AN INTRODUCTION TO P-ADIC AND MOTIVIC ZETA FUNCTIONS AND THE MONODROMY CONJECTURE

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1. INTRODUCTION

Introduced by Weil, the p-adic zeta function associated to a polynomial f over $\mathbb{Z}_p$ was systematically studied by Igusa in the non-archimedean wing of his theory of local zeta functions, which also includes archimedean (real and complex) zeta functions [18][19]. The p-adic zeta function is a meromorphic function on the complex plane, and contains information about the number of solutions of the congruences $f \equiv 0 \mod p^m$ for $m > 0$. Igusa formulated an intriguing conjecture, the monodromy conjecture, stating that, if f is defined over $\mathbb{Z}$, the poles of its p-adic zeta function are closely related to the structure of the singularities of the complex hypersurface defined by f (see Conjecture 3.1 for the precise statement). Special cases of the conjecture have been solved (in particular the case where f is a polynomial in two variables) but the general case remains quite mysterious.

Definition 1.1 (Igusa Poincaré series)**.** *Let p be a prime. For any pair of integers $m, d > 0$, and any polynomial $f \in \mathbb{Z}_p[x_1, \ldots, x_d]$, we denote by $N_m(f)$ the number of solutions of the congruence $f \equiv 0 \mod p^m$ in the ring $(\mathbb{Z}/(p^m))^d$. We denote by $Q(f;t)$ the generating series*

$$Q(f;t) = \sum_{m>0} N_m(f) \cdot t^m \quad \in \mathbb{Z}[[t]]$$

and we call it the Igusa Poincaré series associated to f.

We will see that the series $Q(f;t)$ is rational, i.e. belongs to the subring $\mathbb{Q}(t)$ of $\mathbb{Q}[[t]]$. This was conjectured by Borevich and Shafarevich and proven by Igusa. Igusa's p-adic zeta function $Z(f;s)$ associated to f is given by

$$Z(f;s) = 1 + \frac{(p^{-s} - 1)Q(f; p^{-s-d})}{p^{-s}}$$

Notation 1.2. *If $f \in \mathbb{Z}[x_1, \ldots, x_d]$, we can view f as a p-adic polynomial for any prime $p > 0$, and we write $N_{p,m}(f)$, $Q_p(f;t)$ and $Z_p(f;s)$ for the objects introduced above.*

We adopt the following notation: if f is a polynomial in $\mathbb{Z}[x_1, \ldots, x_d]$ then we denote by V_f the closed subscheme of $\mathbb{A}_{\mathbb{Z}}^d$ defined by f, and by $|V_f(\mathbb{F}_p)|$ the cardinality of its set of $\mathbb{F}_p$-rational points. In down-to-earth terms, this is the number of solutions of the equation $f = 0$ over the field $\mathbb{F}_p$.

Denote by $v_p : \mathbb{Z} \to \mathbb{N} \cup \{\infty\}$ the p-adic valuation and by $\overline{ac}_p : \mathbb{Z}_{\neq 0} \to \mathbb{F}_p$ the reduced angular component map, i.e. $\overline{ac}_p(z)$ is the reduction of $z \cdot p^{-v_p(z)}$ modulo

1

p for any $z \in \mathbb{Z}_{\neq 0}$. By convention, $\overline{ac}_p(0) = 0$. For any integer $m > 0$, the maps v_p and $\overline{ac}_p$ induce maps

$$v_p : \mathbb{Z}/(p^m) \to \{0, \ldots, m-1\} \cup \{\infty\}$$

and $\overline{ac}_p : \mathbb{Z}/(p^m) \to \mathbb{F}_p$ in the obvious way, by composition with an arbitrary section $\mathbb{Z}/(p^m) \to \mathbb{Z}$ of the projection $\mathbb{Z} \to \mathbb{Z}/(p^m)$ which sends $0 \in \mathbb{Z}/(p^m)$ to $0 \in \mathbb{Z}$.

Let us look at some basic examples of p-adic zeta functions, with $d = 2$. For notational convenience we write (x, y) instead of (x_1, x_2). Consider the polynomials $g(x, y) = y - x^2$ and $h(x, y) = y^2 - x^3$ in $\mathbb{Z}[x, y]$, and fix a prime p.

It is obvious that $N_{p,m}(g) = p^m$ for each $m > 0$: we can freely choose a value for x in $\mathbb{Z}/(p^m)$, and this choice determines y. Let us rephrase this a little bit to obtain a formula which will turn out to generalize. For any integer $m > 0$ and any solution u of the equation $f = 0$ in $(\mathbb{Z}/(p^m))^2$, there exist exactly p solutions of the equation $f = 0$ in $(\mathbb{Z}/(p^{m+1}))^2$ which are mapped to u under the projection $(\mathbb{Z}/(p^{m+1}))^2 \to (\mathbb{Z}/(p^m))^2$ (choose an arbitrary lifting of the x-coordinate to $\mathbb{Z}/(p^{m+1})$; this uniquely determines the y-coordinate of the lifting of u). Therefore, we get

$$
\begin{aligned}
Q_p(g; t) &= \frac{|V_g(\mathbb{F}_p)|t}{1 - pt} \\
Z_p(g; s) &= p^{-2}|V_g(\mathbb{F}_p)|(p-1)\frac{p^{-1-s}}{1 - p^{-1-s}} + 1 - p^{-2}|V_g(\mathbb{F}_p)|
\end{aligned}
$$

We have $|V_g(\mathbb{F}_p)| = p$.

The polynomial $h(x, y)$ is more tricky. We exclude the case $p = 3$. Fix an integer $m > 0$. Choose a value a for x in $\mathbb{Z}/(p^m)$. The equation $y^3 = a^2$ has a solution in $\mathbb{Z}/(p^m)$ iff

(1) $2v_p(a) \geq m$, or
(2) $2v_p(a) < m$, $v_p(a)$ is divisible by 3, and $\overline{ac}_p(a^2)$ is a cube in $\mathbb{F}_p$.

In the first case, we can take for y any element b in $\mathbb{Z}/(p^m)$ with $3v_p(b) \geq m$. In the second case, we put $\omega = v_p(a)/3$. For any solution $\overline{b}$ of the equation $y^3 - \overline{ac}_p(a^2) = 0$ in $\mathbb{F}_p$, there exists an element b_0 in $\mathbb{Z}/(p^m)$ such that $\overline{ac}_p(b_0) = \overline{b}$ and such that the set of all solutions b of the equation $y^3 - a^2 = 0$ in $\mathbb{Z}/(p^m)$ with $\overline{ac}_p(b) = \overline{b}$ is given by the coset

$$b_0 + p^{m-4\omega} \cdot \mathbb{Z}/(p^m)$$

Direct computation yields

$$
\begin{aligned}
Q_p(h; t) &= \frac{|V_h(\mathbb{F}_p)|t + p(p-1)t^2 + (p-1)p^6 t^6 - p^8 t^7}{(1 - pt)(1 - p^7 t^6)} \\
Z_p(h; s) &= 1 + \frac{(p^{-s} - 1)(p^{-2}|V_h(\mathbb{F}_p)| + (p-1)p^{-s-3} + (p-1)p^{-5s-6} - p^{-6s-6})}{(1 - p^{-s-1})(1 - p^{-6s-5})}
\end{aligned}
$$

and we have $|V_h(\mathbb{F}_p)| = p$. We leave the case $p = 3$ as an exercise to the reader.

These examples illustrate some general phenomena. If f is a polynomial in $\mathbb{Z}[x_1, \ldots, x_d]$ and p is a prime such that p does not divide f and such that the

equation $f = 0$ defines a smooth subvariety of the affine space $\mathbb{A}_{\mathbb{F}_p}^d$, we have

$$
\begin{aligned}
Q_p(f;t) &= \frac{|V_f(\mathbb{F}_p)|t}{1 - p^{d-1}t} \\
Z_p(f;s) &= p^{-d}|V_f(\mathbb{F}_p)|(p-1)\frac{p^{-1-s}}{1 - p^{-1-s}} + 1 - p^{-d}|V_f(\mathbb{F}_p)|
\end{aligned}
$$

In fact, one can show that for any $m > 0$ and any solution u of the equation $f = 0$ in $(\mathbb{Z}/(p^m))^d$, there exist exactly p^{d-1} solutions of the equation $f = 0$ in $(\mathbb{Z}/(p^{m+1}))^d$ which are mapped to u under the projection $(\mathbb{Z}/(p^{m+1}))^d \to (\mathbb{Z}/(p^m))^d$. For $d = 1$ this statement is well-known as Hensel's Lemma. So in the smooth case, Igusa's zeta function is easy to compute (provided you know the number $|V_f(\mathbb{F}_p)|$, that is!) and only has the so-called trivial poles $s \in \{1 + (2\pi i / \ln p)\mathbb{Z}\}$, each with multiplicity one.

If the equation $f = 0$ defines a smooth subvariety in $\mathbb{A}_{\mathbb{C}}^d$, then the equation $f = 0$ also defines a smooth subvariety in $\mathbb{A}_{\mathbb{F}_p}^d$, for $p \gg 0$. So we see that in this case, the set of poles of $Z_p(f;s)$ is $\{1 + (2\pi i / \ln p)\mathbb{Z}\}$ for $p \gg 0$. This is a (very) special case of Igusa's Monodromy Conjecture. If the closed subscheme of $\mathbb{A}_{\mathbb{C}}^d$ defined by f is not smooth (as in our second example), the behaviour of the p-adic zeta function is much harder to control. According to the Monodromy Conjecture, part of this behaviour can be described by analyzing the singularities of the complex hypersurface defined by f. A powerful method to compute the p-adic zeta function is taking an embedded resolution of singularities for f, which essentially reduces the computation to the case where f is a monomial; see Section 2.4.

A second observation is that our examples show a certain uniformity in p. For fixed f, there exist algebraic varieties $V_1, \ldots, V_r$ defined over $\mathbb{Z}$ and rational functions $G_1, \ldots, G_r$ in $\mathbb{Q}(u, v)$ of the form

$$
G_i(u, v) = \prod_{j=1}^{n_i} \frac{u^{-a_{i,j}} v^{b_{i,j}}}{1 - u^{-a_{i,j}} v^{b_{i,j}}}
$$

with $a_{i,j}, b_{i,j} \in \mathbb{Z}_{>0}$ such that for $p \gg 0$,

$$
Z_p(f;s) = p^{-d} \sum_{i=1}^{r} |V_i(\mathbb{F}_p)| G_i(p, p^{-s})
$$

This is also a general phenomenon: as we will see, there exists a "universal" zeta function $Z_{mot}(f;s)$ associated to f, which is built from algebraic varieties and specializes to the p-adic zeta function for $p \gg 0$ (Theorem 5.5). This zeta function is the so-called motivic zeta function of f.

Further reading. The basic references for the theory of p-adic zeta functions (Sections 2 and 3) are [19] and [8]. For more background on the motivic side of the story (Sections 4 and 5) we refer to [13][27][43].

2. GENERALITIES ON p-ADIC ZETA FUNCTIONS

2.1. **Definitions.** Throughout Sections 2 and 3, we fix the following notations: K is a p-adic field (i.e. a finite extension of $\mathbb{Q}_p$ for some prime p), R its ring of integers,

P the maximal ideal of R, and $k = R/P$ the residue field. We fix a generator π of P, and we denote the cardinality of k by q.

We denote by $v(\cdot)$ the discrete (π-adic) valuation on K, and we put $|z|_K = q^{-v(z)}$ for $z \in K^*$. We denote by $ac(z)$ the angular component $z \cdot \pi^{-v(z)}$ of $z \in K^*$; it is an element of the group of units R^*. We put $|0|_K = 0$ and $ac(0) = 0$. The absolute value $|\cdot|_K$ defines a topology on K and turns it into a locally compact field. We denote by μ the Haar measure on R, with the usual normalization $\mu(R) = 1$. By abuse of notation, we also denote by μ the product measure on the cartesian powers K^d of K. If S is a set, we denote its cardinality by $|S|$. For any complex number α, we denote its real part by $\Re(\alpha)$.

Let $\widetilde{x} = (x_1, \ldots, x_d)$ be a tuple of coordinates, and let f be a polynomial in $K[\widetilde{x}]$. Let $\Phi : K^d \to \mathbb{C}$ be a Schwartz-Bruhat function, i.e. a locally constant function with compact support, and let $\chi : R^* \to \mathbb{C}^*$ be a character of R^*, i.e. a group homomorphism with finite image. We put $\chi(0) = 0$. We'll denote the trivial character by χ_{triv}.

Definition 2.1 (Igusa zeta function). *The Igusa zeta function associated to the triple (f, χ, Φ) is defined by*

$$Z(f, \chi, \Phi; s) = \int_{K^d} \Phi \cdot (\chi \circ ac)(f) \cdot |f|_K^s d\mu$$

for $s \in \mathbb{C}$ with $\Re(s) > 0$.

If Φ is the characteristic function $\mathbf{1}_{R^d}$ of R^d, or if χ is the trivial character χ_{triv}, we'll omit Φ, resp. χ, from the notation. For instance, we write $Z(f; s)$ instead of $Z(f, \chi_{triv}, \mathbf{1}_{R^d}; s)$. It is often convenient to introduce a new variable $t = q^{-s}$; then $Z(f, \chi, \Phi; s)$ is a power series in t with coefficients in $\mathbb{C}$ which converges for $|t|_K < 1$. Observe that $Z(f, \chi, \Phi; s)$ only depends on s modulo $(2\pi i / \ln q)\mathbb{Z}$.

The definition of $Z(f, \chi, \Phi; s)$ depends on the choice of the uniformizer π (through the angular component $ac(\cdot)$). Following Igusa [19, 8.2], it is more natural to consider

$$Z(f, \cdot, \Phi; \cdot)$$

as a function on the complex analytic space $\Omega(K^*)$ of quasi-characters ω of K^* (continuous group homomorphisms $K^* \to \mathbb{C}^*$). We will not use this point of view in this article.

2.2. p-**adic cylinders.** Now we introduce the notion of a cylinder in R^d. Elementary as it may be, it will play a central role in the construction of the motivic measure in Section 4.4.

Definition 2.2 (Cylinder). *We consider, for each $m \geq 0$, the natural projection*

$$\pi_m : R^d \to (R/P^{m+1})^d$$

If S is a subset of $(R/P^{m+1})^d$, we call the subset $(\pi_m)^{-1}(S)$ of R^d the cylinder over S.

We say that a subset C of R^d is a cylinder if it is the cylinder over $\pi_m(C)$ for some $m \geq 0$.

Lemma 2.3. *For any cylinder C in R^d, the series $(q^{-d(m+1)}|\pi_m(C)|)_{m\geq 0}$ is constant for $m \gg 0$, and its limit is equal to the Haar measure of C.*

More precisely, if we choose $m_0 \geq 0$ such that C is the cylinder over $\pi_{m_0}(C)$, then for $m \geq m_0$ we have

$$q^{-d(m+1)}|\pi_m(C)| = q^{-d(m_0+1)}|\pi_{m_0}(C)| = \mu(C)$$

Proof. For $m \geq m_0$, C can be written as a disjoint union

$$C = \bigsqcup_{a \in \pi_m(C)} (a + (P^{m+1}R)^d)$$

By translation invariance of the Haar measure and the fact that $(P^{m+1}R)^d$ has measure $q^{-(m+1)d}$, the measure of C equals

$$q^{-(m+1)d}|\pi_m(C)|$$

$\square$

2.3. **Solutions of congruences.** If f has integer coefficients, i.e. $f \in R[\widetilde{x}]$, then the local zeta function $Z(f; s)$ is closely related to the number of solutions of the congruences $f \equiv 0$ modulo powers of P.

Definition 2.4 (Igusa Poincaré series)**.** *Let f be an element of $R[\widetilde{x}]$. Denote for each $m > 0$ by $N_m(f)$ the cardinality of the set*

$$\{z \in (R/P^m)^d \,|\, f(z) = 0 \text{ in } R/P^m\}$$

The generating series

$$Q(f; t) = \sum_{m > 0} N_m(f) t^m \in \mathbb{Z}[[t]]$$

is called the Igusa Poincaré series associated to f.

Proposition 2.5. *Putting $t = q^{-s}$, we have*

(2.1) $$Q(f; q^{-d}t) = \frac{t}{1-t}(1 - Z(f; s))$$

in $\mathbb{Q}[[t]]$.

Proof. By definition of the zeta function, the coefficient of $t^i = q^{-is}$ in $Z(f; s)$ is equal to the Haar measure of the set

$$S_i = \{z \in R^d \,|\, v(f(z)) = i\}$$

for each $i \geq 0$. So for each $m > 0$, the coefficient c_m of t^m in the right hand side of (2.1) is $1 - \sum_{i=0}^{m-1} \mu(S_i)$. By additivity of the measure and the fact that R^d has measure 1, we get $c_m = \mu(S_{\geq m})$ with

$$S_{\geq m} = \{z \in R^d \,|\, v(f(z)) \geq m\})$$

The set $S_{\geq m}$ is a cylinder over the set

$$C_{m-1} := \{z \in (R/P^m)^d \,|\, f(z) = 0 \text{ in } R/P^m\}$$

whose cardinality equals $N_m(f)$. By Lemma 2.3 we obtain equation (2.1). $\square$

2.4. **Rationality results.** We fix a polynomial $f \in K[\widetilde{x}]$, a Schwartz-Bruhat function Φ on K^d and a character χ on R^*.

Theorem 2.6 (Igusa)**.** *The local zeta function $Z(f, \chi, \Phi; s)$ is rational in $t = q^{-s}$.*

In particular, $Z(f, \chi, \Phi; s)$ has a meromorphic continuation to $\mathbb{C}$. The proof actually yields more information: it specifies a finite subset S of $\mathbb{Q}_{<0}$ such that the poles of $Z(f, \chi, \Phi; s)$ are contained in

$$S + (2\pi i / \ln q)\mathbb{Z}$$

Before giving and proving the precise statement, we need some additional notation.

The polynomial f defines a morphism of K-varieties $f : \mathbb{A}_K^d \to \mathbb{A}_K^1$. We choose an *embedded resolution of singularities* for the morphism f. This is a projective birational morphism of K-varieties

$$h : Y \to \mathbb{A}_K^d$$

such that Y is smooth over K, h is an isomorphism over the complement of the zero locus of f in $\mathbb{A}_K^d$, and the divisor $E = (f \circ h)$ on Y has strict normal crossings. Such an embedded resolution always exists, by [16].

We denote by Jac_h the Jacobian ideal sheaf of h on Y. If ω is a gauge form on an open subvariety U of Y, then on U the Jacobian ideal sheaf Jac_h is generated by the unique regular function g on U satisfying

$$h^*(dx_1 \wedge \ldots \wedge dx_d) = g \cdot \omega$$

Since we assumed that h is an isomorphism on $Y \setminus E$, the divisor (Jac_h) is supported on E.

We denote by E_i, $i \in I$ the irreducible components of E, by N_i the multiplicity of $f \circ h$ along E_i, and by $\nu_i - 1$ the multiplicity of the Jacobian ideal Jac_h of h along E_i. The couples (N_i, ν_i) are called the *numerical data* of the resolution h. Now Theorem 2.6 can be refined as follows.

Theorem 2.7 (Igusa)**.** *The local zeta function $Z(f, \chi, \Phi; s)$ is rational in $t = q^{-s}$. If α is a pole of $Z(f, \chi, \Phi; s)$, then there exists an index $i \in I$ such that the order of χ divides N_i and such that α is contained in*

$$-\frac{\nu_i}{N_i} + \frac{2\pi i}{\ln q}\mathbb{Z}$$

Proof. For the proof of the theorem we assume that the reader is familiar with the theory of integration of differential forms ω of maximal degree on K-analytic manifolds M [19, 7.4]. Recall in particular that ω defines a measure on the set of compact open subsets of M. Since Y is a smooth K-variety, we can view its set of rational points $Y(K)$ as a K-analytic manifold. The change of variables formula for p-adic integrals [19, 7.4.1] yields

$$Z(f, \chi, \Phi; s) = \int_{Y(K)} (\Phi \circ h) \cdot (\chi \circ ac)(f \circ h) \cdot |f \circ h|_K^s h^*(d\widetilde{x})$$

with $d\widetilde{x} = dx_1 \wedge \ldots \wedge dx_d$, and this integral can be computed locally on $Y(K)$ as follows.

Let b be a point of $Y(K)$ and let $\{i_1, \ldots, i_r\}$ be the (possibly empty) set of indices i in I with $b \in E_i$. Since the divisor E has strict normal crossings, there

exist an open neighbourhood U of b in $Y(K)$, and analytic coordinates $y_1, \ldots, y_d$ and nowhere-vanishing K-analytic functions u, v on U, such that

$$f \circ h = u \prod_{j=1}^{r} (y_j)^{N_{i_j}} \text{ and } h^* d\widetilde{x} = v \prod_{j=1}^{r} (y_j)^{\nu_{i_j}-1} dy$$

Since the value group $|K^*|_K$ is discrete, we may even assume that $|u|_K$ and $|v|_K$ are constant on U.

Since h is proper, the support of $\Phi \circ h$ is compact, so we can write $Z(f, \chi, \Phi; s)$ as a finite $\mathbb{C}$-linear combination of finite products with factors of the form $q^{as} \cdot A(e, i)$. Here $e, a \in \mathbb{Z}$, $i \in I$, and

$$A(e, i) := \int_{z \in \pi^e R} \chi^{N_i}(ac(z)) \cdot |z|_K^{N_i s + \nu_i - 1} dz$$

If $\chi^{N_i} \neq 1$ then $A(e, i)$ vanishes, by a standard argument: if $w \in R^*$ then a substitution $z' = wz$ shows that

$$A(e, i) = \chi^{N_i}(w) \cdot A(e, i)$$

If $\chi^{N_i} = 1$ then

$$
\begin{aligned}
A(e, i) &= \sum_{m \geq e} \mu(\{z \in R \mid v(z) = m\}) \cdot q^{-m(N_i s + \nu_i - 1)} \\
&= (1 - \frac{1}{q}) \frac{q^{-e(N_i s + \nu_i)}}{1 - q^{-(N_i s + \nu_i)}}
\end{aligned}
$$

$\square$

Corollary 2.8. *If f has coefficients in R, the Igusa Poincaré series $Q(f; T)$ is rational over $\mathbb{Q}$.*

Proof. This follows from Proposition 2.5. $\square$

The rationality of $Q(f; t)$ was conjectured by Borevich and Shafarevich. Denef gave a vast generalization of Igusa's rationality result in Theorem 2.6, avoiding resolution of singularities but instead using p-adic cell decomposition [6] to prove a general structure theorem on definable p-adic integrals. As a special case, Denef obtained the rationality of the series

$$\widetilde{Q}(f; t) = \sum_{m > 0} \widetilde{N}_m(f) t^m$$

where $\widetilde{N}_m(f)$ denotes the cardinality of the image of the projection

$$\{x \in R^d \mid f(x) = 0\} \to (R/P^m)^d$$

The rationality of this series was conjectured by Serre and Oesterlé. To my knowledge, there is still no "purely" arithmetic or geometric proof which does not use p-adic model theory. If the reduction $\overline{f}$ of f modulo P is non-zero and defines a smooth hypersurface in $\mathbb{A}_k^d$, then $\widetilde{Q}(f; t) = Q(f; t)$, but in the singular case the geometric meaning of its poles is completely unknown. See [12][23][32][33] for some particular cases in the motivic setting.

Igusa's theorem yields a complete set of candidate poles for $Z(f, \chi, \Phi; s)$, but many of these will not be actual poles of the local zeta function. For one thing, the set of candidate poles depends on the embedded resolution h. But even if we take

the intersection of these sets over all resolutions h, the resulting set will in general still be too big.

For instance, in the curve case $d = 2$, there exists a *minimal* embedded resolution, but not all the corresponding candidate poles will be poles of the zeta function. This phenomenon is related to the monodromy conjecture, which puts additional (conjectural, if $d > 2$) restrictions on the poles.

2.5. **Denef's formula.** If the embedded resolution h has good reduction modulo the maximal ideal P of R (in a certain technical sense; see below) then Denef gave a very explicit formula for $Z(f, \chi, \Phi; s)$ in terms of the resolution h [9][7]. If f and h are defined over a number field L, then good reduction holds at almost all finite places $\mathfrak{P}$ of L.

Denef's formula involves the numerical data of h and certain character sums over sets of rational points on the reductions of the exceptional components E_i modulo P. Using the étale Kummer sheaf associated to χ and Grothendieck's trace formula, this yields a cohomological interpretation for the local zeta function [7].

To state Denef's formula, we need some additional notation. A Schwartz-Bruhat function Φ on K^d is called residual if its support is contained in R^d and the value of Φ at a point y of R^d only depends on its residue class in k^d. Let f be an element of $R[\widetilde{x}]$, and $h : Y \to \mathbb{A}_K^d$ an embedded resolution of singularities for f. We fix a closed immersion of Y into projective space $\mathbb{P}_{\mathbb{A}_K^d}^r$ for some $r > 0$. We denote by $E = (f \circ h)$ the divisor on Y defined by $f \circ h$, by E_i, $i \in I$ its irreducible components, and by (N_i, ν_i) the corresponding numerical data. For any integer $e > 0$, we denote by $\mathcal{I}_e$ the set of indices $i \in I$ such that $e | N_i$.

We denote by $\overline{f}$ the reduction of f modulo P. It is an element of $k[\widetilde{x}]$. For any closed subvariety Z of Y we denote by $\overline{Z}$ the reduction modulo P of the Zariski-closure of Z in $\mathbb{P}_{\mathbb{A}_R^d}^r$. The morphism h induces a morphism of k-varieties $\overline{h} : \overline{Y} \to \mathbb{A}_k^d$.

For any subset J of I we put $E_J = \cap_{j \in J} E_j$ and $\overline{E}_J^o = \overline{E_J} \setminus \cup_{i \notin J} \overline{E}_i$. In particular, $E_\emptyset = Y$ and $\overline{E}_\emptyset^o = \overline{Y} \setminus \overline{E}$. We put $m_J = gcd\{N_i \,|\, i \in J\}$. Note that $m_\emptyset = gcd(\emptyset) = 0$. For $i \in I$ we write $\overline{E}_i^o$ instead of $\overline{E}_{\{i\}}^o$.

Following Denef, we say that the resolution h has good reduction modulo P if $\overline{Y}$ and all $\overline{E}_i$ are smooth, $\overline{E}$ is a divisor with strict normal crossings, and $\overline{E}_i$ and $\overline{E}_j$ have no common components if $i \neq j$. In this case, $\overline{E_J} = \cap_{j \in J} \overline{E}_j$ for each subset J of I. We say that the resolution has tame good reduction if, moreover, none of the multiplicities N_i belongs to P. If h has good reduction, J is a subset of I, and a is a point of $\overline{E}_J^o(k)$, then in the local ring $\mathcal{O}_{\overline{Y}, a}$ the element $\overline{f} \circ \overline{h}$ can be written as $u \cdot v^{m_J}$ with u a unit. If χ is a character of R^* which is trivial on $1 + P$, then $\chi(u(a))$ is well-defined since $u(a) \in k^* = R^*/(1 + P)$. If, moreover, e is an element of $\mathbb{Z}_{>0}$ such that $J \subset \mathcal{I}_e$ and χ^e is trivial, then $e | m_J$ and

$$\omega_\chi(a) := \chi(u(a))$$

does not depend on the choice of u and v.

If Φ is a residual Schwartz-Bruhat function on K^d, we denote by $\overline{\Phi}$ the induced function $k^d \to \mathbb{C}$. Let χ be a character of R^*, of order e. If χ is trivial on $1 + P$ and h has good reduction modulo P, then for any subset J of $\mathcal{I}_e$ we put

$$c_{J, \Phi, \chi} = \sum_{a \in \overline{E}_J^o(k)} (\overline{\Phi} \circ \overline{h})(a) \cdot \omega_\chi(a)$$

In particular, if Φ is the characteristic function of R^d and χ is trivial, then

$$c_{J,\Phi,\chi} = |\overline{E}_J^o(k)|$$

for all subsets J of I.

Theorem 2.9 (Denef)**.** *Assume that $\overline{f} \neq 0$. Let Φ be a residual Schwartz-Bruhat function on K^d and χ a character of R^* of order e. If χ is non-trivial on $1 + P$ and h has tame good reduction, then $Z(f, \chi, \Phi; s)$ is constant as a function of s. If χ is trivial on $1 + P$ and h has good reduction, then we have*

$$Z(f, \chi, \Phi; s) = q^{-d} \sum_{J \subset \mathcal{I}_e} \left(c_{J,\Phi,\chi} \prod_{j \in J} \frac{(q-1)q^{-N_j s - \nu_j}}{1 - q^{-N_j s - \nu_j}} \right)$$

If f and h are defined over a number field L, then h has tame good reduction modulo $\mathfrak{P}$ for almost all finite places $\mathfrak{P}$ of L (i.e. all but a finite number). Hence, if we denote by $L_\mathfrak{P}$ the completion of L at $\mathfrak{P}$ and by $\mathcal{O}_\mathfrak{P}$ its ring of integers, then for all but a finite number of $\mathfrak{P}$, Theorem 2.9 gives an explicit expression for the $\mathfrak{P}$-adic zeta function associated to f, any residual Schwartz-Bruhat function Φ on $(L_\mathfrak{P})^d$ and any character χ of $\mathcal{O}_\mathfrak{P}^*$.

3. THE *p*-ADIC MONODROMY CONJECTURE

3.1. **The Milnor fibration.** Let X be a complex manifold, and let

$$g : X \to \mathbb{C}$$

be a non-constant analytic map. We denote by X_s the special fiber of g (i.e. the analytic space defined by $g = 0$), and we fix a point $x \in X_s$.

What does X_s look like in a neighbourhood of x? If g is smooth at x this is easy: X_s is locally a complex submanifold of X. If g is not smooth at x, then the topology of X_s near x can be studied by means of the Milnor fibration [30][15].

Working locally, we may assume that $X = \mathbb{C}^d$. Let $B = B(x, \varepsilon)$ be an open ball around x in $\mathbb{C}^d$ with radius ε, let $D = D(0, \eta)$ be an open disc around the origin 0 in $\mathbb{C}$ with radius η, and put $D^* = D \setminus \{0\}$. For $0 < \eta \ll \varepsilon \ll 1$ the map

$$g_x : g^{-1}(D^*) \cap B \to D^*$$

is a locally trivial fibration, called the Milnor fibration of g at x.

We consider the universal covering space

$$\widetilde{D^*} = \{z \in \mathbb{C} \mid \Im(z) > -\log \eta\} \to D^* : z \mapsto \exp(iz)$$

of D^* and we put

$$F_x = (g^{-1}(D^*) \cap B) \times_{D^*} \widetilde{D^*}$$

This is the universal fiber of the fibration g_x, and it is called the Milnor fiber of g at x. Since g_x is a locally trivial fibration and $\widetilde{D^*}$ is contractible, F_x is homotopy-equivalent to the fiber of g_x over any point of D^*.

If g is smooth at x, then the fibration g_x is trivial. In general, the defect of triviality is measured by the monodromy action on the singular cohomology of F_x, i.e. the action of the group $\pi_1(D^*)$ of covering transformations of $\widetilde{D^*}$ over D^* on $\oplus_{i \geq 0} H_{sing}^i(F_x, \mathbb{Z})$. The action of the canonical generator $z \mapsto z + 2\pi$ is called the monodromy transformation and denoted by M_x. We say that a complex number γ is a monodromy eigenvalue of g at x if γ is an eigenvalue of the monodromy

transformation M_x on $H^i_{sing}(F_x, \mathbb{Z})$ for some $i \geq 0$. These monodromy eigenvalues are roots of unity.

3.2. **The monodromy conjecture.** Now assume that f is a polynomial in the variables $\widetilde{x} = (x_1, \ldots, x_d)$ over some number field L. Then, for any finite place $\mathfrak{P}$, we can view f as a polynomial over the $\mathfrak{P}$-adic completion $L_{\mathfrak{P}}$ of L and consider the associated local zeta functions $Z_{\mathfrak{P}}(f, \chi, \Phi; s)$ for varying Φ and χ. On the other hand, we can view f as a complex polynomial, defining an analytic map

$$f : X = \mathbb{C}^d \to \mathbb{C}$$

and we can consider the singularities of its special fiber X_s. These objects are related by Igusa's Monodromy Conjecture. We denote by $\mathcal{O}_{\mathfrak{P}}$ the ring of integers of $L_{\mathfrak{P}}$.

Conjecture 3.1 (Monodromy Conjecture). *Let L be a number field and f an element of $L[\widetilde{x}]$. For almost all finite places $\mathfrak{P}$ of L, we have the following property:*

if Φ is a Schwartz-Bruhat function on $L_{\mathfrak{P}}$, χ is a character of $\mathcal{O}^*_{\mathfrak{P}}$, and α is a pole of the local zeta function $Z_{\mathfrak{P}}(f, \chi, \Phi; s)$, then $\exp(2\pi i \Re(\alpha))$ is a monodromy eigenvalue of f at some point $x \in X_s$.

There is a stronger version of the conjecture, saying that under the same conditions, $\Re(\alpha)$ is a root of the Bernstein-Sato polynomial $b_f(s)$ of f ; see [19, §4] for the definition of $b_f(s)$. This statement is indeed stronger since it is known that for any root β of $b_f(s)$, the value $\exp(2\pi i \beta)$ is an eigenvalue of monodromy at some point x of X_s [29][21]. It is also known that the roots of $b_f(s)$ are negative rational numbers [20]. If X_s is smooth, then $b_f(s) = s + 1$, so the strong version of the monodromy conjecture is valid in this (very) special case.

For future reference, we state the following particular case of the Monodromy Conjecture.

Conjecture 3.2 (Untwisted Monodromy Conjecture). *Let L be a number field and f an element of $L[\widetilde{x}]$. For almost all finite places $\mathfrak{P}$, we have the following property:*

if α is a pole of the local zeta function $Z_{\mathfrak{P}}(f; s)$ associated to f over $L_{\mathfrak{P}}$ (and to $\Phi = 1_{\mathcal{O}^d_{\mathfrak{P}}}$ and $\chi = \chi_{triv}$), then $\exp(2\pi i \Re(\alpha))$ is a monodromy eigenvalue of f at some point $x \in X_s$.

3.3. **Some evidence.** (1) *The archimedean case.* As we mentioned in the introduction, Igusa's theory of local zeta functions also has a archimedean wing, studying zeta functions over the local field $\mathbb{K}$ with $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. They are defined in a similar way, as functions on the space of quasi-characters of $\mathbb{K}$, associated to a polynomial f over $\mathbb{K}$ and a Schwartz-Bruhat function Φ on $\mathbb{K}^d$ (a $\mathcal{C}^\infty$-function with compact support).

For instance, for $\mathbb{K} = \mathbb{R}$ and $f \in \mathbb{R}[\widetilde{x}]$ with $\widetilde{x} = (x_1, \ldots, x_d)$, we get

$$Z(f, \Phi, \chi; s) = \int_{\mathbb{R}^d} \Phi \cdot \chi(f) \cdot |f|^s d\widetilde{x}$$

for $s \in \mathbb{C}$ with $\Re(s) > 0$, where χ is either the constant function 1 or the sign function $sgn(\cdot)$.

Using the functional equation $D \cdot f^{s+1} = b_f(s) f^s$ for the Bernstein-Sato polynomial (with $D \in K[\widetilde{x}, \partial_{\widetilde{x}}, s]$) and integration by parts, it is not hard to show that $Z(f, \Phi; s)$ has a meromorphic continuation to $\mathbb{C}$, and that its poles α are of the form

$\beta - j$ with $b_f(\beta) = 0$ and $j \in \mathbb{N}$ [19, 5.3]. In particular, $\exp(2\pi i \alpha)$ is an eigenvalue of monodromy[1].

However, integration by parts does not make sense in the *p*-adic setting. Therefore, it is quite surprising that we still get (at least conjecturally) a similar relation between poles of the zeta function and roots of the Bernstein polynomial.

(2) *A'Campo's formula.* Let g be a polynomial in $\mathbb{C}[x_1, \ldots, x_d]$ and denote by Y_s the complex hypersurface defined by g. The eigenvalues of monodromy of g at the points of $Y_s(\mathbb{C})$ can be computed on an embedded resolution of singularities $h : Y \to \mathbb{A}^d_{\mathbb{C}}$ of g. Denote by E_i, $i \in I$ the irreducible components of the divisor $E = (g \circ h)$ and by (N_i, ν_i) the corresponding numerical data. For each $i \in I$ we put $E_i^o = E_i \setminus \cup_{j \neq i} E_j$.

If x is a point of $Y_s(\mathbb{C})$, then the monodromy zeta function $\zeta_{g,x}(T)$ is defined as the alternating product of the characteristic polynomials of the monodromy transformation on the singular cohomology spaces of the Milnor fiber F_x of g at x :

$$\zeta_{g,x}(T) = \prod_{i \geq 0} \det(1 - T \cdot M_x \,|\, H^i_{sing}(F_x, \mathbb{Z}))^{(-1)^{i+1}}$$

Using Leray's spectral sequence and an explicit description of the stalks of the complex of nearby cycles of $g \circ h$, A'Campo [1] proved the following formula:

$$\zeta_{g,x}(T) = \prod_{i \in I} (1 - T^{N_i})^{-\chi_{top}(E_i^o \cap h^{-1}(x))}$$

Here $\chi_{top}(.)$ is the topological Euler characteristic.

Moreover, using the perversity of the nearby cycles complex, Denef [9] observed that any eigenvalue of monodromy at a point $x \in Y_s(\mathbb{C})$ occurs as a zero or pole of $\zeta_{g,y}(T)$ for some (possibly different) point $y \in Y_s(\mathbb{C})$.

Now let L be a number field, and f a polynomial in $L[x_1, \ldots, x_d]$, and fix an embedded resolution of singularities for f. In principle, using Denef's explicit formula (Theorem 2.9) and A'Campo's formula, one can compute on this embedded resolution the residues of the candidate poles of the $\mathfrak{P}$-adic zeta function of f (for almost all finite places $\mathfrak{P}$ of L), and all eigenvalues of monodromy. By this procedure, one eliminates fake candidate poles and one obtains a list of eigenvalues of monodromy, and when all remaining candidate poles induce eigenvalues, the monodromy conjecture is proven. For instance, using A'Campo's formula, one easily sees that 1 and $\exp(\pi i / 3)$ are monodromy eigenvalues of the polynomial $g(x, y) = y^2 - x^3$ at the origin. Hence, the monodromy conjecture holds for $g(x, y)$, since the computations in Section 1 show that for $p \gg 0$ the poles of $Z_p(g, s)$ have real part -1 or $-5/6$.

Unfortunately, this strategy requires a strong control over the configuration of the exceptional components E_i and their numerical data (one has to prove that configurations inducing "bad" poles cannot occur in the exceptional locus of a resolution). It led, for instance, to a proof of the conjecture in the following cases:

- $d = 2$ [25][38]
- $d = 3$ and f homogeneous [39] [2]
- superisolated surface singularities [2]

---

[1] In spite of this result, it remains a challenging problem to determine which candidate poles actually occur, and to determine their multiplicity; see for instance [14].

More generally, Veys obtained very nice results on possible configurations of exceptional divisors, thus gathering strong evidence for the conjecture, especially for $d = 3$ [41][42].

In other settings, some combinatorial description of the singularities yields an expression for the local zeta function and/or the monodromy zeta function, and the conjecture can be proven from this expression; e.g. for

- f non-degenerate w.r.t. its Newton polyhedron, with an additional technical assumption [26]
- f quasi-ordinary [3]

(3) *Prehomogeneous vector spaces.* Finally, we should mention the case where f is a relative invariant of a prehomogeneous vector space (G, X). This was one of the first classes of examples where Igusa made extensive computations (exploiting the group structure), and these computations led to several conjectures, including the monodromy conjecture. The monodromy conjecture was proven in [22] for (G, X) irreducible and reduced, using Igusa's group-theoretic expression for a complete list of candidate-poles of the local zeta function.

It does not seem probable that any of these techniques can be used to deal with the general case, because the geometric complexity becomes quite hard to control in higher dimensions; a more intrinsic relation between the local zeta function and monodromy should be discovered.

Recently, Sebag and the author introduced a geometric object whose rational points are related to the (motivic) zeta function, and whose étale cohomology (with Galois action) coincides with the cohomology of the Milnor fiber (with monodromy action) [35][31]. This object is a non-archimedean analytic space over the field $\mathbb{C}((t))$ of complex Laurent series; we called it the analytic Milnor fiber. Its geometric/arithmetic properties are closely related to the structure of the singularities of f, and many of its invariants have a natural interpretation in singularity theory. We hope that the study of this object will lead to new insights into the monodromy conjecture.

4. Motivic integration

4.1. **From $\mathbb{Z}_p$ to $k[[t]]$.** Kontsevich invented motivic integration to strengthen the following result by Batyrev [4].

Theorem 4.1 (Batyrev)**.** *If two complex Calabi-Yau varieties are birationally equivalent, then they have the same Betti numbers.*

Batyrev proved this result using p-adic integration and the Weil Conjectures. Kontsevich observed that Batyrev's proof could be "geometrized", avoiding the passage to finite fields and yielding a stronger result: equality of Hodge numbers. The key was to replace the p-adic integers by $\mathbb{C}[[t]]$, and p-adic integration by motivic integration.

Kontsevich presented these ideas at a famous "Lecture at Orsay" in 1995, but never published them. The theory was developed and generalized in the following directions:

- Denef and Loeser [11] developed a theory of *geometric* motivic integration on arbitrary algebraic varieties over a field of characteristic zero. They also

created a theory of *arithmetic* motivic integration [12], with good special-
ization properties to *p*-adic integrals in a general setting, using the model
theory of pseudo-finite fields. The motivic integral appears here as a uni-
versal integral, specializing to the *p*-adic ones for almost all *p*.
- Loeser and Sebag constructed a theory of motivic integration on formal
schemes [40] and rigid varieties [28], working over an arbitrary complete
discrete valuation ring with perfect residue field.
- Cluckers and Loeser built a very general framework for motivic integration
theories, based on model theory [5]. A different model-theoretic approach
was developed by Hrushovksi and Kazhdan [17].

We will only discuss the so-called "naïve" geometric motivic integration on smooth
algebraic varieties (but the adjective "naïve" does no right to the stunning vision
behind the constructions). We start by explaining the basic ideas.

In Section 2.1, we introduced the notion of cylinder in $(\mathbb{Z}_p)^d$ using the projection
maps $\pi_m : (\mathbb{Z}_p)^d \to (\mathbb{Z}/p^{m+1})^d$ for $m \geq 0$, and we saw that the Haar measure of
a cylinder C can be computed from the cardinality of the projection $\pi_m(C)$ for
$m \gg 0$. If we identify $\mathbb{Z}_p$ with the ring of Witt vectors $W(\mathbb{F}_p)$, then the map π_m
simply corresponds to the truncation map

$$W(\mathbb{F}_p) \to W_{m+1}(\mathbb{F}_p) : (a_0, a_1, \ldots) \mapsto (a_0, a_1, \ldots, a_m)$$

The idea behind the theory of motivic integration is to make a similar construc-
tion, replacing $W(\mathbb{F}_p)$ by the ring of formal power series $k[[t]]$ over some field k,
and the map π_m by the truncation map

$$k[[t]]^d \to (k[t]/(t^{m+1}))^d : (\sum_{i \geq 0} a_{1,i} t^i, \ldots, \sum_{i \geq 0} a_{n,i} t^i) \mapsto (\sum_{i=0}^{m} a_{1,i} t^i, \ldots, \sum_{i=0}^{m} a_{n,i} t^i)$$

The problem is to give meaning to the expression $|\pi_m(C)|$ if C is a "cylinder"
in $k[[t]]^d$ for infinite fields k, and to find a candidate to replace p. But interpret-
ing the coefficients of a power series as affine coordinates, the set $(k[[t]]/(t^{m+1}))^d$
gets the structure of the set of k-points on an affine space $\mathbb{A}_k^{(m+1)d}$, and if we re-
strict to cylinders C such that $\pi_m(C)$ is constructible in $\mathbb{A}_k^{(m+1)d}$, we can use the
Grothendieck ring of varieties as a universal way to "count" points on constructible
subsets of an algebraic variety. The cardinality p of $\mathbb{F}_p$ is replaced by the "number"
of points on the affine line $\mathbb{A}_k^1$; this is the *Lefschetz motive* $\mathbb{L}$.

The price to pay is that we leave classical integration theory since our value ring
will be an abstract object (the Grothendieck ring of varieties) instead of $\mathbb{R}$.

In the following sections, we will make these ideas precise.

4.2. The Grothendieck ring of varieties. Let k be any field. A k-variety is a
reduced separated k-scheme of finite type.

Definition 4.2 (Grothendieck ring of k-varieties)**.** *As an abelian group, $K_0(Var_k)$
has the following presentation:*
- *generators: isomorphism classes $[X]$ of separated k-schemes of finite type
X*
- *relations: $[X] = [X \setminus Y] + [Y]$ if Y is a closed subscheme of X ("scissor
relations")*

*We endow $K_0(Var_k)$ with the unique ring multiplication such that $[X_1] \cdot [X_2] =
[X_1 \times_k X_2]$ for any pair X_1, X_2 of separated k-schemes of finite type.*

We put $\mathbb{L} = [\mathbb{A}_k^1]$ and we denote by $\mathcal{M}_k$ the localized Grothendieck ring

$$\mathcal{M}_k = K_0(Var_k)[\mathbb{L}^{-1}]$$

The Grothendieck ring and its localization are still quite mysterious. It is known that $K_0(Var_k)$ is not a domain if k is a field of characteristic zero [37]. It is not known if the localization morphism $K_0(Var_k) \to \mathcal{M}_k$ is injective, i.e. if $\mathbb{L}$ is a zero divisor in $K_0(Var_k)$, or if $\mathcal{M}_k$ is a domain if k is algebraically closed. For related questions and results, see for instance [24] and [34].

Remark. If X is any k-scheme of finite type and X_{red} is the maximal reduced closed subscheme of X, then the closed immersion $X_{red} \to X$ is a bijection and the scissor relations imply that $[X] = [X_{red}]$ in $K_0(Var_k)$. Hence, we get the same Grothendieck ring if we replace "separated k-scheme of finite type" by "k-variety", and the reader who is unfamiliar with the formalism of schemes can stick to this definition. One word of warning, though: if k is imperfect and X_1 and X_2 are k-varieties, $X_1 \times_k X_2$ need not be reduced. $\qquad\square$

A subset of a k-variety X is called *locally closed* if it is open in its closure in X w.r.t. the Zariski topology on X. Such a subset carries a unique structure of subvariety of X. A subset C of X is called *constructible* if it is a finite union of locally closed subsets of X. Then we can always write C as a finite disjoint union of locally closed subsets $U_1, \ldots, U_r$ and the scissor relations in the Grothendieck ring imply that the class

$$[C] := \sum_{i=1}^r [U_i]$$

in $K_0(Var_k)$ does not depend on the choice of the partition.

Let F be a k-variety. A morphism of k-varieties $Y \to X$ is a Zariski-locally trivial fibration with fiber F if any point of X has an open neighbourhood U such that $Y \times_X U$ is isomorphic to $F \times_k U$ as a U-scheme. In this case, we have $[Y] = [X] \cdot [F]$ in $K_0(Var_k)$. Indeed, using the scissor relations and Noetherian induction we can reduce to the case where the fibration is trivial.

By its very definition, the Grothendieck ring is the universal additive multiplicative invariant of k-varieties: whenever $\chi(\cdot)$ is an invariant of k-varieties taking values in a ring A and such that $\chi(\cdot)$ is additive w.r.t. closed immersions and multiplicative w.r.t. the fiber product over k, it will factor through a unique morphism of rings

$$\chi : K_0(Var_k) \to A$$

with $\chi([X]) = \chi(X)$ for any k-variety X. So in a way, taking the class $[X]$ of X in the Grothendieck ring is the most general way to "count points" on, or "measure the size" of, the variety X.

Here are some important specialization morphisms.

(1) For any finite field $\mathbb{F}_q$, consider the invariant $\sharp$ which associates to a $\mathbb{F}_q$-variety X the number of $\mathbb{F}_q$-rational points on X. This invariant is additive and multiplicative and hence defines a ring morphism $\sharp : K_0(Var_{\mathbb{F}_q}) \to \mathbb{Z}$. It localizes to a ring morphism $\sharp : \mathcal{M}_{\mathbb{F}_q} \to \mathbb{Z}[q^{-1}]$.

(2) For $k = \mathbb{C}$, we can consider the invariant χ_{top} which associates to a $\mathbb{C}$-variety X the topological Euler characteristic of $X(\mathbb{C})$ w.r.t. the complex topology. Again,

this invariant defines a ring morphism $\chi_{top} : K_0(Var_{\mathbb{C}}) \to \mathbb{Z}$ which localizes to a ring morphism $\chi_{top} : \mathcal{M}_{\mathbb{C}} \to \mathbb{Z}$ since $\chi_{top}(\mathbb{A}_{\mathbb{C}}^1) = 1$.

If k is any field we can consider the ℓ-adic Euler characteristic χ_{top} instead, with ℓ a prime invertible in k. It is known that χ_{top} does not depend on the choice of ℓ.

(3) For $k = \mathbb{C}$, we can consider the Hodge-Deligne polynomial $HD(X; u, v) \in \mathbb{Z}[u, v]$ of a $\mathbb{C}$-variety X. It is defined by

$$HD(X; u, v) = \sum_{p,q \geq 0} \sum_{i \geq 0} (-1)^i h^{p,q}(H_c^i(X(\mathbb{C}), \mathbb{C})) u^p v^q$$

where $h^{p,q}(H_c^i(X(\mathbb{C}), \mathbb{C}))$ denotes the dimension of the (p, q)-component of Deligne's mixed Hodge structure on $H_c^i(X(\mathbb{C}), \mathbb{C})$. One can show that $HD(\cdot; u, v)$ is additive and multiplicative, so there exists a unique ring morphism

$$HD : K_0(Var_{\mathbb{C}}) \to \mathbb{Z}[u, v]$$

mapping $[X]$ to $HD(X; u, v)$ for each complex variety X. We have

$$HD(\mathbb{A}_{\mathbb{C}}^1; u, v) = uv$$

and HD localizes to a ring morphism

$$HD : K_0(Var_{\mathbb{C}}) \to \mathbb{Z}[u, u^{-1}, v, v^{-1}]$$

4.3. **Arc spaces.** Let X be a variety over k. For each integer $n \geq 0$, we consider the functor

$$F_n : (k - alg) \to (Sets) : A \mapsto X(A[t]/(t^{n+1}))$$

which sends a k-algebra A to the set of points on X with coordinates in $A[t]/(t^{n+1})$.

Proposition 4.3. *The functor F_n is representable by a separated k-scheme of finite type $\mathcal{L}_n(X)$, called the n-th jet scheme of X. If X is affine, then so is $\mathcal{L}_n(X)$.*

The proposition means that there exists for any k-algebra A a bijection $\phi_n(A)$ between the set of points on $\mathcal{L}_n(X)$ with coordinates in A and the set $F_n(A)$, with the property that the diagram

$$
\begin{array}{ccc}
\mathcal{L}_n(X)(A) & \xrightarrow{\phi_n(A)} & F_n(A) \\
\downarrow & & \downarrow \\
\mathcal{L}_n(X)(B) & \xrightarrow{\phi_n(B)} & F_n(B)
\end{array}
$$

commutes for any morphism of k-algebras $A \to B$. By Yoneda's Lemma, this property determines $\mathcal{L}_n(X)$ as a k-scheme, up to canonical isomorphism.

Instead of giving a proof of Proposition 4.3, we look at an example.

Example. Let X be the closed subvariety of $\mathbb{A}_k^2 = \operatorname{Spec} k[x, y]$ defined by the equation $x^2 - y^3 = 0$. Then a point of $\mathcal{L}_2(X)$ with coordinates in some k-algebra A is a couple

$$(x_0 + x_1 t + x_2 t^2, y_0 + y_1 t + y_2 t^2)$$

with $x_0, \ldots, y_2 \in A$ such that

$$(x_0 + x_1 t + x_2 t^2)^2 - (y_0 + y_1 t + y_2 t^2)^3 \equiv 0 \mod t^3$$

Working this out, we get the equations

$$\begin{cases} (x_0)^2 - (y_0)^3 & = & 0 \\ 2x_0 x_1 - 3(y_0)^2 y_1 & = & 0 \\ (x_1)^2 + 2x_0 x_2 - 3y_0(y_1)^2 - 3(y_0)^2 y_2 & = & 0 \end{cases}$$

and if we view $x_0, \ldots, y_2$ as affine coordinates, these equations define $\mathcal{L}_2(X)$ as a closed subscheme of $\mathbb{A}_k^6$.   □

For any pair of integers $m \geq n \geq 0$ and any k-algebra A, the truncation map

$$A[t]/t^{m+1} \to A[t]/t^{n+1}$$

defines a natural transformation of functors $F_m \to F_n$, so by Yoneda's Lemma we get a natural truncation morphism of k-schemes

$$\pi_n^m : \mathcal{L}_m(X) \to \mathcal{L}_n(X)$$

This is the unique morphism such that for any k-algebra A, the square

$$\begin{array}{ccc} X(A[t]/t^{m+1}) & \longrightarrow & X(A[t]/t^{n+1}) \\ \phi_m(A) \downarrow & & \downarrow \phi_n(A) \\ \mathcal{L}_m(X)(A) & \xrightarrow{\pi_n^m(A)} & \mathcal{L}_n(X)(A) \end{array}$$

commutes.

Since the schemes $\mathcal{L}_n(X)$ are affine for affine X, and $\mathcal{L}_n(\cdot)$ takes open covers to open covers, the morphisms π_n^m are affine for any k-variety X. This property guarantees that the projective limit

$$\mathcal{L}(X) = \varprojlim_n \mathcal{L}_n(X)$$

exists in the category of k-schemes. The scheme $\mathcal{L}(X)$ is called the arc scheme of X. It is not Noetherian, in general. It comes with natural projection morphisms

$$\pi_n : \mathcal{L}(X) \to \mathcal{L}_n(X)$$

For any field F over k, we have a natural bijection

$$\mathcal{L}(X)(F) = X(F[[t]])$$

and the points of these sets are called F-valued arcs on X. The morphism π_n maps an arc to its truncation modulo t^{n+1} in $X(F[t]/(t^{n+1}))$. In particular, π_0 sends an element ψ of $X(F[[t]])$ to the element $\psi(0)$ of $X(F)$ obtained by putting $t = 0$ in the coordinates of ψ. We call $\psi(0)$ the origin of the arc ψ. An arc should be seen as a two-dimensional infinitesimal disc on X with origin at $\psi(0)$.

It follows immediately from the definition that $\mathcal{L}_0(X) = X$ and that $\mathcal{L}_1(X)$ is the tangent scheme of X. A morphism of k-varieties $h : Y \to X$ induces morphisms

$$\begin{array}{rcl} \mathcal{L}(h) & : & \mathcal{L}(Y) \to \mathcal{L}(X) \\ \mathcal{L}_n(h) & : & \mathcal{L}_n(Y) \to \mathcal{L}_n(X) \end{array}$$

which commute with the truncation maps.

If X is smooth over k, of pure dimension d, then the morphisms π_n^m are Zariski-locally trivial fibrations with fiber $\mathbb{A}_k^{d(m-n)}$. To see this, note that by smoothness, X can be covered with open subvarieties U which admit an étale morphism $h : U \to$

$\mathbb{A}_k^d$. Since $A[t]/(t^{m+1}) \to A[t]/(t^{n+1})$ is a nilpotent immersion for any k-algebra A, the infinitesimal lifting criterion for étale morphisms implies that the square

$$
\begin{array}{ccc}
\mathcal{L}_m(U) & \xrightarrow{\mathcal{L}_m(h)} & \mathcal{L}_m(\mathbb{A}_k^d) \\
\pi_n^m \downarrow & & \downarrow \pi_n^m \\
\mathcal{L}_n(U) & \xrightarrow{\mathcal{L}_n(h)} & \mathcal{L}_n(\mathbb{A}_k^d)
\end{array}
$$

is Cartesian. In intuitive terms, arcs are (étale-)local objects on X and any smooth variety of pure dimension d looks (étale-)locally like an open subvariety of $\mathbb{A}_k^d$; but an element of

$$
\mathcal{L}_n(\mathbb{A}_k^d)(A) = \mathbb{A}_k^d(A[t]/t^{n+1})
$$

is simply a d-tuple of elements in $A[t]/t^{n+1}$.

If X is singular, the schemes $\mathcal{L}_n(X)$ and $\mathcal{L}(X)$ are still quite mysterious. They contain a lot of information on the singularities of X. Motivic integration provides a powerful way to extract interesting invariants of the singularities from the arc schemes $\mathcal{L}(X)$.

Example. We continue our previous Example. For any k-algebra A, a point on $\mathcal{L}(X)$ with coordinates in A is given by a couple

$$
(x(t) = x_0 + x_1 t + x_2 t^2 + \ldots, y(t) = y_0 + y_1 t + y_2 t^2 + \ldots)
$$

with $x_i, y_i \in A$, such that $x(t)^2 - y(t)^3 = 0$.

Working this out yields an infinite number of polynomial equations in the variables x_i, y_i and these realize $\mathcal{L}(X)$ as a closed subscheme of the infinite-dimensional affine space

$$
\mathbb{A}_k^\infty = \operatorname{Spec} k[x_0, y_0, x_1, y_1, \ldots]
$$

The truncation map

$$
\pi_n : \mathcal{L}(X) \to \mathcal{L}_n(X)
$$

sends $(x(t), y(t))$ to

$$
(x_0 + \ldots + x_n t^n, y_0 + \ldots + y_n t^n)
$$

and (if A is a field) the origin of $(x(t), y(t))$ is simply the point (x_0, y_0) in $X(A)$. $\square$

If k has characteristic zero, one can give an elegant construction of the schemes $\mathcal{L}_n(X)$ and $\mathcal{L}(X)$ using differential algebra. Assume that X is affine, say given by polynomial equations

$$
f_1(x_1, \ldots, x_r) = \ldots = f_\ell(x_1, \ldots, x_r)
$$

in affine r-space $\mathbb{A}_k^r = \operatorname{Spec} k[x_1, \ldots, x_r]$. Consider the k-algebra

$$
B = k[y_{1,0}, \ldots, y_{r,0}, y_{1,1}, \ldots]
$$

and the unique k-derivation $\delta : B \to B$ mapping $y_{i,j}$ to $y_{i,j+1}$ for each i, j. Then $\mathcal{L}(X)$ is isomorphic to the closed subscheme of $\operatorname{Spec} B$ defined by the equations

$$
\delta^{(i)}(f_q(y_{1,0}, \ldots, y_{r,0})) = 0
$$

for $q = 1, \ldots, \ell$ and $i \in \mathbb{N}$. The point with coordinates $y_{i,j}$ corresponds to the arc

$$
\left(\sum_{j \geq 0} \frac{y_{1,j}}{j!} t^j, \ldots, \sum_{j \geq 0} \frac{y_{r,j}}{j!} t^j \right)
$$

See for instance [36] for details.

4.4. **Motivic integrals.** Copying the notion of cylinder and the description of its Haar measure to the setting of arc spaces, we can define a *motivic measure* on a class of subsets of the arc space $\mathcal{L}(X)$. From now on, we assume that X is smooth over k, of pure dimension d.

Definition 4.4. *A cylinder in $\mathcal{L}(X)$ is a subset C of the form $(\pi_m)^{-1}(C_m)$, with $m \geq 0$ and C_m a constructible subset of $\mathcal{L}_m(X)$.*

Note that the set of cylinders in $\mathcal{L}(X)$ is a Boolean algebra, i.e. it is closed under complements, finite unions and finite intersections.

Lemma-Definition 4.5. *Let C be a cylinder in $\mathcal{L}(X)$, and choose $m \geq 0$ such that $C = (\pi_m)^{-1}(C_m)$ with C_m constructible in $\mathcal{L}_m(X)$. The value*

$$\mu(C) := [\pi_m(C)]\mathbb{L}^{-d(m+1)} \quad \in \mathcal{M}_k$$

does not depend on m, and is called the motivic measure $\mu(C)$ of C.

Proof. This follows immediately from the fact that the truncation morphisms π_m^n are Zariski-locally trivial fibrations with fiber $\mathbb{A}_k^{d(n-m)}$. $\qquad\square$

Here is an elementary but important example.

Example. If $C = \mathcal{L}(X)$, then $\mu(C) = \mathbb{L}^{-d}[X]$. $\qquad\square$

The normalization factor $\mathbb{L}^{-d}$ is added in accordance with the p-adic case, where the ring of integers gets measure one (rather then the cardinality of the residue field). For geometric applications, it would be more natural to omit it (and this is often done in literature). Note that the motivic measure μ is additive w.r.t. finite disjoint unions.

Remark. In the general theory of motivic integration on algebraic varieties [11], one constructs a much larger class of measurable subsets of arc spaces of (possibly singular) k-varieties, and one defines the motivic measure via approximation by cylinders. This necessitates replacing $\mathcal{M}_k$ by a certain "dimensional completion" $\widehat{\mathcal{M}}_k$. $\qquad\square$

The following definition suggests itself.

Definition 4.6. *We say that a function*

$$\alpha : \mathcal{L}(X) \to \mathbb{N} \cup \{\infty\}$$

is integrable if it takes finitely many values, and if the fiber $\alpha^{-1}(i)$ is a cylinder for each $i \in \mathbb{N}$.

We define the motivic integral of α by

$$\int_{\mathcal{L}(X)} \mathbb{L}^{-\alpha} = \sum_{i \in \mathbb{N}} \mu(\alpha^{-1}(i))\mathbb{L}^{-i} \quad \in \mathcal{M}_k$$

We will need the following generalization of the definition.

Definition 4.7. *Let s be a formal variable, and consider functions*

$$\alpha, \beta : \mathcal{L}(X) \to \mathbb{N} \cup \{\infty\}$$

We say that $\alpha \cdot s + \beta$ is integrable if $\alpha^{-1}(i)$ and $\beta^{-1}(i)$ are cylinders for every $i \in \mathbb{N}$, and if β takes only a finite number of values on each fiber $\alpha^{-1}(i)$ with $i \in \mathbb{N}$.

We define the motivic integral of $\alpha \cdot s + \beta$ by

$$\int_{\mathcal{L}(X)} \mathbb{L}^{-(\alpha \cdot s + \beta)} = \sum_{i,j \in \mathbb{N}} \mu(\alpha^{-1}(i) \cap \beta^{-1}(j))\mathbb{L}^{-(is+j)} \quad \in \mathcal{M}_k[[\mathbb{L}^{-s}]]$$

The condition that β assumes only finitely many values on the fibers of α guarantees that the coefficient of each $\mathbb{L}^{-is}$ in the definition is a finite sum.

4.5. **Change of variables.** The central and most powerful tool in the theory of motivic integration is the change of variables formula. It has various profound applications in birational geometry and singularity theory. For its precise statement, we need some auxiliary notation. For any k-variety Y, any ideal sheaf $\mathscr{J}$ on Y and any arc

$$\psi : \operatorname{Spec} F[[t]] \to Y$$

on Y, we define the order of $\mathscr{J}$ at ψ by

$$ord_t \, \mathscr{J}(\psi) = \min\{ord_t \psi^* f \mid f \in \mathscr{J}_{\psi(0)}\}$$

where $ord_t \psi^* f$ is the t-adic valuation of $\psi^* f \in F[[t]]$. In this way, we obtain a function

$$ord_t \, \mathscr{J} : \mathcal{L}(Y) \to \mathbb{N} \cup \{\infty\}$$

whose fibers over $\mathbb{N}$ are cylinders. Note that $ord_t \, \mathscr{J}(\psi) = \infty$ iff the image of ψ is contained in the support of $\mathscr{J}$.

Theorem 4.8 (Denef-Loeser; Change of variables formula)**.** *Assume that k is perfect. Let $h : Y \to X$ be a proper birational morphism, with Y smooth over k, and denote by Jac_h the Jacobian ideal sheaf of h. Let $\alpha \cdot s + \beta$ be an integrable function on $\mathcal{L}(X)$, and assume that $ord_t Jac_h$ takes only finitely many values on each fiber of $\alpha \circ \mathcal{L}(h)$ over $\mathbb{N}$. Then*

$$\int_{\mathcal{L}(X)} \mathbb{L}^{-(\alpha \cdot s + \beta)} = \int_{\mathcal{L}(Y)} \mathbb{L}^{-((\alpha \circ \mathcal{L}(h)) \cdot s + \beta \circ \mathcal{L}(h) + ord_t Jac_h)}$$

in $\mathcal{M}_k[[\mathbb{L}^{-s}]]$.

Remark. In [11], Denef and Loeser stated the change of variables formula in the case where k has characteristic zero, but this assumption is not necessary; see [40]. $\square$

The proof of the change of variables formula goes beyond the scope of this introduction. The very basic idea behind the formula is the following: if we denote by V the closed subscheme of Y defined by the Jacobian ideal Jac_h, and by U its image under h, then the morphism $h : Y - V \to X - U$ is an isomorphism. Combined with the properness of h, this implies that

$$\mathcal{L}(h) : \mathcal{L}(Y) - \mathcal{L}(V) \to \mathcal{L}(X) - \mathcal{L}(U)$$

is a bijection; but $\mathcal{L}(V)$ and $\mathcal{L}(U)$ have measure zero in $\mathcal{L}(Y)$, resp. $\mathcal{L}(X)$ (w.r.t. a certain more refined motivic measure) so it is reasonable to expect that there exists a change of variables formula.

The jet schemes $\mathcal{L}_n(Y)$, however, are "contracted" under the morphism

$$\mathcal{L}_n(h) : \mathcal{L}_n(Y) \to \mathcal{L}_n(X)$$

and this affects the motivic measure of cylinders. The "contraction factor" is measured by the Jacobian.

Example. Put $X = \mathbb{A}_k^3 = \operatorname{Spec} k[x, y, z]$ and let $h : Y \to X$ be the blow-up of X at the origin O. Let $\mathscr{J}$ be the ideal sheaf on X defining the origin. It corresponds to the ideal (x, y, z) in $k[x, y, z]$. Denote by C the subset of $\mathcal{L}(X)$ where the function $ord_t \mathscr{J}$ takes the value 1. It consists of the arcs

$$(x_0 + x_1 t + x_2 t^2 + \ldots, y_0 + y_1 t + y_2 t^2 + \ldots, z_0 + z_1 t + z_2 t^2 + \ldots)$$

on $\mathbb{A}_k^3$ with $x_0 = y_0 = z_0 = 0$ and $(x_1, y_1, z_1) \neq (0, 0, 0)$, so we see that it is a cylinder over the constructible subset $\pi_1(C)$ of

$$\mathcal{L}_1(X) = \operatorname{Spec} k[x_0, x_1, y_0, y_1, z_0, z_1]$$

defined by the same conditions. Since $\pi_1(C)$ is isomorphic to $\mathbb{A}_k^3 \setminus \{(0, 0, 0)\}$, the motivic measure of C is equal to

$$\mu(C) = \mathbb{L}^{-6}[\pi_1(C)] = \mathbb{L}^{-6}(\mathbb{L}^3 - 1)$$

in $\mathcal{M}_k$.

The pull-back $\mathscr{J}' = \mathscr{J}\mathcal{O}_Y$ of the ideal sheaf $\mathscr{J}$ to Y is the defining ideal sheaf of the exceptional divisor $E = h^{-1}(O)$. The jacobian ideal Jac_h of h is the square of $\mathscr{J}'$. The inverse image C' of C under the morphism $\mathcal{L}(h)$ is the cylinder in $\mathcal{L}(Y)$ where the function

$$ord_t \mathscr{J}' = ord_t \mathscr{J} \circ \mathcal{L}(h)$$

takes value 1. This implies that $ord_t Jac_h$ takes value 2 on C'.

The set C' is a cylinder over its image $\pi_1(C') \cong E \times \mathbb{A}_k^2 \times_k \mathbb{G}_{m,k}$ in $\mathcal{L}_1(C')$. Indeed: if u is any closed point of E and we choose local coordinates (x', y', z') on an open neighbourhood U of u (i.e. an étale morphism $U \to \mathbb{A}_k^3 = \operatorname{Spec} k[x', y', z']$ mapping u to the origin) such that $E \cap U$ is defined by the equation $x' = 0$, then $C' \cap (\pi_0)^{-1}(U)$ is determined by the conditions $x_0' = 0$ and $x_1' \neq 0$.

The motivic measure of C' is equal to

$$\mu(C') = \mathbb{L}^{-6}(\mathbb{L} - 1)\mathbb{L}^2[E]$$

in $\mathcal{M}_k$. The change of variables formula tells us that

$$\mu(C) = \mu(C')\mathbb{L}^{-2}$$

which is indeed the case since E is isomorphic to $\mathbb{P}_k^2$ and $[E] = \mathbb{L}^2 + \mathbb{L} + 1$ in $\mathcal{M}_k$. $\square$

4.6. **A (cheating) proof of Kontsevich' theorem.** As we already mentioned, Kontsevich created motivic integration to prove the following theorem. Recall that a Calabi-Yau variety over $\mathbb{C}$ is a smooth, proper, connected $\mathbb{C}$-variety X of dimension d such that $\Omega_{X/\mathbb{C}}^d$ is isomorphic to $\mathcal{O}_X$, i.e. such that X admits a nowhere vanishing differential form of maximal degree.

Theorem 4.9 (Kontevich)**.** *If X and Y are birationally equivalent Calabi-Yau varieties over $\mathbb{C}$, then X and Y have the same Hodge numbers.*

To prove this result, we need the refined version of motivic integration taking values in a certain completion $\widehat{\mathcal{M}}_{\mathbb{C}}$ of $\mathcal{M}_{\mathbb{C}}$. Nevertheless, it is possible to indicate at least the general ideas. Denote by d the dimension of X and Y. The fact that X and Y are birationally equivalent implies that there exists a smooth and proper $\mathbb{C}$-variety Z together with proper birational morphisms

$$X \xleftarrow{\ f\ } Z \xrightarrow{\ g\ } Y$$

We can express $\mathbb{L}^{-d}[X] \in \mathcal{M}_{\mathbb{C}}$ by the motivic integral

$$\int_{\mathcal{L}(X)} 1$$

(where we should really write $\mathbb{L}^{-0}$ instead of 1) and the analogous expression holds for $\mathbb{L}^{-d}[Y]$. Now we can compute both motivic integrals on Z, using the change of variables formula. This yields

$$(4.1) \qquad \mathbb{L}^{-d}[X] \quad = \quad \int_{\mathcal{L}(Z)} \mathbb{L}^{-ord_t Jac_f}$$

$$(4.2) \qquad \mathbb{L}^{-d}[Y] \quad = \quad \int_{\mathcal{L}(Z)} \mathbb{L}^{-ord_t Jac_g}$$

The alert reader may have noticed that this is the place where we're cheating, since $ord_t Jac_f$ and $ord_t Jac_g$ take infinitely many values on $\mathcal{L}(Z)$ if f and g are not isomorphisms, so that the integrals on the right hand side become infinite sums. However, the general formalism of motivic integration can deal with such expressions, replacing $\mathcal{M}_{\mathbb{C}}$ by $\widehat{\mathcal{M}}_{\mathbb{C}}$.

One can show that the specialization morphism HD from Section 4.2 factors through the image of $\mathcal{M}_{\mathbb{C}}$ in $\widehat{\mathcal{M}}_{\mathbb{C}}$, so to prove the theorem it suffices to show that (4.1) and (4.2) coincide. Choose nowhere vanishing differential forms ω_X and ω_Y of degree d on X, resp. Y. Since X is normal and Y is proper, the rational map $g \circ f^{-1}$ is defined on an open subvariety of X whose complement has codimension ≥ 2 in X, and the pullback $(g \circ f^{-1})^* \omega_Y$ extends uniquely to a degree d differential form on X. Hence, there exists a unique regular function a on X with $(g \circ f^{-1})^* \omega_Y = a \cdot \omega_X$. Pulling back to Z we get $g^* \omega_Y = f^* a \cdot f^* \omega_X$. Symmetrically, there exists a unique regular function b on Y such that $(f \circ g^{-1})^* \omega_X = b \cdot \omega_Y$ and on Z we have $f^* \omega_X = g^* b \cdot g^* \omega_Y$. Hence, $f^* a$ and $g^* b$ are units on Z and belong to $\mathbb{C}$. This means that the ideal sheaves Jac_f and Jac_g coincide.

A similar proof yields a stronger result: we say that two proper, smooth, connected $\mathbb{C}$-varieties X and Y are K-equivalent if there exists a smooth, proper, connected $\mathbb{C}$-variety Z and proper birational morphisms $f : Z \to X$ and $g : Z \to Y$ such that the Jacobian divisors (Jac_f) and (Jac_g) are linearly equivalent (i.e. Jac_f and Jac_g are isomorphic as invertible sheaves on Z). One can show that this automatically implies that the ideal sheaves Jac_f and Jac_g coincide, and the above arguments show that $[X] = [Y]$ in $\widehat{\mathcal{M}}_{\mathbb{C}}$.

5. MOTIVIC ZETA FUNCTIONS

5.1. **Definitions.** Using the ideas in the previous section, we can transfer the definition of the *p*-adic local zeta function to the motivic framework. We will only consider the case where Φ is the characteristic function of R^d, and the character χ is trivial, i.e. we'll construct the motivic counterpart of

$$Z(f; s) = \int_{R^d} |f|_K^s d\mu$$

with $f \in K[\widetilde{x}] = K[x_1, \ldots, x_d]$. There is also a more delicate definition of the motivic zeta function for non-trivial characters χ and more general Φ; see [10].

Let k be any field, and let f be a polynomial in $k[x_1, \ldots, x_d]$. The integration space R^d is replaced by the arc scheme $\mathcal{L}(\mathbb{A}_k^d)$ with the motivic measure. What should $|f|_K^s$ be?

For $g \in K[\widetilde{x}]$ and $x \in R^d$, $|g(x)|_K$ equals $q^{-v(g(x))}$, with q the number of rational points on the affine line over the residue field of R. So in the motivic setting, the natural candidate to replace $|f(x)|_K$ for some k-field F and

$$x \in \mathcal{L}(\mathbb{A}_k^d)(F) = F[[t]]^d$$

is $\mathbb{L}^{-ord_t f(x)}$ where ord_t denotes the t-adic valuation on $F[[t]]$. This leads to the following definition.

Definition 5.1. *Let k be any field, let X be a smooth k-variety of pure dimension, and let*

$$f : X \to \mathbb{A}_k^1$$

be a k-morphism to the affine line. We define the motivic zeta function $Z_{mot}(f; s)$ of f by

$$Z_{mot}(f; s) = \int_{\mathcal{L}(X)} \mathbb{L}^{-ord_t f \cdot s} \quad \in \mathcal{M}_k[[\mathbb{L}^{-s}]]$$

Here $ord_t f = ord_t(f)$, the function $\mathcal{L}(X) \to \mathbb{N} \cup \{\infty\}$ associated to the ideal sheaf (f) on X generated by f.

Denote by V_f the closed subscheme of X defined by (f). As in the p-adic case, a simple transformation rule relates this zeta function to the motivic Poincaré series

$$Q_{mot}(T) = \sum_{m \geq 0} [\mathcal{L}_m(V_f)] T^{m+1} \quad \in K_0(Var_k)[[T]]$$

with $T = \mathbb{L}^{-s}$. This is the motivic counterpart of Igusa's Poincaré series, since for any k-field F, the set $\mathcal{L}_m(V_f)(F)$ is the space of solutions of the congruence $f \equiv 0$ mod t^{n+1} in $X(F[t]/(t^{n+1}))$.

Theorem 5.2 (Denef-Loeser). *Suppose that k has characteristic zero. The motivic zeta function $Z_{mot}(f; s)$ is rational in $T = \mathbb{L}^{-s}$ over $\mathcal{M}_k$. More precisely, there exists a finite subset S of $\mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$ such that $Z_{mot}(f; s)$ belongs to*

$$\mathcal{M}_k \left[\frac{\mathbb{L}^{-as-b}}{1 - \mathbb{L}^{-as-b}} \right]_{(a,b) \in S} \subset \mathcal{M}_k[[\mathbb{L}^{-s}]]$$

If $h : Y \to X$ is an embedded resolution of singularities for f, then we can take for S the set of numerical data of the resolution. In fact, by a similar computation as in the p-adic case, using the change of variables formula, one obtains an explicit expression for $Z_{mot}(f; s)$ in terms of h. Note that Theorem 4.8 applies to the morphism h: since we always assume that h is an isomorphism over the complement of the zero locus of f in X, the Jacobian divisor (Jac_h) is supported on $(f \circ h)$, which implies that $ord_t Jac_h$ takes only finitely many values on each fiber of $ord_t(f \circ h) = ord_t f \circ \mathcal{L}(h)$.

Denote by E_i, $i \in I$ the irreducible components of the divisor $E = (f \circ h)$, and by (N_i, ν_i) the corresponding numerical data. For any non-empty subset J of I we put $E_J = \cap_{j \in J} E_j$ and $E_J^o = E_J \setminus (\cup_{i \notin J} E_i)$. In particular, $E_\emptyset = Y$ and $E_\emptyset^o = Y \setminus E$. We denote by d the dimension of X.

Theorem 5.3 (Denef-Loeser)**.**

$$Z_{mot}(f;s) = \mathbb{L}^{-d} \sum_{J \subset I} [E_J^o] \prod_{j \in J} \frac{(\mathbb{L}-1)\mathbb{L}^{-N_i s - \nu_i}}{1 - \mathbb{L}^{-N_i s - \nu_i}} \quad \in \mathcal{M}_k[[\mathbb{L}^{-s}]]$$

5.2. **The motivic monodromy conjecture.** If we want to translate the statement of the monodromy conjecture to the motivic setting, there is a technical complication: one should be careful when speaking about poles of the zeta function, since $K_0(Var_k)$ is not a domain.

Conjecture 5.4 (Denef-Loeser; Motivic monodromy conjecture)**.** *Suppose that k is a subfield of $\mathbb{C}$. Let X be a smooth k-variety and $f : X \to \mathbb{A}^1_k$ a morphism of k-varieties, and denote by X_s the complex analytic space defined by the equation $f = 0$ on the analytic space X^{an} associated to the complex variety $X \times_k \mathbb{C}$.*

There exists a finite subset S of $\mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$ such that

$$Z_{mot}(f;s) \in \mathcal{M}_k \left[\mathbb{L}^{-s}, \frac{1}{1 - \mathbb{L}^{-as-b}} \right]_{(a,b) \in S}$$

and such that for each $(a,b) \in S$, the value $\exp(-2\pi i b/a)$ is a monodromy eigenvalue of f at some point of X_s.

We will see that the motivic monodromy conjecture implies the p-adic one.

5.3. **Specialization to the p-adic world.** We'll now explain how the motivic zeta function relates to the p-adic one. Let L be a number field, and let f be an element of $L[x_1, \ldots, x_d]$. The polynomial f defines a morphism of L-varieties $f : \mathbb{A}^d_L \to \mathbb{A}^1_L$. We'll denote by $\mathcal{O}_L$ the ring of integers of L, and by $\mathrm{Msp}\,\mathcal{O}_L$ its maximal spectrum. For each finite place $\mathfrak{P} \in \mathrm{Msp}\,\mathcal{O}_L$, we denote its residue field by $k_{\mathfrak{P}}$.

On the one hand, we can associate to f its motivic zeta function

$$Z_{mot}(f;s) \in \mathcal{M}_L[[\mathbb{L}^{-s}]]$$

On the other hand, for each finite place $\mathfrak{P}$, we can consider the $\mathfrak{P}$-adic zeta function $Z_{\mathfrak{P}}(f;s)$.

Consider the ring

$$\mathscr{Z} = \left(\prod_{\mathfrak{P} \in \mathrm{Msp}\,\mathcal{O}_L} \mathbb{Q} \right) \Big/ \left(\bigoplus_{\mathfrak{P} \in \mathrm{Msp}\,\mathcal{O}_L} \mathbb{Q} \right)$$

For any variety X over L, we can choose a model over $\mathcal{O}_L$, and count rational points on the reduction modulo $\mathfrak{P}$, for each finite place $\mathfrak{P}$. The outcome may depend on the chosen model, but all these values together yield a well-defined element of $\mathscr{Z}$. Moreover, since this operation is additive (w.r.t. closed immersions) and multiplicative (w.r.t. the fiber product over L), we obtain a morphism of rings $K_0(Var_L) \to \mathscr{Z}$ which induces a morphism of rings

$$\mathscr{N} : \mathcal{M}_k \left[\frac{\mathbb{L}^{-as-b}}{1 - \mathbb{L}^{-as-b}} \right]_{(a,b) \in \mathbb{Z}_{>0} \times \mathbb{Z}_{>0}} \to \mathscr{Z}'$$

with

$$\mathscr{Z}' = \left(\prod_{\mathfrak{P} \in \mathrm{Msp}\,\mathcal{O}_L} \mathbb{Q} \left[\frac{|k_{\mathfrak{P}}|^{-as-b}}{1 - |k_{\mathfrak{P}}|^{-as-b}} \right]_{(a,b) \in \mathbb{Z}_{>0} \times \mathbb{Z}_{>0}} \right) \Big/ (\oplus \ldots)$$

Theorem 5.5 (Denef-Loeser). *The motivic zeta function $Z_{mot}(f; s)$ specializes to $Z_{\mathfrak{P}}(f; s)$ for almost all finite places $\mathfrak{P}$, in the following sense: the image of $Z_{mot}(f; s)$ under the morphism $\mathcal{N}$ is the quotient class of the tuple*

$$(Z_{\mathfrak{P}}(f; s))_{\mathfrak{P} \in \mathrm{Msp}\, \mathcal{O}_L}$$

Proof. Combine the expressions in Theorem 2.9 and Theorem 5.3. $\qquad\square$

Corollary 5.6. *The motivic monodromy conjecture for $k = L$ and $f \in L[\widetilde{x}]$ implies the untwisted p-adic monodromy conjecture for f (Conjecture 3.2).*

In fact, Denef and Loeser formulated a more general motivic monodromy conjecture, involving motivic zeta functions which are twisted by characters χ. A similar specialization result holds in that setting; see [10].

In virtually all of the cases where the p-adic monodromy conjecture is proven, the same strategy yields a proof for the motivic version of the conjecture.

5.4. **Why motivic zeta functions?** The motivic zeta function appears in Theorem 5.5 as a universal zeta function, with the p-adic zeta functions as its avatars. It captures the geometric structure which explains the uniform behaviour of the p-adic zeta functions for $p \gg 0$. In this sense it fully deserves the name "motivic".

Although more general than the p-adic one, the motivic mondromy conjecture seems more accessible, since we never leave the equicharacteristic zero world. The arc spaces appearing in its definition are closely related to the infinitesimal structure of the morphism f, and hence to its singularities, so one can certainly believe that something like the motivic monodromy conjecture should hold. The connection becomes even stronger when we consider the so-called monodromic motivic zeta function [13, 3.2], where we actually see the monodromy appear in the form of an action of the profinite group $\hat{\mu}$ of roots of unity on the coefficients of the zeta function. Intriguingly, by taking a formal limit of this monodromic zeta function one gets a motivic incarnation of the nearby cycles of f, and the monodromy eigenvalues can be read from this object [13, 3.5]. Nevertheless, at this moment the conjecture still seems far out of reach.

The motivic zeta functions are also interesting in their own right, independently of the relation with p-adic zeta functions and the monodromy conjecture. They provide very fine invariants of hypersurface singularities, which can be explicitly computed on a resolution of singularities. On the other hand, since motivic zeta functions are defined intrinsically by means of a motivic integral, they show that certain invariants of a resolution of singularities are actually independent of the chosen resolution. See for instance [27, § 1] for the genesis of these ideas.

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