must lie in $W_2^{\curlyvee}$ and since $W_2 = \{x = 0\}$, there are no equiconstant points in the x-chart. At the origin of the z-chart we have weak transform $I_3' = I_3^{\curlyvee} = (x^2 + yz)$. This is hence an equiconstant point, and $W_2' = W_2^{\curlyvee} = \{x = 0\}$ is osculating for $P_3' = I_3'$. As $E_3' = \emptyset$ at points which are equiconstant for I_3, we have $Q_3' = 1$ and $K_3' = P_3' = I_3'$. The coefficient ideal J_2' of K_3' in W_2' is $(yz) = (z) \cdot (y)$ with factors $M_2' = (z)$ and $I_2' = I_2^{\curlyvee} = (y)$. Hence the order of I_2' has dropped at this point. The first components of the invariant will be

$$i_{a'}(\mathcal{M}') = (2, 2, (0, 0), 1, \ldots)$$

which implies — as we will see in a moment — that the origin of the z-chart does not lie in the top locus of $\mathcal{M}'$. As we are only interested in finding this top locus in order to know the next center, we need not compute the further components of the invariant at this point. The points outside the origin of the z-chart lie all in the y-chart, so we may restrict ourselves from now on to this chart.

In the y-chart, the weak transform of I_3 equals $I_3' = I_3^{\curlyvee} = P_3' = (x^2 + yz^2)$. The origin is the only point of Y' in this chart where the order of I_3 has remained constant (the other points lie on the line $\{x = z = 0\}$ and are thus outside Y'). Let us therefore compute the invariant at the origin a' of this chart. We have $W_2' = W_2^{\curlyvee} = \{x = 0\}$ osculating for $P_3' = I_3'$, and as $E_3' = \emptyset$ and $Q_3' = 1$ we get $K_3' = P_3' = (x^2 + yz^2)$. The coefficient ideal J_2' of K_3' in W_2' equals $(yz^2) = (y) \cdot (z^2)$ with factors $M_2' = (y)$ and $I_2' = I_2^{\curlyvee} = P_2' = (z^2)$. As the order of I_2 has decreased at a', we will have $E_2' = Y' = \{y = 0\}$, so that $Q_2' = (y)$ and $K_2' = (yz^2)$. A new osculating hypersurface W_1' of W_2' has to be chosen for P_2' (and therefore E_2' had to be chosen equal to Y'). We will take of course $W_1' = \{z = 0\}$ (which is only by chance transversal to Y'). The coefficient ideal J_1' of K_2' in W_1' equals $(y^3) = (y^3) \cdot 1$ with factors $M_1' = (y^3)$ and $I_1' = 1$. The invariant of $\mathcal{M}'$ at the origin a' of the y-chart thus equals

$$i_a(\mathcal{M}) = (2, 2, (0, 0), 2, 3, (0, 0), 0, 0, (3, *))$$

where $(3, *)$ denotes the combinatorial invariant of M_1'. Outside the exceptional component the invariant of $\mathcal{M}'$ equals along the line $\{x = z = 0\}$ the value of the invariant of $\mathcal{M}$ at points in the y-axis of W outside the origin, say

$$i_a(\mathcal{M}) = (2, 2, (0, 0), 2, 2, (0, 0), 0, 0, (0, 0)).$$

This shows that the top locus of $\mathcal{M}'$ consists of one point, the origin a' of the y-chart. It will be the next center Z'.

> Here, the center is so small because — as $Q'_2 = (y)$ appears as a factor in K'_2 — we will have to separate Y' and W'_1 first, according to our strategy for how to treat transversality problems. Once this is done, a larger center will be chosen.

Let $W'' \to W'$ be the induced blowup. For simplicity, we shall only compute the value of the invariant of $\mathcal{M}''$ at the origin a'' of the y-chart. It will turn out to be locally at a'' a line, the y-axis $\{x = z = 0\}$; hence it will globally be a regular curve.

At the origin of the y-chart we will have $I''_3 = P''_3 = (x^2 + yz^2)$ with osculating hypersurface $W''_2 = (W'_2)^\curlyvee = \{x = 0\}$. As $E''_3 = \emptyset$ we get $K''_3 = P''_3$ with coefficient ideal $J''_2 = (yz^2) = (y)(z^2)$ in W''_2. The factors are $M''_2 = (y)$ and $I''_2 = (I'_2)^\curlyvee = (z^2)$. As the order of I'_2 has remained constant, we will have $E''_2 = (E'_2)^\curlyvee = (Y')^\curlyvee$. As Y' was given at $Z' = \{a'\}$ by $y = 0$, this divisor does not pass through a''. Hence $Q''_2 = 1$ at a'' (in contrast to what has happened in W and W'), which implies that $K''_2 = P''_2 = (z^2)$ is bold regular. The order of K'_2 (and hence the complexity of the transversality problem) has improved. Actually, Y' and W'_1 have been separated by the last blowup $W'' \to W'$. Taking $W''_1 = \{z = 0\}$ we get that $J''_1 = 1$ (by definition of the descent in dimension and since the coefficient ideal of K''_2 would be zero). This gives for the invariant

$$i_{a''}(\mathcal{M}'') = (2, 2, (0, 0), 2, 2, (0, 0), 0, 0, (0, 0)).$$

Now the invariant is constant along the line $\{x = z = 0\}$, which will therefore form our next center Z'' (which coincides with the top locus of I''_3). So it took us two auxiliary blowups to arrive at a situation where the invariant chooses the desired line as center. In the next blowup, the order of I'''_3 will be at most one at all points, so there will be no more equiconstant points for I''_3. The underlying scheme will be regular. We leave it to the reader to compute the whole invariant at the points of W''' and to complete the monomialization of I_3.

Example 5. More general surface singularity. Let J be the ideal in $W = \mathbb{A}^3$ generated by the polynomial $f = x^3 - 3x^2y + (y^2 + z^3)^2$. This is already quite complicated. The mobile $\mathcal{M}$ we associate to J will consist of the ideal J, the control $c_+ = 1$ and empty handicaps D and E. The factorization $J = M \cdot I = 1 \cdot I$ of J is trivial with $I = J$. In order to determine the first center of blowup we have to compute a punctual setup for $\mathcal{M}$. As before, we add indices to distinguish ideals in various dimensions. Thus $J_3 = I_3 = J = I$, $M_3 = 1$ and $W = W_3$. We place ourselves at the origin of $\mathbb{A}^3$. The first thing to do is choose an osculating hypersurface for f. The order of f at 0 is 3, and its tangent cone equals $x^3 + 6x^2y$ in the given coordinates. The minimal number of variables appearing in the tangent cone is 2, because terms involving y cannot be eliminated. Both $x = 0$ and $y = 0$ are adjacent hypersurfaces, but $\{x = 0\}$ is not osculating.

Expanding f with respect to x, we wish to eliminate the coefficient of $x^{3-1} = x^2$ by a coordinate change (recall here the definition of osculating hypersurfaces). Clearly the change $x \to x + y$ will do the job. In the new coordinates we have $f = x^3 + 3xy^2 + y^3 + (y^2 + z^3)^2$ and $V = W_2 = \{x = 0\}$ is osculating. The companion ideal P_3 of I_3 equals I_3, and the transversality ideal Q_3 is trivial equal to 1, so that $K_3 = P_3 \cdot Q_3 = I_3$. We now have to take the coefficient ideal J_2 of

K_3 in W_2. It is generated by $y^{3/1}$ and $y^3 + (y^2 + z^3)^2$, hence by y^3 and $2y^2z^3 + z^6$. Thus $J_2 = (y^3, 2y^2z^2 + z^6) = I_2 = P_2 = K_2$. The tangent cone of P_2 is generated by y^3, and $W_1 = \{y = 0\}$ is osculating for P_2. The next coefficient ideal J_1 of P_2 is generated by z^4. The invariant at $a = 0$ will be

$$i_a(\mathcal{M}) = (3, 3, (0, 0), 3, 3, (0, 0), 4, 4, (0, 0)),$$

because all combinatorial components will be $(0, 0)$. It is checked that the top locus of $i_a(\mathcal{M})$ will be the origin, which is hence our first center. Let $W' \to W$ be the corresponding blowup. The exceptional component Y' is isomorphic to projective space $\mathbb{P}^2$. Among the many points to consider in Y', we pick up the most interesting ones, namely the equiconstant points with respect to I_3. These lie in the weak transform $V' = W_2'$ of our hypersurface $V = W_2$; hence we may restrict ourselves to the y- and z-charts.

> Outside W_2', the order o_3 has dropped. There D_2' and D_1' will be set empty, and D_3' will be chosen equal to Y' so that $J_3' = J_3^* \cdot I(Y')^{-1}$ factors into $J_3' = M_3' \cdot I_3'$ with $I' = I_3^\curlyvee$ and $M' = I_W(D_3')$. The transversal handicap E' is given by $E_3' = Y'$ (since a new W_2' has to be chosen and it may not be transversal to W_3') and $E_2' = E_1' = \emptyset$.

In the y-chart we get total transform $f^* = x^3y^3 + 3xy^3 + y^3 + (y^2 + y^3z^3)^2 = y^3(x^3 + 3x + 1 + y(1 + yz^3)^2)$. The order of the weak transform $f^\curlyvee = (x^3 + 3x + 1 + y(1 + yz^3)^3)$ at the origin of this chart has dropped from 3 to 0; hence it is not an equiconstant point (though it lies in $W_2' = \{x = 0\}$). The handicaps are therefore defined analogously as before.

The complement of the origin of the y-chart in $W_2' \cap Y'$ lies entirely in the z-chart, and we may restrict w.l.o.g. to points of these charts. The total transform of f equals there $f^* = x^3z^3 + 3xy^2z^3 + y^3z^3 + (y^2z^2 + z^3)^2 = z^3(x^3 + 3xy^2 + y^3 + z(y^2 + z)^2)$. The weak transform $f' = x^3 + 3xy^2 + y^3 + z(y^2 + z)^2$ has order 3 at the origin of this chart, and only there. This is the only equiconstant point for J_3. Outside we set $D_3' = \{z^3 = 0\}$, $E_3' = Y'$ and the remaining handicaps equal to $\emptyset$.

At the origin of the z-chart the hypersurface $W_2' = \{x = 0\}$ is osculating for $I_3' = P_3'$. We set $D_3' = \{z^3 = 0\}$ and $E_3' = \emptyset$, because W_2' is the weak transform of W_2 and hence automatically transversal to Y' (note here that this coincides with the definition $E_3' = E_3^\curlyvee$ of E_3' at equiconstant points). For the remaining handicaps we have to compute the coefficient ideal J_2' of K_3' in W_2'. We have $J_3' = I(y')^3 \cdot I_3'$, and as $o_3' = 3$ and the controls are 1, the companion ideal P_3' equals $P_3^\curlyvee = I_3'$ (cf. "Commutativity"). From $Q_3' = 1$ we get $K_3' = P_3' = I_3' = (x^3 + 3xy^2 + y^3 + z(y^2 + z)^2)$ of order $k_3' = k_3 = 3$. The control for J_2' is therefore 3. The coefficient ideal of K_3' in W_2' is $J_2' = J_2^! = J_2^* \cdot I_{W_2'}(Y' \cap W_2')^{-3} = (y^3, 2y^2z^2 + z^3)^* \cdot z^{-3} = (y^3z^3, 2y^2z^4 + z^6) \cdot z^{-3} = (y^3, 2y^2z + z^3)$. We see that $J_2' = I_2' = I_2^\curlyvee$ with $M_2' = 1$ according to the definition $D_2' = D_2^* \cdot I_{W_2'}(Y' \cap W_2')^{o_2 - c_3} = \emptyset$ at points where $k_3' = k_3$. The order o_2' of I_2' has remained constant equal to 3, and $P_2' = K_2' = I_2'$. The control for J_1' is 3, and $W_1' = W_1^\curlyvee$ yields coefficient ideal $J_1' = z^3 \cdot 1$ with $I_1' = I_1^\curlyvee = 1$. Thus the invariant has dropped to

$$i_{a'}(\mathcal{M}') = (3, 3, (0, 0), 3, 3, (0, 0), 0, 0, (3, *)).$$

Here the components $(3, *)$ indicate the combinatorial invariant in dimension 1, given by the order of $M_1' = (z^3)$ and its label (which we do not specify). The remaining handicaps are $D_1' = \{z^3 = 0\}$ and $E_2' = E_1' = \emptyset$. The center will be the origin of the z-chart.

The reader will realize that even though systematic, the computation of the invariant becomes rather involved. We leave it as an exercise (tedious, as we admit, but instructive) to complete the resolution for this mobile on a blank sheet. Some reader will complain that the author was too lazy to type the details; at any rate, it is better to do the computation on his own.

Example 6 (Centers outside strict transform)**.** While resolving mobiles, the centers are always chosen inside the support of the ideal J and its weak transforms. However, they need not lie inside the support of the strict transforms of the ideal, and the intersection may even be singular, as shown in the following example of S. Encinas. Nevertheless, the centers always map to the singular locus of the original ideal.

Take the ideal $J = J_3 = I_3 = (x^2 + y^3, z^3 - w^3)$ in $W = \mathbb{A}^4$ and the control $c_+ = 1$. The first center will be the origin. Let $W' \to W$ be the induced blowup, and let us look at the origin of the y-chart. The weak transform of I_3 equals $I_3' = I_3^\curlyvee = (x^2 + y, y(z^3 - w^3))$, whereas the strict transform would be $I_3^{st} = (x^2 + y, z^3 - w^3)$.

We have $P_3 = J_3$. As the order of I_3 has dropped, we have $E_3' = Y'$ and $Q_3' = (y)$, so that $K_3' = I_3' \cdot Q_3' = (x^2 y)$. The next center will be the plane $\{x = y = 0\}$ in W'. It does not lie in the support of the strict transform I_3^{st}, and the intersection is even singular, defined in $\{x = y = 0\}$ by $z^3 - w^3 = 0$.

13. Resolution of schemes. Now that we have discussed in detail the construction of mobiles and their setups, we shall return to our original object of interest, singularities of schemes. The resolution of mobiles can be used in various ways to construct a strong resolution of reduced singular subschemes X of W. One possibility goes as follows (for a somewhat different reasoning, see [EV 3]).

We may assume that X is different from W and that W is equidimensional. Let $\mathcal{J}$ be the ideal of X in W. Associate to it the mobile $\mathcal{M} = (\mathcal{J}, c, D, E)$ with control $c = 1$ and empty handicaps D and E (we omit here the index $+$ in c). We shall look closely at the various stages of the resolution process for $\mathcal{M}$. Its final goal will be to monomialize the total transform of $\mathcal{J}$. At any stage W' of the resolution of $\mathcal{M}$ the controlled transform $\mathcal{J}'$ of $\mathcal{J}$ defines a subscheme of W' formed by the strict transform X' of X and some components inside the exceptional locus.

As the final controlled transform of $\mathcal{J}$ equals 1 (since we will decrease the order of the controlled transforms of $\mathcal{J}$ below $c = 1$), there corresponds to each component of X a (uniquely determined) stage where the strict transform of this component has become regular and has been taken locally as the center of the next blowup. Let X_1 denote the union of those components of X which reach this stage first. The corresponding strict transform X_1' of X_1 at the indicated stage is regular and transversal to the exceptional locus.

Write $X' = X_1' \cup X_2'$ with X_2' the strict transform of the remaining components of X. We stop here the resolution process of the mobile $(\mathcal{J}, c, D, E)$ and define a new mobile whose resolution will be given by a sequence of blowups which separates X_2' from X_1'. Omitting primes, let $\mathcal{K}$ be the ideal of X_2 in W. Let $\mathcal{J}$ be the coefficient ideal of $\mathcal{K}$ in X_1 with control c set equal to the maximum on X_1 of the order of $\mathcal{K}$ in W. Set all handicaps D_i and E_i empty with the exception of the first member E_n

of E (n is the dimension of X_1) for which we take the exceptional locus produced so far.

Resolve the mobile $(\mathcal{J}, c, D, E)$. The controlled transforms of $\mathcal{J}$ are the coefficient ideals of the weak transforms of $\mathcal{K}$ as long as the maximum of the order of $\mathcal{K}$ in W along X_1 remains constant. Therefore the resolution of $(\mathcal{J}, c, D, E)$ will make this maximum drop. Hence also the maximum of the order of the strict transform of $\mathcal{K}$ in W along X_1 drops. Iterating this process the final strict transform X_2' of X_2 will be separated from the weak transform X_1' of X_1. Now induction on the number of components applies to construct a sequence of blowups which makes X_2' regular and transversal to the exceptional locus. Thus X has become a regular scheme.

The properties *embeddedness* and *equivariance* of a strong resolution of schemes follow from the specific resolution of mobiles we have constructed. The reader is invited to prove this with all details. The restriction to X of the resolution $W' \to W$ of X does not depend on the embedding of X in W since, under embeddings of W into some W^+, the restriction of $i(\mathcal{M}^+)$ to W equals $i(\mathcal{M})$ (cf. this with example 3 from above). This proves *excision*.

The sequence of blowups $W' \to W$ we have constructed for the singular scheme X via well chosen mobiles thus satisfies all properties of a *strong resolution* of X. The proof of the Hironaka Theorem on Resolution of Singularities is now completed.

This concludes the main body of the paper. Readers who have gotten this far may judge whether at least two of the three objectives mentioned in the introduction were met: Easy reading and good understanding. If so, the reader should now be in perfect shape to answer affirmatively his neighbour's question:

Do you know how to prove resolution of singularities in characteristic zero?

14. Problems in positive characteristic. To emphasize the limitations of the induction argument used to establish resolution in characteristic zero, we now address the difficulties in positive characteristic which prevent extending the above proof to this case. We will describe two examples. The first exhibits a sequence of equiconstant points which leaves any regular hypersurface accompanying the resolution process. This shows that hypersurfaces of permanent contact need not exist in positive characteristic and that the accompanying hypersurfaces have to be changed from time to time.

The second example illustrates what can happen if one has to replace the accompanying hypersurface at a certain stage so as to contain after the next blowup the subsequent equiconstant points or so that weak maximal contact is ensured. As a matter of fact, the invariant we have constructed may increase when choosing instead of osculating hypersurfaces (which may not exist but would persist under blowup) hypersurfaces of weak maximal contact (which always exist but need not persist under blowup and therefore have to be changed in the course of the resolution). This increase destroys the vertical induction on the resolution invariant. It is *not* a counterexample to the existence of resolution of singularities in positive characteristic; it only shows that the proof of characteristic 0 does not go through without applying substantial modifications.

Example 1 (Narasimhan [Na 1], [Na 2], [Mu])**.** In positive characteristic, the top locus of an ideal may not be contained locally in a regular hypersurface. Take K of characteristic 2 and $f = x^2 + yz^3 + zw^3 + y^7w$ of order 2 at 0. Check that $\mathrm{top}(f) = V(f, z^3 + y^6 w, yz^2 + w^3, zw^2 + y^7)$ and that the parametrized curve $(t^{32}, t^7, t^{19}, t^{15})$ in $\mathbb{A}^4$ has image equal to $\mathrm{top}(f)$. From this it follows that there cannot exist locally at 0 a regular hypersurface V of $\mathbb{A}^4$ which contains $\mathrm{top}(f)$.

Take now a regular hypersurface V passing through $a = 0$. We claim that for any sequence of point blowups whose first center is the origin, the sequence of equiconstant points above a will leave eventually the strict transforms of V. Indeed, as the point blowups keep $\mathrm{top}(f)$ unchanged outside 0, the order of the transforms of f will remain constant equal to 2 at points above points of $\mathrm{top}(f)$ outside 0. The strict transform of the locus $\mathrm{top}(f)$ will therefore consist of points of order 2 for f, by the upper semicontinuity of the order. In particular, the points above 0 which lie in these strict transforms will all be equiconstant points above 0.

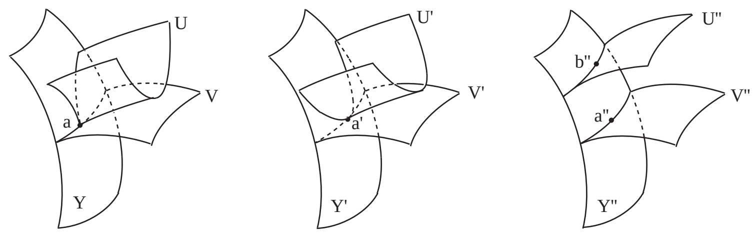

FIGURE 14. *Failure of permanent contact*: Any regular hypersurface V may lose equiconstant points such as b''. A new hypersurface U'' has to be chosen. Its image in W may be singular.

But by a sequence of point blowups, the curve $\mathrm{top}(f)$ will always be separated from the hypersurface V and its strict transforms (since it is not contained in V). Combining both observations we conclude that the equiconstant points above 0 will eventually leave the strict transforms of V (see Figure 14).

We complement the discussion of positive characteristic by the example of an ideal and a sequence of blowups in centers contained in its top locus such that the characteristic zero invariant associated to mobiles increases.

Moh was the first to give an example where the maximum of the order of the first coefficient ideal with respect to local hypersurfaces increases at an equiconstant point of the original ideal [Mo 2, ex. 3.2], [Ha 2, ex. 16]. For a comprehensive description of how to construct such examples in positive characteristic, see [Ha 5].

Example 2 (Hauser, [Ha 5])**.** Consider a sequence of three local blowups $W^3 \to W^2 \to W^1 \to W$ at points a^i in W^i with $W = W^0$ a regular scheme of dimension three. All blowups are point blowups and will be considered locally at specified points. For given local coordinates x, y, z in W at $a = 0$, the first map $W^1 \to W$ is the blowup of W with center the origin, considered at the origin a^1 of the y-chart.

The second $W^2 \to W^1$ is the blowup of W^1 with center a^1, considered at the origin a^2 of the z-chart. Hence, a^1 and a^2 will be the origins of the respective charts, and a^2 lies in the intersection of the two exceptional components in W^2 having occurred so far.

The third blowup $W^3 \to W^2$ is no longer monomial and involves also a translation. Its center is the origin a^2 of the z-chart of W^2, but the blowup is considered in the z-chart of W^3 at the point a^3 with coordinates $(0, 1, 0)$. Said differently, this blowup is the composition of the monomial point blowup at the origin of the z-chart

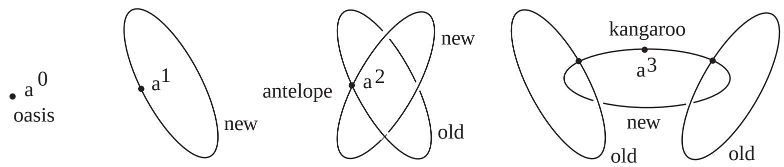

FIGURE 15. The picture shows the configuration of the exceptional components inside the hypersurface $V = \{x = 0\} \subset W$ through the sequence of blowups. For the notions of oasis, antelope and kangaroo point, see [Ha 5].

followed by the translation $y \to y + 1$. Hence a^3 belongs to the new exceptional component Y^3 in W^3, but lies outside the strict transforms of the two exceptional components through a^3 (see Figure 15).

We choose now a specific principal ideal J in W at 0 and look at its various transforms together with the respective coefficient ideals. Take for J the ideal in W generated by the polynomial $f = f^0 = x^2 + y^7 + yz^4$ and let V be the hypersurface of W defined by $x = 0$. The resulting sequence of strict transforms f^i of f and V^i of V is

$$\begin{aligned}
f^0 &= x^2 + 1 \cdot (y^7 + yz^4), && V^0 : x = 0,\\
f^1 &= x^2 + y^3 \cdot (y^2 + z^4), && V^1 : x = 0,\\
f^2 &= x^2 + y^3 z^3 \cdot (y^2 + z^2), && V^2 : x = 0,\\
f^3 &= x^2 + z^6 (y+1)^3 \cdot ((y+1)^2 + 1) && \\
&= x^2 + z^6 \cdot ((y^2+1)(y+1)y^2), && V^3 : x = 0.
\end{aligned}$$

Here, the monomial factors in front of the parentheses denote exceptional components of the restriction of f^i to V^i (more precisely, of the coefficient ideal of f^i in V^i). The order of f^i at a^i has remained constant equal to 2 for all i. The hypersurface $V = \{x = 0\}$ has weak maximal contact with f at 0, and the same holds for its strict transforms V^1 and V^2.

But the hypersurface V^3 no longer has weak maximal contact with f^3: The coefficient ideal of f^3 in V^3 equals $z^6 \cdot ((y^2+1)(y+1)y^2) = z^6 \cdot (y^5 + y^4 + y^3 + y^2)$. After deleting the exceptional factor z^6, its order at $a^3 = 0$ is 2. In characteristic 2, this is not the maximal possible value. Indeed, the hypersurface $U^3 = \{x + yz^3 = 0\}$ in W^3 yields coefficient ideal $z^6 \cdot (y^5 + y^4 + y^3)$, which, after deletion of z^6, has order 3 at 0.

We compute the first two components $(o_n, o_{n-1}) = (o, o_-)$ of our invariant (we neglect here the transversality problem and the other components k_i and m_i of the invariant) along the sequence of local blowups. Let o^i be the order of f^i at a^i, and let o_-^i denote the maximal value of the order of the coefficient ideal of f^i in a regular hypersurface through a^i, diminished by the exceptional multiplicity. Then

$$\begin{aligned}(o^0, o^0_-) &= (2,5),\\ (o^1, o^1_-) &= (2,4),\\ (o^2, o^2_-) &= (2,2),\\ (o^3, o^3_-) &= (2,3).\end{aligned}$$

Hence the invariant has increased in the last blowup. Moreover, the hypersurface V^0 of weak maximal contact with f^0 at a^0 preserved weak maximal contact only until a^2. In W^3, its transform V^3 had no longer weak maximal contact, and therefore V^3 had to be replaced by a new hypersurface U^3 to ensure weak maximal contact with f^3 (see Figure 16).

Let us look at whether U^3 stems from a regular hypersurface U^0 in W. Blowing it down to W^2 and W^1 yields $U^2 = \{x+yz^3+z^4 = 0\}$ and $U^1 = \{x+yz^3+z^5 = 0\}$. This last has singular image $U^0 = \{xy^4 + y^3z^3 + z^5 = 0\}$ in W. Therefore U^3 is not the strict transform of a regular hypersurface in W.

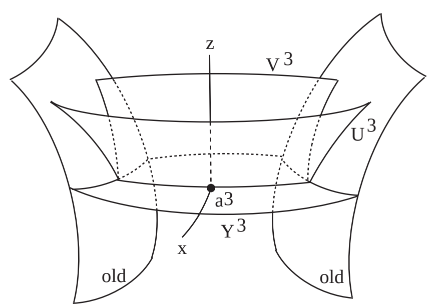

FIGURE 16. The picture shows the configuration of exceptional components at a^3 and the position of the new hypersurface U^3.

But possibly we can modify U^3 slightly to a hypersurface $\tilde{U}^3$ which still has weak maximal contact with f^3 and which does stem from a regular hypersurface $\tilde{U}^0$ in W. It is easy to see that the linear term of the equation of $\tilde{U}^0$ must be x (up to a constant factor). So let us write $g^0 = x + \sum g_{jk}y^jz^k$ for the equation of $\tilde{U}^0$ in W. We get

$$\begin{aligned}g^0 &= x + \sum g_{jk}y^jz^k,\\ g^1 &= x + \sum g_{jk}y^{j+k-1}z^k,\\ g^2 &= x + \sum g_{jk}y^{j+k-1}z^{j+2k-2},\\ g^3 &= x + \sum g_{jk}(y+1)^{j+k-1}z^{2j+3k-4}.\end{aligned}$$

This yields a monomial yz^3 (which is the monomial of f^3 which has to be eliminated by the local isomorphism mapping V^3 onto $\tilde{U}^3$ in order to increase the order of the coefficient ideal) in its expansion if and only if, for some j, k, the sum $j+k$ is even (recall that we are in characteristic 2) and $2j+3k = 7$. From the last equality follows $k = 1$ and $j = 2$, for which $j+k$ is odd. Hence no regular $\tilde{U}^0$ exists in W whose transform $\tilde{U}^3$ in W^3 has weak maximal contact with f^3 at a^3.

On first view, the example above looks quite artificial, having no internal structure or general pattern. But there is some "rule" behind it. The first two monomial blowups in opposite charts are needed to produce two exceptional components and a point a^2 in their intersection. The third and last blowup is characterized by the "disappearance of the two exceptional components" when passing from a^2 to a^3. It is here that the key phenomenon occurs, namely the increase of the order of the coefficient ideal $y^3z^3 \cdot (y^2+z^2)$ of f^2 in V^2 (after having factored from it the exceptional monomial y^3z^3).

It seems that this type of construction is necessary to produce an example for the failure of the persistence of weak maximal contact in positive characteristic. It turns out that the above construction produces counterexamples with increasing invariant if and only if the exponents and the coefficients of f are chosen in a very specific manner. The necessary conditions are as follows:

- The residues modulo p of the exceptional multiplicities of f^2, i.e., of the exponents of the monomial factors in front of the parentheses, must satisfy a prescribed arithmetic inequality (for surfaces, both must be positive and their sum must not exceed p).
- The order of the coefficient ideal of f^2 in V^2 must be a multiple of the characteristic.
- The coefficients of the weighted tangent cone of f^2 must satisfy precise linear relations. They are uniquely determined up to coordinate changes and explicitly related to the position of the point a^3 on the exceptional divisor.

The given example is the simplest one with these properties. For higher order examples, the coefficients of the weighted tangent cone of f are also unique up to coordinate changes. The actual values of the coefficients of the defining equations of the singularity seem to play a decisive role in positive characteristic. All this and more is explained in the forthcoming paper [Ha 5].

APPENDIX

Appendix A: Order of ideals. In the five sections of the appendix we collect some basic facts from commutative algebra and the theory of blowups which are constantly used in the article. In addition, all concepts of this paper will be properly defined and the notations will be listed for quick reference. For further reading we refer to [Hi 1], [ZS], [EV 2], [BM1].

Let I be a coherent ideal sheaf on a regular ambient scheme W of finite type over a field K. For $a \in W$ a closed point let I_a denote the stalk of I at a, and m_a the maximal ideal of a in the stalk of the structure sheaf $\mathcal{O}_{W,a}$ of W. Let Z be a closed subscheme of W with defining ideal $I(Z)$. The order of I along Z is the maximal power of $I(Z)$ containing I in the localization $\mathcal{O}_{W,Z}$,

$$\mathrm{ord}_Z = \max\,\{k,\ I \subset I(Z)^k \text{ in } \mathcal{O}_{W,Z}\}.$$

The order of I at a coincides with the maximal power of m_a containing the stalk I_a. The zero ideal has infinite order. We have

$$\mathrm{ord}_Z = \min_{a \in Z}\ \mathrm{ord}_a\, I$$

where it suffices to take the minimum over the closed points a of Z. Denote by $\mathrm{ord}\ I : W \to \mathbb{N} \cup \{\infty\},\ a \to \mathrm{ord}_a\, I$ the order function on W. It satisfies various functorial properties.

The order is invariant with respect to field extensions, passage to the completion and local isomorphisms, more generally, with respect to smooth morphisms $W^- \to W$. It does not increase under localization. This has been proven by Zariski-Nagata using resolution of curves [Hi 1, Thm. 1, p. 218]. It does not increase under blowup when passing from I to the weak transform $I^\curlyvee$, provided the order of I is constant along the center [Hi 1, Lemma 8, p. 217]. It is upper semicontinuous , so that the locus of points $\mathrm{top}(I,c) = \{a \in W,\ \mathrm{ord}_a I \geq c\}$ is closed in W for any constant $c \in \mathbb{N}$. This can be seen as follows.

By the transitivity of the Zariski-topology under field extensions, we may assume that K is algebraically closed. Let $x_1, \ldots, x_n$ be a regular system of parameters of $\mathcal{O}_{W,a}$. By Cohen's structure theorem, the completion $\hat{\mathcal{O}}_{W,a}$ is isomorphic to the formal power series ring $K[[x]]$.

We assume for simplicity that $W = \mathbb{A}^n$ and place ourselves at the origin of $\mathbb{A}^n$. If I is a principal ideal generated by some f with expansion $f(x) = \sum_{\alpha\in\mathbb{N}^n} c_\alpha x^\alpha$ in $\hat{\mathcal{O}}_{W,a}$, the expansion at a is given by $f(x) = \sum_{\alpha\in\mathbb{N}^n} d_\alpha(a)(x-a)^\alpha$ where $f(x+a) = \sum d_\alpha(a)x^\alpha$. Then $\mathrm{ord}_a f \geq c$ if and only if all $d_\alpha(a)$ with $|\alpha| < c$ are zero. By binomial expansion of $f(x+a)$, the d_α are polynomials in a whose coefficients are $\mathbb{Z}$-linear combinations of the c_α's. This shows that the locus of points a where $\mathrm{ord}_a f \geq c$ is closed in $\mathbb{A}^n$. For arbitrary ideals I, the locus of order $\geq c$ is the intersection of the corresponding loci of a generator system of I. The assertion follows. You may also consult [BM1, p. 233], [EV 1, p. 208] and [EV 2].

As all our schemes will be assumed to be noetherian, the order of I takes only finitely many values on W.

If X is a subscheme of W of ideal I, we call $\mathrm{ord}_a X = \mathrm{ord}_a I$ the order of X at a. It clearly depends on the embedding of X in W. To see this, just embed X into another ambient scheme W^+ via an inclusion $W \subset W^+$, and the new order of X at a will be 1 if $\dim W^+ > \dim W$.

Exercise 1. Show that $\mathrm{ord}_a I$ coincides for algebraically closed fields with the minimum of the orders of the Taylor expansion of elements of I at a. Use this to prove that the ideal $\overline{I}$ of $K[[x]]$ generated by an ideal I of $K[x]$ has the same order at 0 as I. Moreover, the order of an ideal does not depend on the choice of affine or formal coordinates.

Exercise 2. Show that for $Z \subset \mathbb{A}^n$ a closed and reduced subscheme, the order of I along Z is the minimum of the orders of I at points a of Z.

Appendix B: Computation of top loci. The upper semicontinuity of the order of an ideal implies that any closed subscheme X of W admits a finite stratification given by the order of its defining ideal. More precisely, there exist finitely many locally closed subschemes X_i of X which form a partition of X and such that the order of the defining ideal I of X in W equals i at each point of X_i. The stratum X_o with o the maximal value of the order of X on W is closed and will be called the top locus of X (or of I) in W. The closure $\overline{X}_i$ of X_i decomposes into $\overline{X}_i = X_i \cup X_{i+1} \cup \ldots \cup X_o = \mathrm{top}(X, i)$.

Exercise 3. Let $f(x,y,z) = x^3 + y^k z^m$. Determine according to the values of k and m the locus of points $a \in \mathbb{A}^3$ where f has order ≥ 2 respectively equal to 3.

Exercise 4. Let X be the union of the four coordinate hyperplanes in $\mathbb{A}^4$. Determine the strata of constant order of X. Do the same for Y defined in $\mathbb{A}^4$ by $x^k y^l z^m w^n$ with $k, l, m, n \in \mathbb{N}$.

Exercise 5. Let K be algebraically closed. Show that the support of a (not necessarily reduced) hypersurface $X = V(f)$ equals $\mathrm{top}(f)$ if and only if $f = x^k$ is locally at each point a of X a (positive) power of a coordinate x. Moreover, the number k is locally constant along X.

If the order of $f \in K[x]$ at a is o, the local top locus $\mathrm{top}_a(f)$ is the zero-set of all derivatives of f up to order $o-1$ (including the 0-th derivative).

Example 1. Let $f = x^{2p} + y^{3p} = (x^2 + y^3)^p$ be given in characteristic $p > 3$. Then f has order $2p$ at 0 and $\mathrm{top}(f) = V(\partial_x^\alpha f,\, |\alpha| \leq 2p-1) = V(f)$ is the cusp.

Example 2. Let $f = 0$ define a hypersurface X in $\mathbb{A}^n$. If K has characteristic zero, $\mathrm{top}(X) = \mathrm{top}(f)$ is contained locally at each point in a regular hypersurface H of $\mathbb{A}^n$. To see this, let o be the order of f at $a = 0$. Then the Taylor expansion of f at 0 has order o; hence there is a higher order derivative ∂_x^α with $|\alpha| = o$ such that $\partial_x^\alpha f(0) \neq 0$. Let $\gamma \in \mathbb{N}^n$ be obtained from α by decreasing one positive entry by 1. Then ∂_x^γ has order $o-1$ and $\partial_x^\gamma f$ has order 1 at 0. Thus $\partial_x^\gamma f$ is regular at 0 and its zero-set contains $\mathrm{top}_0(f)$.

Exercise 6. Let $f(x,y) = \sum_i a_i(y) x^i$ be the Taylor expansion of f at 0 with respect to one coordinate, say $x = x_1$, of a regular system of parameters $(x,y) = (x_1, x_2, \ldots, x_n)$ of $\mathcal{O}_{W,a}$, where the coefficients $a_i(y)$ live in $\mathcal{O}_{V,a}$ with $V = \{x = 0\}$. Let o be the order of f at 0, and assume that the characteristic is zero. Then $\mathrm{top}_0(f)$ is the locus of points in $\{x = 0\}$ where the coefficients $a_i(y)$ have order $\geq o - i$ for all i.

Appendix C: Local coordinates for blowups. Let W be a regular scheme of dimension n, and let Z be a closed regular subscheme of dimension d (both schemes will be assumed for simplicity to be equidimensional). Let $\pi : W' \to W$ be the induced blowup with center Z and exceptional component Y', and let $(W', a') \to (W, a)$ denote the corresponding local blowup for any pair of points $a \in Z$ and $a' \in Y'$ above a. We shall assume that the ground field is algebraically closed. As the resolution invariant associated to a mobile is upper semicontinuous, we may and will restrict to closed points a and a'. Let V be a local regular hypersurface of W at a containing Z locally.

Any choice of local coordinates in W at a induces local coordinates in W' at a' and $n-d$ affine charts for W'. We order the coordinates by decreasing indices and thus write $x_n, \ldots, x_1$.

Assume we are given an ideal K in W at a, with coefficient ideal $J = \mathrm{coeff}_V K$ in V, with given factorization $J = M \cdot I$, where M defines a normal crossings divisor in V at a. Let $c = \mathrm{ord}_a K$ and $c' = \mathrm{ord}_{a'} K'$ with $K' = K^\curlyvee$ the weak transform of K in W'. Assume that V has weak maximal contact with K (i.e., maximizes the order of J), and that Z is transversal to the divisor D of V defined by M. For many proofs on the transfom of ideals under blowups it is useful to work in local coordinates for which the blowup has particularly simple form. This is ensured by the following assertions.

There exist local coordinates $x = (x_n, \ldots, x_1)$ of W at a, i.e., a regular system of parameters of $\mathcal{O}_{W,a}$, such that

(1) $a = (0, \ldots, 0)$ with respect to x in W at a.

(2) V is defined in W by $x_n = 0$.

(3) Z is defined in W by $x_n = \ldots = x_{d+1} = 0$.

(4) M is defined in V by $x_{n-1}^{q_{n-1}} \cdots x_1^{q_1}$ for some $q \in \mathbb{N}^{n-1}$.

(5) The weak transform V' of V is given in the coordinates in W' by $x_n = 0$.

(6) The points a' of W' where the order of the weak transform $K^\curlyvee$ of K has remained constant, $c' = c$, are contained in V'.

(7) a' lies in the x_{n-1}-chart of W' with components $(0, 0, a'_{n-2}, \ldots, a'_{j+1}, 0, \ldots, 0)$ with respect to the induced coordinates x in W' at a', where $j - d$ is the number of components of D whose transforms pass through a'.

(8) The blowup $(W', a') \to (W, a)$ is the composition of the linear map ℓ_t : $x \to (x + tx_{n-1})$ in W at a where $t = (0, 0, t_{n-2}, \ldots, t_{j+1}, 0, \ldots, 0)$ has components $t_i = a'_i$ and the monomial blowup π of W in Z considered in the x_{n-1}-chart which maps $(x_n, \ldots, x_1)$ to $(x_n, x_{n-1}, x_{n-2}x_{n-1}, \ldots, x_{d+1}x_{n-1}, x_d, \ldots, x_1)$. Thus $t_i \neq 0$ for $n - 2 \geq i \geq j + 1$ and $t_i = 0$ for $j \geq i \geq 1$ and $i = n - 1$. The map ℓ_t preserves Z and V and the factorization $J = M \cdot I$, but destroys the monomiality of M as in (4) with respect to the given coordinates.

(9) If condition (4) is not required, the coordinates can be chosen so that a' is the origin of the x_{n-1}-chart and that $(W', a') \to (W, a)$ is the monomial blowup given by $(x_n, \ldots, x_1)$ mapping to $(x_n, x_{n-1}, x_{n-2}x_{n-1}, \ldots, x_{d+1}x_{n-1}, x_d, \ldots, x_1)$.

The assertions can be proven as follows. It is clear that $(x_n, \ldots, x_1)$ can be chosen satisfying (1) to (3), and (4) follows immediately from the fact that D and Z are transversal. As for (7), we know by (3) that the exceptional component Y' is covered by the charts corresponding to $x_n, \ldots, x_{d+1}$. As $c' = c$ and x_n is supposed to appear in the tangent cone of K, we conclude that a' cannot lie in the x_n-chart. Hence a' lies in the other charts and satisfies there $a'_n = 0$. A permutation of $y_m, \ldots, y_{d+1}$ allows us to assume that a' lies in the x_{n-1}-chart. This permutation does not alter (2) and (3). As Y' is given in the x_{n-1}-chart by $x_{n-1} = 0$ and as $a' \in Y'$, we get $a'_m = 0$. From $a_d = \ldots = a_1 = 0$ it follows that $a'_d = \ldots = a'_1 = 0$. After a permutation of $x_{n-2}, \ldots, x_{d+1}$ we may assume that $a'_i \neq 0$ for $n-2 \geq i \geq j+1$ and $a'_i = 0$ for $j \geq i \geq 1$ and $i = n-1$ with $n-1-j-1$ the number of non-zero components of a'. This establishes (7).

Properties (8), (9) and (5) are immediate. As for (6), let f be an element of K of order c at a which lies in the weighted tangent cone of K. As $V = \{x_n = 0\}$ maximizes the order of the coefficient ideal of K in V, x_n appears as a variable in the homogeneous tangent cone of f. Computing K' in the affine coordinates of the various charts shows that its order drops at all points of the x_n-chart. Hence the points of W' with $c' = c$ are contained in the hypersurface $x_n = 0$. This proves the claimed assertions.

Appendix D: Résumé of definitions. We fix a regular ambient scheme W and a regular locally closed n-dimensional subscheme V of W. A *divisor* in W is an effective Weil divisor. A closed subscheme D of W has *normal crossings* if it can be defined locally by a monomial ideal. The subscheme V meets D *transversally* if the product of the defining ideals of V and D defines a normal crossings scheme.

A *local flag* in V at a is a decreasing sequence $W_n \supset \ldots \supset W_1$ of closed i-dimensional regular subschemes W_i of a neighborhood U of a in V. An ideal K in V is *bold regular* if it is a power of a regular principal ideal in V. A *stratified ideal* in V is a collection of coherent ideal sheaves, each of them defined on a stratum of a stratification of V by locally closed subschemes. A *stratified divisor* is defined by a stratified principal ideal.

A map $Q_b \to (Q_b)^\sharp$ associating to stalks of ideals Q_b in an open subscheme U of W stalks of ideals $(Q_b)^\sharp$ in V is *tuned* along the stratum S of a stratification of V through a point b of V if Q_b and $(Q_b)^\sharp$ admit locally at any point b of V coherent representatives $\overline{Q_b}$ on U and $\overline{(Q_b)^\sharp}$ on V so that the stalks $((\overline{Q_b})_a)^\sharp$ and $(\overline{(Q_b)^\sharp})_a$ at a coincide along S. This is abridged by saying that the ideals $(Q_b)^\sharp$ are tuned along S.

A *shortcut* of a (stratified) normal crossings divisor M in W is a divisor N obtained from M by deleting on each stratum of the underlying stratification $\mathrm{strat}(M)$ of M some components of M. The divisor M is *labelled* if each shortcut N comes with a different non-negative integer lab N, its *label*. The empty shortcut has label 0. A shortcut N of a normal crossings divisor M is *tight at a of order $\geq c$* if it has order $\geq c$ at a and if any proper shortcut of N has order $< c$ at a. It is *maximal tight at a* if M is labelled and if $(\mathrm{ord}_a N, \mathrm{lab}\ N)$ is lexicographically maximal among the tight shortcuts of M of order $\geq c$ at a.

A *handicap* on W is a sequence $D = (D_n, \ldots, D_1)$ of stratified normal crossings divisors D_i of W. The *truncation* of D at index i is ${}^iD = (D_n, \ldots, D_i)$.

A *singular mobile* in W is a quadruple $\mathcal{M} = (\mathcal{J}, c, D, E)$ with $\mathcal{J}$ a coherent nowhere zero ideal sheaf on V, c a non-negative constant associated to V, and D and E handicaps in W with D labelled and E reduced. The number c is the *control* of $\mathcal{J}$, and D and E form the *combinatorial* and *transversal handicap* of $\mathcal{M}$. The truncation ${}^i\mathcal{M}$ at index i of $\mathcal{M}$ is $(\mathcal{J}, c, {}^iD, {}^iE)$.

A *strong resolution of a mobile* $\mathcal{M} = (\mathcal{J}, c, D, E)$ in W with $\mathcal{J}$ a nowhere zero ideal in V is a sequence of blowups of W in regular closed centers Z such that the ideal $\mathcal{J}'$ of the final transform $\mathcal{M}' = (\mathcal{J}', c', D', E')$ of $\mathcal{M}$ has order $< c$. We require that the centers are transversal to the exceptional loci and that the resolution is equivariant.

The *top locus* of an upper semicontinuous function t on V is the reduced closed subscheme $\mathrm{top}(t)$ of points of V where t attains its maximum. The *order* at a of an ideal J of V is the largest power $o = \mathrm{ord}_a J$ of the maximal ideal of $\mathcal{O}_{V,a}$ containing the stalk of J at a. We set $\mathrm{top}(J) = \mathrm{top}(\mathrm{ord}\ J)$ and denote by $\mathrm{top}(J, c)$ the locus of points in V where the order of J is at least c. For closed subschemes of V, the analogous loci are associated to the defining ideals. When working locally at a point a, $\mathrm{top}(t)$ also denotes the local top locus of t in a neighborhood of a.

Let $W' \to W$ be the blowup of W with center Z inside V and exceptional component Y'. The *total* and *weak transform* of an ideal J of V are the inverse image J^* of J under the induced blowup $V' \to V$ and the ideal $J^\curlyvee = J^* \cdot \mathcal{I}(Y' \cap V')^{-o}$ with $o = \mathrm{ord}_Z J$. The *controlled transform* of J with respect to $c \leq o$ is the ideal $J^! = J^* \cdot \mathcal{I}(Y' \cap V')^{-c}$ in V'. The *strict transform* of a closed subscheme X of V in V' is the closure X^{st} of the pullback of $X \setminus Z$ in V'. It is defined by the ideal J^{st} in V' generated by all $f^* \cdot \mathcal{I}(Y' \cap V')^{-\mathrm{ord}_Z f}$ with f in the ideal J of X in V.

The *companion ideal* P of a product $J = M \cdot I$ of ideals in V at a with respect to a control $c \leq \mathrm{ord}_a J$ on V is the ideal P in V at a given by

$$\begin{aligned} P &= I + M^{\frac{o}{c-o}} && \text{if } 0 < o = \mathrm{ord}_a I < c, \\ P &= I && \text{otherwise.} \end{aligned}$$

The *transversality ideal* Q in V of a normal crossings divisor E of W is the ideal

$$Q = I_V(E \cap V)$$

defining $E \cap V$ in V. The *composition ideal* K in V of a product $J = M \cdot I$ of ideals in V with respect to a control c and a normal crossings divisor E in W is

$$K = P \cdot Q \qquad \text{if } I \neq 1,$$
$$K = 1 \qquad \text{if } I = 1,$$

with P the companion ideal of J and c, and Q the transversality ideal of E in V.

The *tag* of an ideal J in V at a with control c and normal crossings divisors D and E in W such that $J = M \cdot I$ for $M = I_V(D \cap V)$ with D labelled and transversal to V is the vector

$$t_a(J) = (o, k, m) \in \mathbb{N}^4,$$

equipped with the lexicographic order. Here, $o = \operatorname{ord}_a I$ and $k = \operatorname{ord}_a K$ with $K = P \cdot Q$ the composition ideal of (J, c, E, D). We set $m = (0, 0)$ if $o > 0$, and $m = (\operatorname{ord}_a N, \operatorname{lab} N)$ otherwise with N the maximal tight shortcut of M at a of order $\geq c$.

The *coefficient ideal* of an ideal K of W at a with respect to V is defined as follows. Let x, y and y be regular systems of parameters of $\mathcal{O}_{W,a}$ and $\mathcal{O}_{V,a}$ so that $x = 0$ defines V in W. For f in K denote by $a_{f,\alpha}$ the elements of $\mathcal{O}_{V,a}$ so that $f = \sum_\alpha a_{f,\alpha} \cdot x^\alpha$ holds after passage to the completion. Then set

$$\operatorname{coeff}_V K = \sum_{|\alpha| < c} (a_{f,\alpha}, f \in K)^{\frac{c}{c - |\alpha|}}.$$

The *junior ideal* J in V of an ideal K of W at a is the coefficient ideal $\operatorname{coeff}_V K$ of K in V if K is not bold regular or 1, and is set equal to 1 otherwise.

A point a' of the blowup W' of W with center Z is an *equiconstant point* for an ideal I in W at a if a' maps to a and if the order of the weak transform $I^\curlyvee$ of I at a' equals the order of I at a.

The subscheme V of W is *adjacent* to I at a if its strict transform V^{st} in W' contains all equiconstant points a' of I above a. It has *permanent contact* with I at a if the successive strict transforms of V under any sequence of blowups with centers inside the top loci of I and of its weak transforms contain all successive equiconstant points of I above a. We say that V has *weak maximal contact* with I at a if V maximizes the order of the coefficient ideal $\operatorname{coeff}_V I$ of I in V at a. It is *osculating* for I if there is an $f \in I$ with $\operatorname{ord}_a f = \operatorname{ord}_a I$ and $\operatorname{ord}_a \operatorname{coeff}_V f = \operatorname{ord}_a \operatorname{coeff}_V I$ such that $a_{f,\alpha} = 0$ for all α with $|\alpha| = \operatorname{ord}_a I - 1$.

Let $\mathcal{M} = (\mathcal{J}, c, D, E)$ be a singular mobile in W with $\mathcal{J}$ a coherent ideal in a locally closed regular n-dimensional subscheme V. Write J_n for the stalk of $\mathcal{J}$ at a point a of V. A *punctual setup* of $\mathcal{M}$ at a is a sequence $(J_n, \dots, J_1)$ of stalks of ideals J_i in a local flag $(W_n, \dots, W_1)$ of V at a satisfying for all $i \leq n$:

– $J_i = M_i \cdot I_i$ with $M_i = I_{W_i}(D_i \cap W_i)$ and I_i an ideal in W_i at a.

– M_i defines a normal crossings divisor in W_i at a.

– W_{i-1} has weak maximal contact at a with the composition ideal K_i in W_i of (J_i, c_{i+1}, D_i, E_i). Here, c_{i+1} is the control of J_i on W_i. It is given for $i < n$ as the order of K_{i+1} in W_{i+1} at a, and $c_{n+1} = c$.

– J_{i-1} is the junior ideal of K_i in W_{i-1}.

The *invariant* $i_a(\mathcal{M})$ of a mobile $\mathcal{M} = (\mathcal{J}, c, D, E)$ in W admitting locally on V setups $(J_n, \dots, J_1)$ is the vector

$$i_a(\mathcal{M}) = (t_n, \dots, t_1) \in \mathbb{N}^{4n}$$

with $t_i = (o_i, k_i, m_i)$ the tag of (J_i, c_{i+1}, D_i, E_i) at a.

Appendix E: Table of notations. Script capitals $\mathcal{M}$, $\mathcal{J}$ denote stratified global objects or sheaves; roman capitals J, I, M, P, Q, K stalks of ideals or sufficiently small representatives of them. Subscripts refer to the embedding dimension or, for handicaps, to the relevant dimension; left superscripts to truncations. Primes denote objects after blowup, analogous to their sisters below. Minus sign subscripts denote objects in one dimension less, analogous to their cousins without subscript.

$\mathcal{M} = (\mathcal{J}, c, D, E)$	singular mobile in regular ambient scheme W
$\mathcal{J}$	coherent ideal sheaf in n-dimensional regular subscheme V of W
c	positive integer constant, the control
$D = (D_n, \ldots, D_1)$	combinatorial handicap with D_i normal crossings divisor in W
$E = (E_n, \ldots, E_1)$	transversal handicap with E_i normal crossings divisor in W
$(J_n, \ldots, J_1)$	punctual setup of $\mathcal{M}$ at a with J_i coherent ideal in W_i at a
${}^i\mathcal{M} = (\mathcal{J}, c, {}^iD, {}^iE)$	truncated mobile with ${}^iD = (D_n, \ldots, D_i)$, ${}^iE = (E_n, \ldots, E_i)$
$(J_n, \ldots, J_i)$	truncated punctual setup of ${}^i\mathcal{M}$ at a
J, J_n	stalk of $\mathcal{J}$ at a
$J_i = M_i \cdot I_i$	factorization of J_i with $M_i = I_{W_i}(D_i \cap W_i)$
I_i	factor of J_i which passes under blowup to weak transform
J_-, M_-, I_-	ideals in one lower dimension playing the same role as J, M, and I
$W_n, \ldots, W_1$	local flag of i-dimensional regular subschemes W_i of W at a
W_i	osculating hypersurface for P_{i+1} in W_{i+1}
c_{i+1}	control for J_i in W_i at a, equal to the order of K_{i+1} in W_{i+1} at a
$P_i = I_i + M_i^{c_{i+1}/(c_{i+1}-o_i)}$, resp. $P_i = I_i$	companion ideal of $J_i = M_i \cdot I_i$
$Q_i = I_{W_i}(E_i \cap W_i)$	transversality ideal of E_i in W_i
$K_i = P_i \cdot Q_i$	composition ideal of $J_i = M_i \cdot I_i$, c_{i+1} and E_i
$J_{i-1} = \mathrm{coeff}_{W_{i-1}}(K_i)$	coefficient ideal of K_i in W_{i-1}
N_i	maximal tight shortcut of M_i of order $\geq c_{i+1}$
$\mathrm{top}(I_i)$	top locus of I_i of points where I_i has maximal order in W_i
$\mathrm{top}(J_i, c_{i+1})$	locus of J_i of points where J_i has order $\geq c_{i+1}$ in W_i
o_i	order of I_i at a in W_i
k_i	order of K_i at a in W_i
$m_i = (\mathrm{ord}_a N_i, \mathrm{lab}\, N_i)$	combinatorial tag of M_i
$t_i = (o_i, k_i, m_i)$	tag of $J_i = M_i \cdot I_i$, c_{i+1}, E_i at a
$i_a(\mathcal{M}) = (t_n, \ldots, t_1)$	local invariant of mobile $\mathcal{M}$ at a
Z	center of blowup in W
W'	blowup of W in Z
Y'	new exceptional component in W'
$V^\curlyvee$, $J^\curlyvee$	weak transform of V, resp. J in W'
V^{st}, J^{st}	strict transform of V, resp. J in W'
J^*	total transform (= pullback) of J in W'
$J^!$	controlled transform of J in W' w.r.t. a control c
W_i'	weak transform of W_i at equiconstant points of P_{i+1}'
W_i'	newly chosen osculating hypersurface for P_{i+1}' outside equiconstant points
J_i^*	total transform of J_i under blowup of W_i in Z
$J_i^! = J_i^* \cdot I(Y')^{-c_{i+1}}$	controlled transform of J_i w.r.t. c_{i+1}
$I_i^\curlyvee = I_i^* \cdot I(Y')^{-\mathrm{ord}_Z I_i}$	weak transform of I_i under blowup of W_i in Z
D_i', E_i'	transforms of D_i and E_i
$M_i' = I_{W_i'}(D_i' \cap W_i')$	exceptional monomial factor of J_i'
$\mathcal{M}' = (\mathcal{J}', c', D', E')$	transform of $\mathcal{M}$ under blowup

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