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$GL_m \quad M_mF$

$GL_n \quad M_nF$

$$GL_mF \otimes GL_nF \le GL_{mn}F$$

${}_kU, {}_kV$

$\varphi_1, \varphi_2 \in End_k U$

$\psi_1, \psi_2 \in End_k V$

$$(\varphi_1 \otimes \psi_1)(\varphi_2 \otimes \psi_2) = \varphi_1\varphi_2 \otimes \psi_1\psi_2$$

$$(A \otimes B)(C \otimes D) = AC \otimes BD$$

$$GL_mF \otimes GL_mF \le GL_{m^2}F$$

$\tau \in Aut(GL_mF \otimes GL_mF)$

$$A \otimes B \longmapsto B \otimes A$$

$\boxed{\text{Теорема}}$ $n > r \geq 2 \qquad X \le GL_{nr}F$

$$SL_nF \otimes SL_rF \le X \Rightarrow \left[\begin{array}{l} X \le M \\ X \trianglerighteq SL_{nr}F \end{array}\right.$$

$(r > 2) \vee (\operatorname{char} F \neq 2)$

$$M = N(SL_mF \otimes SL_nF) = \left[\begin{array}{ll} GL_mF \otimes GL_nF, & m \neq n \\ \langle GL_mF \otimes GL_nF, \tau \rangle & m = n \end{array}\right.$$

$\boxed{\text{Лемма 1}}$

R – асс. с 1

$\mathcal{D}$ – подкольцо R – совпадающее с адд. группой, порожденной $\mathcal{D}^*$

$\Lambda = \{r \in R^* \mid r\mathcal{D}r^{-1} \subseteq \mathcal{D}\}$

$n \geq 3 \quad \Gamma := \Lambda \cdot GL_n\mathcal{D}$

$\exists g \in GL_nR \setminus \Gamma : \; g_{1j} = 0 \;\; \forall j \neq 1$

$\Rightarrow \langle SL_n\mathcal{D}, {}^gSL_n\mathcal{D} \rangle \ni t_{1j}(c)$, где $c \in L$ – подкольцо R

$L \supsetneq \mathcal{D}$

$\boxed{\text{Лемма 2}}$ $n \geq 3$. $SL_nF \otimes SL_rF \le X \le GL_{nr}F$

$\exists g \notin M \quad g_{1j} = 0 \in M_rF, \; j \geq 2$

$\Rightarrow X \trianglerighteq SL_{nr}F \qquad (r > 2 \vee \operatorname{char} F \neq 2)$

$\square$ $R := Mat_rF \quad \mathcal{D} = k \cdot 1_r, \; \Lambda = GL_rF, \; \Gamma = \Lambda GL_n\mathcal{D} = GL_nF \otimes GL_rF$

$C_1 = T_{21}(1)\cdot C\cdot T_{21}(-1) - C \in L$

C такого, что $C_{12}=1$ $\quad ((C \rightsquigarrow \lambda P C P^{-1})$

$$C_1 = \begin{pmatrix} -1 & 0 & 0 \\ * & 1 & * \\ & & \\ & & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & & \\ 1 & 1 & \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & \\ & C \end{pmatrix}\begin{pmatrix} 1 & & \\ -1 & 1 & \\ & & 1 \end{pmatrix}$$

если $\operatorname{char} F \neq 2$

$C_2 = T_{21}(-1/2)\cdot C_1 T_{21}(\frac{1}{2}) - C_1$ — матр. единица

$\operatorname{char} F = 2$

$C_2 = T_{23}(1)\, C_1 T_{23}(-1) - C_1$

$C_3 = T_{31}(1)\, C_2 T_{31}(-1) - C_2$

Лемма 3 $\quad n \geq 3$

$\exists A_0$ — трансвекция в $SL_n F$

$T_0 = A_0 \otimes E_r$

$\exists g \in GL_{nr}F : T_1 = g(A_0 \otimes E_r)^{-1} g \in M$

$\rightsquigarrow$ (1) $T_1 = E_n \otimes B \qquad (B-E_r)^2 = 0$

$\qquad\qquad\qquad\qquad\qquad rk(B-E_r) = \frac{r}{n}$

(2) $T_1 = A \otimes E_r \qquad rk(A-E_n) = 1$

(3) $T_1 = (E_n \otimes B)\tau(E_n \otimes B)^{-1}$

$n = r = 3$

$\square$ $(T_1 - E_{nr})^2 = 0, \quad rk(T_1 - E_{nr}) = r$

Случай $n \neq r \qquad M = GL_n F \otimes GL_r F$

$T_1 = A \otimes B$

(1) $A = aE_n \Rightarrow T_1 = E_n \otimes \frac{1}{a} B \rightarrow$ случай (1)

(2) $A \neq aE_n \rightsquigarrow \exists P \in GL_n F : PAP^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ & * & \end{pmatrix} =: \widetilde{A}$

$T_2 = (P \otimes E_r) T_1 (P \otimes E_r)^{-1}$

$rk(T_2 - E_{nr}) = rk\begin{pmatrix} -E_n & B & 0 & 0 \\ & & & \end{pmatrix}$

$$\operatorname{rk}(T_2 - E_{nr}) = \operatorname{rk}\begin{pmatrix} -E_r & B & 0 & 0 \\ & & \tilde{A}_{33}B - E_r & \end{pmatrix}$$

$$\geq$$

$$\underbrace{\operatorname{rk} B}_{=\, r} + \operatorname{rk}(\tilde{A}_{33} B - E_r)$$

$\rightsquigarrow \tilde{A}_{33} B = E_r \rightsquigarrow B$ — центральна

$T_1 = A \cdot a \otimes E_r$

$\boxed{\text{Лемма 4}}$ $n \geq 3$, $n \geq r$

$g_1 \in GL_{nr}F \setminus M$ но $g_1 (t_{n1}(s) \otimes e_r) g_1^{-1} \in M$

$T_1 :=$ $g_1 (t_{n1}(s) \otimes e_r) g_1^{-1}$ $\in GL_nF \otimes GL_rF$

($T_1 \in M_0$, если $n = r = 3$ и $\#F = 2$)

$\Rightarrow \langle SL_nF \otimes SL_rF, g_1 \rangle = Y$

$\exists g$ $\quad g \in St(e_1 \otimes \langle e_1 \dots e_r \rangle) \setminus M$

$\square$ $n \neq r$, $n \vee r > 3$ м. считать, что $T_1 \in GL_nF \otimes GL_rF$

по лемме 3 1) $T_1 = A \otimes E_n$ или 2) $\underbrace{T_1 = E_n \otimes A}_{\text{тогда } n = r}$ $\operatorname{rk}(A - E_n) = 1$

(1) $\exists P_1 : P_1 A P_1^{-1} = t_{n1}(s)$

тогда $g := (P_1 \otimes E_r) \cdot g_1 \in Y \setminus M$

и $g (t_{n1}(s) \otimes E_r) g^{-1} = t_{n1}(1) \otimes E_r$

$\rightsquigarrow g = \begin{pmatrix} * & 0 \\ & * & 0 \\ & & * \end{pmatrix} \in Y \rightsquigarrow \exists g \in Y \cap St(e_1 \otimes \langle e_1, \dots, e_r \rangle)$

Случай (2) ($n = r$, $T_1 = E_n \otimes A$)

$P_1 A P_1^{-1} = t_{n1}(1)$ $\quad \tilde{g} = (P_1 \otimes E_r)^{-1} g \notin M$

$t_{n1}(1) \otimes E_r = \tilde{g} (t_{n1}(s) \otimes e) \tilde{g}^{-1} \rightsquigarrow \tilde{g}$ — искомый

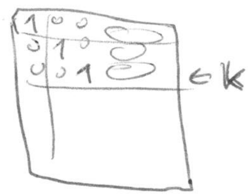
в $Y \in t_{n1}(C^{(n)}) \in M$ нецентральные

□ (Теоремы) $n > r$

$X \not\subseteq M \rightsquigarrow \exists\, k$ – наибольшее число такое, что

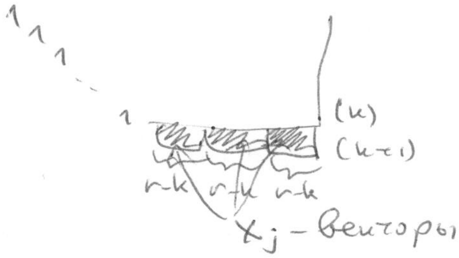
$\exists\, g \in X \setminus M : g \in St(e_1 \otimes \langle e_1, \dots, e_k \rangle)$

$k \in \overline{0,r}$



Хотим: $k = r$

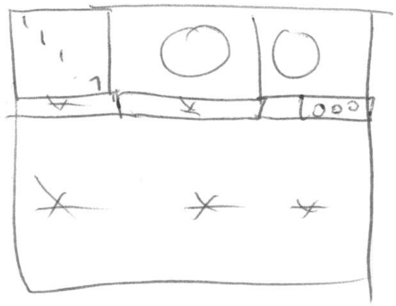
$j \in \{k+1, \dots, n\}$



X_j размера $r-k$, их $n-k$

$\rightsquigarrow \exists \sum_{i=k+1}^{n} \lambda_i x_i = 0$

$$g_1 = g\left(\begin{pmatrix} E_k & 0 \\ 0 & \begin{matrix} * & \begin{matrix}\lambda_{k+1} \\ \vdots \\ \lambda_n\end{matrix}\end{matrix} \end{pmatrix} \otimes E_r\right) =$$



$g_2 = g_1(t_{n1}(s) \otimes E_r)\, g_1^{-1}$ имеет нужный в-д

$\rightsquigarrow \overset{\in X}{g_2} \in St(e_1 \otimes \langle e_1, \dots, e_{k+1} \rangle) \Rightarrow g_2 \notin M$

$\overset{\text{л.ч.}}{\rightsquigarrow}$ в $\overset{\subseteq X}{Y}$ лежит элемент

$g \in St(e_1 \otimes \langle e_1, \dots, e_r \rangle)$