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Stefan Gille

Chow motives $\qquad$ k – field

Psm_k = cat. of smooth proj k-schemes

$$\begin{tikzcd} Psm_k \arrow[d, "\text{Dream}", "\text{„Linearization"}"'] \\ \text{ab. Cat.} \end{tikzcd}$$

$\times$ Dream is over

(1) $Corr_k$ = cat. of correspondances of degree 0 with coeff in R (comm. ring)

$Ob = Psm_k$, $\quad Mor(X,Y) = \bigoplus_{i\in I} CH_{\dim X_i}(X_i \times Y) \otimes_k R$

X_i – irred comp of X

$$X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z$$

$$\begin{tikzcd} & X\times Y\times Z \arrow[dl, "p_{X,Y}"'] \arrow[d, "p_{X,Z}"] \arrow[dr, "p_{Y,Z}"] & \\ X\times Y & X\times Z & Y\times Z \end{tikzcd}$$

$$\beta\circ\alpha = p_{XZ*}\left(p_{XY}^*(\alpha) \cap p_{YZ}^*(\beta)\right)$$

(для кон. числа точек это умн. матриц)

(2) Idempotent completion $\rightsquigarrow Chow_k^R$

$Obj = (X,p)$ $\quad p$ – correspond. with $p\circ p = p$

Morph. $(X,p) \longrightarrow (Y,q)$

$\alpha : X \longrightarrow Y$

$$\begin{tikzcd} X \arrow[r, "\alpha"] \arrow[d, "p"] & Y \arrow[d, "q"] \\ X \arrow[r, "\alpha"'] & Y \end{tikzcd} \Big/ q\alpha = \alpha p = 0$$

id_X = class of Diagonal

$\boxed{Ex}$. $\mathbb{P}^1_k$ $\quad$ Diag $= \underbrace{pt\times\mathbb{P}^1_k + \mathbb{P}^1_k\times pt}_{\text{idempotents}}$ $\rightsquigarrow$ decomposition of $\mathbb{P}^1$

in $Chow_k^R$: $\mathbb{P}^1_k = (\mathbb{P}^1_k, \overset{\mathbb{A}_1}{pt\times\mathbb{P}^1}) \oplus (\mathbb{P}^1, \overset{pt}{\mathbb{P}^1\times pt})$

Tate motive (or Lefschetz) — $R(1)$

$\mathcal{P}Sm_k \longrightarrow Chow_k^R$ $\qquad\qquad$ $\boxed{h = k}$

$X \longmapsto (X, id_X) =: X$

$f \longmapsto \Gamma_f$ = Graph of f.

q – quad. form

$\sum_{i=0}^{n} a_i x_i^2 \qquad X_q \hookrightarrow \mathbb{P}_k^n$ – corr. proj. quadric: $q=0$

Assume $X_q(h) \neq \emptyset \Rightarrow q \simeq X \cdot Y + \sum_{i=2}^{n} b_i \cdot X_i^2$

$$\rightsquigarrow X_q = \underbrace{(X \neq 0)}_{\cong\, \mathbb{A}_k^{n-1}} \cup \underbrace{(X=0) \setminus \{Y=1\}}_{\cong\, X_{q'} \times \mathbb{A}_k^1} \cup \{[0:1:0 \ldots :0]\}$$

$\rightsquigarrow$ decomposition of Chow motive

$$X_q = \mathbb{B}(n-1) \oplus \underbrace{X_{q'}(1)}_{= X_{q'} \otimes \underline{R}(1)} \oplus \underline{R} \qquad R(\ell) = R(1)^{\otimes \ell}$$

$$(X, p) \otimes (Y, q) = (X \times Y, p \times q)$$

(Rost)

Rost nilpotence

E/k $\quad \rightsquigarrow X \longmapsto X \times_k E = X_E$

Field

induces a functor $\quad res_{E/k}$ $\quad Chow_k^R \longrightarrow Chow_E^R$

$M \longmapsto M_E$

$\alpha \longmapsto \alpha_E$

$Hom_k(M, N)_R$ – morph $\quad$ in $Chow_k^R$

$End_k(M)_R$ – endomorph

Rost nilpotence is $\quad$ Sw. $M \in Chow_k^R$ if (and only if)

the kernel of $res_{E/k}: End_k(M) \longmapsto End_E(M_E)$

consists of nilpotent elements for all

field extensions E/k

Rost: RN is true for quadrics

– split quadrics are very well known

$$End((X_q)_{\bar{k}}) = CH_{\dim X_q}((X_q \times X_q)_{\bar{k}}) \qquad \text{– RN means that you can lift idempotents}$$

$$\Big\uparrow res_{\bar{k}/k}$$

$$\underbrace{CH_{\dim X_q}(X_q \times X_q)}$$

i.e. if $\alpha \in End_k(X_q)$ has property $\alpha_{\bar{k}}$ is idempotent ~~is idempotent~~

$\Rightarrow \exists \tilde{\alpha} \in End_k(X_q)$ – idempotent and $\tilde{\alpha}_{\bar{k}} = \alpha_{\bar{k}}$.

Rost uses RN to give a decomposition of the norm quadric $\langle\langle a_1,\dots,a_{n-1}\rangle\rangle \perp \langle -a_n\rangle$ of the symbol $\{a_1,\dots,a_n\} \in K_n^M(k)/2$ which plays a very crucial role in Voevodsky's proof of Milnor conjecture

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Conj (Vishik) (inofficial) R.N. is true for all smooth projective schemes

Where it's proven

- If $R = \mathbb{Q}$ – RN is true for all motives in Chow_k
- R arbitrary, proj. homogenous varieties over semisimple groups
- R arbitrary, X – smooth proj. scheme s.t. $\underset{\text{generically split}}{X_{k(X)}} \simeq \bigoplus_{i=0}^{S} \underline{R}(i)^{n_i}$ (Vishik–Zainoulline)

All proofs are based on a lemma by Rost:

$X \in \mathcal{P}Sm_k$, $\alpha \in \mathrm{End}_k(X)_R$

Assume $\alpha_{k(x)*}(CH_{\mathrm{codim}\, x}(X_{k(x)})) = 0 \quad \forall x \in X$

Then $\alpha^{\dim X+1} = 0$

$$\alpha^{\dim X+1} = \underbrace{\alpha \circ \dots \circ \alpha}_{\dim X+1 \text{ times}}$$

$CH_i(X_{k(x)}) = \mathrm{Hom}(k(x)(i), X_{k(x)})$

$\alpha_{k(x)*}(f)$

$\alpha_{k(x)} \circ f$

Now $R = \mathbb{Q}$ – easy exercise from this lemma P. Brosnan

Now $R = \mathbb{Z}$

Thm Let S be a geometrically rational k-surface, then R.N. is true for S in $\mathrm{Chow}_k^{\mathbb{Z}} =: \mathrm{Chow}_k$ (recall char $k=0$)

k-surface – a smooth projective k-scheme, dim $=2$ + geom. integral, i.e. $S_{\bar k}$ integral

geom. rational k-surface = k-surface with $S_{\bar k}$ birationally isomorphic to $\mathbb{P}^2_{\bar k}$

Coombes Th (86) $\Rightarrow S_{k_s}$ bir $\cong \mathbb{P}^2_{k_s}$.

$\Rightarrow \exists L|k_s$ finite Galois S_L is L-rational

$$\begin{tikzcd} S \arrow[rr, dashed, "\cancel{\phi}"', "\text{nat. map}"] & & S' \\ & X \arrow[ul, "\psi"] \arrow[ur, "\varphi"'] & \end{tikzcd}$$

$\to \exists$ ψ — surfaces

commutes (where it makes sense)

ψ, φ – iterations of blow-ups

Implies: If S – k-rational, then
$\operatorname{Pic} S$ is ~~free~~ abelian of finite rank
$\operatorname{Pic} S = CH_1(S)$ (by blow-up formula for Chow groups)

$\Rightarrow$ (Hochschild Sense)
$\operatorname{Pic} S$ is ~~free~~ abelian for S – geom. rat.
$(\operatorname{Pic} S_{k_s})^G$

$\boxed{\text{Thm 1}}$ S – k-rational, $\operatorname{Pic} S \cong \operatorname{Pic} S_{\bar{k}}$ — additional
Then $S \simeq \mathbb{Z} \oplus \mathbb{Z}(1)^{\operatorname{rk} \operatorname{Pic} S} \oplus \mathbb{Z}(2)$
(consequence of blow-up formula)

$\operatorname{Hom}(\mathbb{Z}(2), S) = CH_2(S) = \mathbb{Z}$
$\operatorname{Hom}(\mathbb{Z}(1), S) = CH_1(S) = \operatorname{Pic} S$
$\operatorname{Hom}(\mathbb{Z}, S) = CH_0(S) = \mathbb{Z}$

$L|k$ is called a <u>splitting field</u> if
the motive of S_L splits, i.e. is $\simeq \mathbb{Z} \oplus \mathbb{Z}(1)^{\operatorname{rk} \operatorname{Pic} S_L} \oplus \mathbb{Z}(2)$

Proof of RN for geometrically rational surfaces
$\alpha \in \operatorname{End}_k(S) = CH_2(S \times S)$
$\alpha_E = 0$ for some $E|k$ $\Rightarrow \alpha_L = 0$ $(L = k_s = \bar{k})$

$\alpha_{k(x)*} \underbrace{CH_{\operatorname{codim} x}}_{=1 \,\to\, \text{free abelian}}(S) = 0 \quad \forall x \in S$

$\to$ enough to show $\to$ no torsion
$\alpha^2_{k(S)*} CH_0(S_{k(S)}) = 0$

$CH_0(S) = \mathbb{Z} \oplus \underbrace{(\text{kernel of degree-map}: CH_0(S) \to \mathbb{Z})}_{A_0(S)}$

$A_0(S)$ — torsion part of $CH_0(S)$

⇝ enough to show $\alpha_{k(S)*} A_0(S_{k(S)}) = 0$

<u>Bloch map</u> ϕ_S (~ 1980)

splitting field

$G = \mathrm{Gal}(L|k)$

$$C_\bullet(S) : K_2(k(S)) \longrightarrow \bigoplus_{x \in S_{(1)}} k(x)^* \longrightarrow \bigoplus_{x \in S_{(0)}} \mathbb{Z}$$

$$A_0(S) \xrightarrow{\ \phi_S\ } H^1(G, H_1(C_\bullet(S_L)))$$

injective if $S(k) \neq \emptyset$

$\boxed{\text{Thm}}$ $\forall \alpha \in \mathrm{End}_k(S)$

$$\begin{tikzcd}
A_0(S) \arrow[r, "\phi_S"] \arrow[d, "\alpha_*"'] & H^1(G, H_1(C_\bullet(S_L))) \arrow[d, "H^1(G, \alpha_{L*})"] \\
A_0(S) \arrow[r, "\phi_S"] & H^1(G, H_1(C_\bullet(S_L)))
\end{tikzcd}$$

$\alpha = \sum [Z_i]$ $\quad Z_i \subset S \times S$, dim 2

→ we can assume $\alpha = [Z]$

resolution of singularities

$$\begin{tikzcd}
Y \arrow[r, "p"] \arrow[dd, bend right, "g"'] \arrow[ddrr, bend left, "f"] & Z \arrow[d, phantom, "\cap"] & \\
& S \times_k S \arrow[dl, "pr_1"'] \arrow[dr, "pr_2"] & \\
S & & S
\end{tikzcd}$$

⇝ $[Z]_* = f_* \circ g^*$

→ Y — smooth proj. dim$=2$

$$\begin{tikzcd}
& Y \arrow[dl, "g"'] \arrow[dr, "f"] & & \\
S & & S \arrow[r] & D_0(S)
\end{tikzcd}$$

$$K_2(k(S)) \longrightarrow \bigoplus_{x \in S_{(1)}} k(x)^* \xrightarrow{\ d_1^S\ } \bigoplus \mathbb{Z} \xrightarrow{\ \mathrm{Deg}\ } \mathbb{Z}$$

$D_0(S) \subset \bigoplus \mathbb{Z}$

$(n_x) \longmapsto \sum n_x [k(x):k]$

$\mathrm{Im}\, d_1^S \subseteq D_0(S)$

Kernel Deg $= D_0(S)$

if S is rational then

$\mathrm{Im}\, d_1^S = D_0(S)$ ~~Deg is surjective~~

L = splitting field of S ; S_L - rational motive of S_L is split

⇝ exact sequence $\qquad G = Gal(L|k)$

$$\begin{tikzcd} Z(S_L) \arrow[r, tail] \arrow[d, equal] & C_1(S_L) \arrow[r, two heads] & D_0(S_L) \\ \text{Kernel } d_1^{S_L} & & \end{tikzcd}$$

⇝ take Galois cohomology sequence: $\qquad$ Faddeev-Shapiro's lemma

$$\begin{tikzcd} C_1(S_L)^G \arrow[r] \arrow[d, equal] & D_0(S_L)^G \arrow[r, two heads] \arrow[d, equal] & H^1(G, Z(S_L)) \\ C_1(S) & D_0(S) & \end{tikzcd}$$

⇝ nat. isomorphism

$$A_0(S) \simeq H^1(G, Z(S_L))$$

$$\rightsquigarrow Z(S_L) \longrightarrow H_1(C_\bullet(S_L))$$

$= \Phi_S$ (Bloch map)

$$A_0(S) \simeq H^1(G, Z(S_L)) \longrightarrow H^1(G, H_1(L_\bullet(S_L)))$$

$$\begin{tikzcd} & Z(S_L) \arrow[r, tail] \arrow[dl] \arrow[d, "f_{L*}g_{L\tau}^*"] & C_1(S_L) \arrow[r, two heads] \arrow[d, "f_{L*}g_{L\tau}^*"] & D_0(S_L) \arrow[d, "f_{L*}g_{L\tau}^*"] \\ H_1(C_\bullet(S_L)) \arrow[d, "f_{L*}g_L^*"] & Z(S_L) \arrow[r, tail] \arrow[dl] & C_1(S_L) \arrow[r, two heads] & D_0(S_L) \\ H_1(C_\bullet(S_L)) & & & \end{tikzcd}$$

if it is flat /S ⇝ no choices for τ.

$$\begin{tikzcd} & Y \arrow[dl, "\boxed{g}"'] \arrow[dr, "f"] & \\ X & & X \end{tikzcd}$$

$$(f_{L*} g_{L\tau}^*)^G = f_* g_\tau^*$$

$\boxed{\text{Thm}}$ k = perfect field

⇝ RN is birational invariant in the cat. of all surfaces