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$\boxed{\text{Thm 1}}$ R – module finite over its center.
G is a classical-like group of rank $n \geq 2$
E – elementary subgroup of G

Then $E(R) \trianglelefteq G(R)$ and the standard sandwich classification of ~~...~~ E-normal subgroups holds

i.e. subgroup H of $G(R)$ is normalized by $E(R)$
$\Leftrightarrow$ it is a member of some sandwich

$$E(R, \mathfrak{a}) \subset H \subset G'(R, \mathfrak{a})$$

for a unique ideal $\mathfrak{a}$ where
$G'(R, \mathfrak{a})$ is the inverse image of center$(G(R/\mathfrak{a}))$

$\boxed{\text{Thm I}}$ R – an associative ring
G – a classical-like group of rank n
E – its elementary subgroup

Suppose $n >$ local stable dimension of R + 1.
Then we get the same conclusion as in thm 1.

$\boxed{\text{Thm 2}}$ Assume the hypotheses in thm 1. Then
$G'(R, \mathfrak{a})/E(R, \mathfrak{a})$ is a nilpotent-by-abelian group of nilpotent class $\leq$ Bass–Serre dimension of R
($\leq$ Jack-Krull dim (R))

$\boxed{\text{Thm II}}$ Assume the hyp. of thm I
$\rightarrow$ $G'(R, \mathfrak{a})/E(R, \mathfrak{a})$ is nilpotent by abelian of nilpotent class $\leq$ „universal dimension" (R) providing this dim. is finite

R - ass. ring with 1

A multiplicative spectrum on R is a collection Σ of right denominator sets T in R such that:

if X is any subset of R : $X \cap T \neq \emptyset \ \forall T \in \Sigma$

$\rightsquigarrow XR =$ (right submodule of R generated by X) $= R$

(Right) Denominator set $T \subseteq R$

is a multiplicatively closed set in R s.t. the following holds:

$\exists$ a ring Q and a ring homomorphism $R \xrightarrow{\varphi_T} Q$ such that

1. each element $\underset{\hat{Q}}{q} = \varphi_T(r)\varphi_T(t)^{-1}$ for some $r \in R$, $t \in T$
2. $\ker \varphi_T = \{r \in R \mid rt = 0 \text{ for some } t \in T\} = \mathrm{Ass}(T)$

The ring Q is called the localization of R at T and is often denoted by RT^{-1}

Denom set $\iff$ 1. Ore (right Ore, actually)
2. Image of T in $R/\mathrm{ass}(T)$ consists of regular elements

$\forall s \in T \, \forall b \in R \ \exists a \in R, t \in T : sa = bt$

Defn A denominator set T is called <u>semi-primitive</u> if the canonical homomorphism from $R \longrightarrow RT^{-1}/\mathrm{Jac}(RT^{-1})$ is surjective

A semi-primitive <u>Spectrum</u> (multiplicative) is a multiplicative spectrum in which each denominator set T is semi-primitive

Def. <u>stable dimension</u> $(\Sigma) = \sup\limits_{T \in \Sigma} \left(sd\left(RT^{-1}/\mathrm{Jac}(RT^{-1})\right)\right) = sd(RT^{-1})$

semi-primitive mult. spectrum

$sd(R) \stackrel{\text{df}}{=} sr(R) - 1$

$\ell sd(R) = \inf\limits_{\substack{\Sigma\text{-semi-primitive} \\ \text{mult. spectrum on } R}} \{ sd(\Sigma) \}$ ~~semi~~

Why semi-primitive?

Let T be a denominator set

$$J(T,R) = J(T) = \ker\left(R \longrightarrow RT^{-1}/Jac(RT^{-1})\right) \text{ — ideal}$$

$\boxed{\text{Prop}}$ Equivalent:

① $R \longrightarrow RT^{-1}/Jac(RT^{-1})$ is surjective

② $J(T)$ is semi-primitive, i.e. is an intersection of primitive ideals

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Two semi-primitive spectra Σ, Σ' are called equivalent if $\{J(R,T) \mid T \in \Sigma\} = \{J(R,T') \mid T' \in \Sigma'\}$

$\boxed{\text{Lemma}}$ if Σ and Σ' are equivalent then

$$sd(\Sigma) = sd(\Sigma')$$

$\square$ Obvious, because sd depends only on ~~sd~~ of each R $(R/J(R,T))$

$$\left(RT^{-1}/Jac(RT^{-1})\right)$$

$\boxed{\text{Lemma}}$ Let Σ be a collection of semi-primitive localizable ideals....

$\boxed{\text{Lemma}}$ if $\mathfrak{a}$ is self-localizing, then its primitive closure $J(\mathfrak{a})$ is also self-localizing.

Let T denote a semi-prim. denominator set.

Let $U(T,R) = U(J(T),R)$ denote $\{r \in R \mid r \text{ is a unit in } R/J(T,R)\}$ $\left(RT^{-1}\right)$

$\boxed{\text{Prop}}$ T – semi-primitive

$\to$ any multiplicative set S s.t. $T \subseteq S \subseteq U(T,R)$ is a denominator set,

~~and~~ $J(S,R) = J(T,R)$

and $1 + J(T,R)$ is also a denominator set s.t.

$$J(1 + J(T,R)) = J(T,R)$$

~~Defn~~ An ideal $\mathfrak{a}$ is called self-localizing if $1 + \mathfrak{a}$ is a denominator set

Ideal-theoretic definition of primitive spectrum

A collection Ω of ideals $\mathfrak{a}$ of R is called a semi-primitive spectrum, if

1. Each $\mathfrak{a}$ is semi-primitive
2. Each $\mathfrak{a}$ is self-localizing
3. Each prim. ideal of R contains some $\mathfrak{a} \in \Omega$

$\boxed{\text{Theorem}}$ There is a 1-1 correspondence between classes of semi-primitive multiplicative spectra on R and semi-primitive spectra

$$\underset{\text{multipl. spectra}}{\Sigma} \longmapsto \Omega = \{ \mathfrak{I}(T,R) \mid T \in \Sigma \}$$

$$\Omega \longmapsto \Sigma = \{ 1 + \mathfrak{a} \mid \mathfrak{a} \in \Omega \}$$

$$R \longrightarrow R'$$

$$\rightsquigarrow sd(R') \leq sd(R)$$

Example

Suppose k is a field

$A_n(k) =$ Weyl algebra

$lsd(A_n(k)) \leq n+1$

$\overset{\shortparallel}{sd}(A_n(k))$ — no change, but ...

k - commutative Jacobson ring (every ideal is semi-primitive)

$A_n(k) =$ Weyl algebra

$lsd(A_n(k)) \leq n+1$

$sd(A_n(k)) \leq$ Krull dim.$(k) + n + 1$

comm. Jacobson ring

R-constructible k-alg

$$\rightsquigarrow lsd(AR) \leq \sup_{m \in Max(k)} (sd(R/mR))$$