

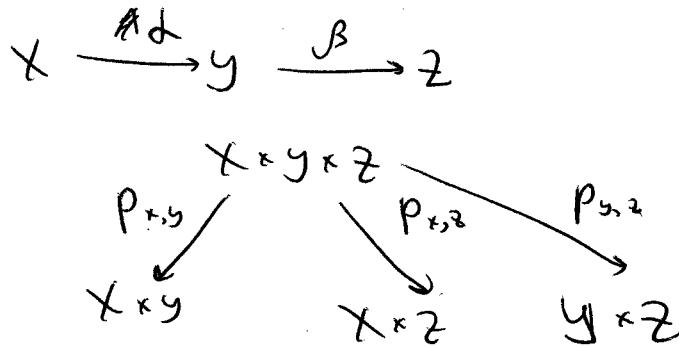
Chow motives k -field

$PSm_k = \text{cat. of smooth proj } k\text{-schemes}$

\int Dream "Linearization"
 \downarrow
 ab. cat. X Dream is over

① $Corr_k = \text{cat. of correspondences of degree 0 with coeff in } R \text{ (comm. ring)}$

$Ob = PSm_k, \quad Mon(X, Y) = \bigoplus_{i \in I} CH_{dim X_i}(X_i \times Y) \otimes_k R$
 X_i - irred comp of X



$$\beta \circ \alpha = P_{X,Z} \circ (P_{X,Y}^* \alpha \cap P_{Y,Z}^* \beta)$$

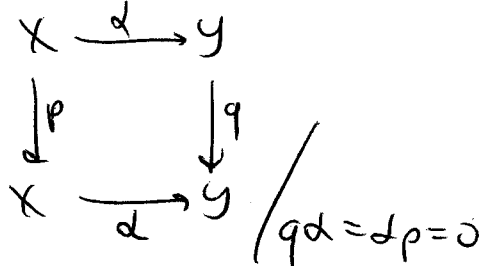
(this non-commutative ring is not symmetric)

② Idempotent completion $\rightsquigarrow (h\text{ow}_k^R)$

$Obj = (X, p)$ p -correspond. with $p \circ p = p$

Morph. $(X, p) \longrightarrow (Y, q)$

$$d: X \longrightarrow Y$$



$id_X = \text{class of Diagonal}$

Ex. IP_k^1 $Diag = pt \times IP_k^1 + IP_k^1 \times pt \rightsquigarrow \text{decomposition of } IP^2$
 $\swarrow \searrow$
 idempotents
 in $(h\text{ow}_k^R: IP_k^1 = (IP_k^1 \times pt + IP_k^1 \times pt) \oplus (IP_k^1 \times pt))$
 (or Lefschetz) Tate motive $R(N)$

$$\mathbb{P}S_{m,k} \longrightarrow \text{Chow}_{k,R}$$

$$k \quad \boxed{k = k}$$

$$X \longmapsto (X, id_X) =: X$$

$$f \longmapsto \Gamma_f = \text{Graph of } f.$$

q - quad. form

$$\sum_{i=0}^n a_i x_i^2$$

$$X_q \hookrightarrow \mathbb{P}_x^n \text{ - con. proj. quadric: } q=0$$

$$\text{Assume } X_q(k) \neq \emptyset \Rightarrow q \simeq x \cdot y + \sum_{i=2}^n \beta_i \cdot x_i^2$$

$$\sim X_q = (X \neq 0) \cup (X=0) \setminus \{Y=1\} \cup [0:1:0:\dots:0]$$

$$\cong \mathbb{R}^n / A_k$$

$$\cong X_{q'} \times A_k^1$$

decomposition of Chow motive

$$X_q = B(n-1) \oplus X_{q'}(1) \oplus \underline{\mathbb{R}} \quad R(e) = R(1)^{\otimes e}$$

$$\cong X_{q'} \otimes R(1)$$

$$(X, p) \otimes (Y, q) = (X+Y, p+q)$$

(Rost)

Rost nilpotence

$$\begin{matrix} \mathbb{F}/k \\ \uparrow \\ \text{Field} \end{matrix}$$

$$X \longmapsto X \times_k \mathbb{F} = X_{\mathbb{F}}$$

induces a function $\text{res}_{\mathbb{F}/k}$

$$\text{Chow}_k^R \longrightarrow \text{Chow}_{\mathbb{F}}^R$$

$$\text{Hom}_k(M, M)_{\mathbb{R}} \text{ - morph}$$

$$M \longmapsto M_{\mathbb{F}}$$

$$\text{End}_k(M)_{\mathbb{R}} \text{ - endomorph}$$

in Chow_k^R

$$\alpha \longmapsto \alpha_{\mathbb{F}}$$

Rost nilpotence is

$\{w \cdot M \in \text{Chow}_k^R \text{ if (and only if)}$

the kernel of $\text{res}_{\mathbb{F}/k}: \text{End}_k(M) \longrightarrow \text{End}_{\mathbb{F}}(M_{\mathbb{F}})$

consists of nilpotent elements for all

field extensions \mathbb{F}/k

Rost: RN is true for quadrics

- split quadrics are very well known

$$\left\{ \text{End}((X_q)_{\bar{k}}) \right\} = \text{CH dim } X_q \times_{\bar{k}} ((X_q \times X_q)_{\bar{k}})$$

$\uparrow \text{res}_{\bar{k}/k}$

$$\text{CH dim } X_q (X_q \times X_q)$$

i.e. if $\alpha \in \text{End}_k(X_q)$ has property $\alpha_{\bar{k}}$ is idempotent

$\Rightarrow \alpha \in \text{End}_k(X_q)$ - idempotent and $\alpha_{\bar{k}} = \bar{\alpha}$

- RN means that you can lift idempotents

Rost uses RN to give a decomposition of the norm quadric $\langle\langle a_1, \dots, a_{n-1} \rangle\rangle \perp \langle -a_n \rangle$ of the symbol $\{a_1, \dots, a_n\} \in K_n^M(k)/2$ which plays a very crucial role in Voevodsky's proof of Milnor conjecture

Conj (Vishik) (unofficial) R.N. is true for all smooth projective schemes

Where it's proven

- If $R = \mathbb{Q}$ - RN is true for all motives in Chow_k
- R arbitrary, proj. homogenous varieties over semisimple groups
- R arbitrary, X -smooth proj. scheme s.t. $X_{k(x)} \simeq \bigoplus_{i=0}^s R(i)^{n_i}$ (Vishik-Zainullina)

↑
generically split

All proofs are based on a lemma by Rost:

$$X \in \mathbb{P}S_{m,k}, d \in \text{End}_k(X) \otimes R$$

$$\text{Assume } d_{k(x)*}(\text{CH}_{\text{codim } x}(X_{k(x)})) = 0 \quad \forall x \in X$$

$$\text{Then } d_{\text{odd dim } X+1} = 0$$

$$\underbrace{d_0 \dots d_d}_{\dim X + 1 \text{ times}}$$

$$\text{CH}_i(X_{k(x)}) = \text{Hom}(k(x)(i), X_{k(x)})$$

$$d_{k(x)*}(f)$$

$$d_{k(x)} \circ f$$

P. Brosnan

Now $R = \mathbb{Q}$ - easy exercise from this lemma

Now $R = \mathbb{Z}$

Thm Let S be a geometrically rational k -surface, then R.N. is true for S in $\text{Chow}_k^{\mathbb{Z}} =: \text{Chow}_k$ (recall char $k = 0$)

k -surface - a smooth projective k -scheme, $\dim = 2$

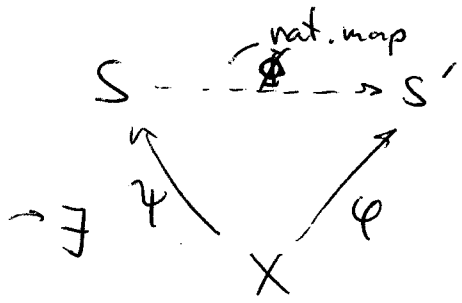
+ geom. integral, i.e. $S_{\bar{k}}$ integral

geom. rational k -surface = k -surface with $S_{\bar{k}}$ birationally isomorphic to \mathbb{P}_k^2

Coombes

Thm (86) $\Rightarrow S_{k_S}$ bin $\cong \mathbb{P}_{k_S}^2$

$\Rightarrow \exists L|k_S$ finite Galois S_L is L -rational



- surfaces

commutes (where it makes sense)

ψ, φ - iterations of blow-ups

Implies: If S - k -rational, then

$\text{Pic } S$ is preabelian of finite rank

$\text{CH}_1(S)$ (by blow-up formula for Chow groups)

(Hochschild Sense)

\Rightarrow

$(\text{Pic } S_{k_s})^{\oplus}$

$\text{Pic } S$ is preabelian for S -geom. rat.

Thm 1

S - k -rational, $\text{Pic } S \cong \text{Pic } S_{\bar{k}}$ - additional

Then $S \simeq \mathbb{Z} \oplus \mathbb{Z}(1)^{\text{rk Pic } S} \oplus \mathbb{Z}(2)$

(consequence of blow-up formula)

$$\text{Hom}(\mathbb{Z}(2), S) = \text{CH}_2(S) = \mathbb{Z}$$

$$\text{Hom}(\mathbb{Z}(1), S) = \text{CH}_1(S) = \text{Pic } S$$

$$\text{Hom}(\mathbb{Z}, S) = \text{CH}_0(S) = \mathbb{Z}$$

\mathbb{L}/k is called a splitting field if

the motive of $S_{\mathbb{L}}$ splits, i.e. is $\simeq \mathbb{Z} \oplus \mathbb{Z}(1)^{\text{rk Pic } S_{\mathbb{L}}} \oplus \mathbb{Z}(2)$

Proof of RV for geometrically rational surfaces

$$\alpha \in \text{End}_k(S) = \text{CH}_2(S \times S)$$

$$\alpha_E = 0 \text{ for some } E/k \Rightarrow \alpha_L = 0 \quad (L = \overset{\tau_k}{\underset{\parallel}{k_s}}$$

$$\alpha_{k(x)} * \text{CH}_{\text{codim } x}(S) = 0 \quad \forall x \in S$$

$= 1 \rightarrow$ free abelian

\rightarrow no torsion

\rightarrow enough to show $\text{CH}_0(S) = 0$

$$\text{CH}_0(S) = \mathbb{Z} \oplus \underbrace{(\text{kernel of degree map: } \text{CH}_0(S) \rightarrow \mathbb{Z})}_{A_0(S)}$$

torsion part of $\text{CH}_0(S)$

→ enough to show $\alpha_{k(s)^*} A_0(S) = 0$

Bloch map: $\phi_S (\sim 1980)$

$G = \text{Gal}(L/k)$ ← splitting field

$$C_0(S): k_2(k(S)) \longrightarrow \bigoplus_{x \in S^{(1)}} k(x)^* \longrightarrow \bigoplus_{x \in S^{(0)}} \mathbb{Z}$$

$$A_0(S) \xrightarrow{\phi_S} H^2(G, H_1(C_0(S)))$$

injective if $S(k) \neq \emptyset$

Thm $\forall d \in \text{End}_k(S)$

$$A_0(S) \xrightarrow{\phi_S} H^1(G, H_1(C_0(S)))$$

$$d_* \downarrow$$

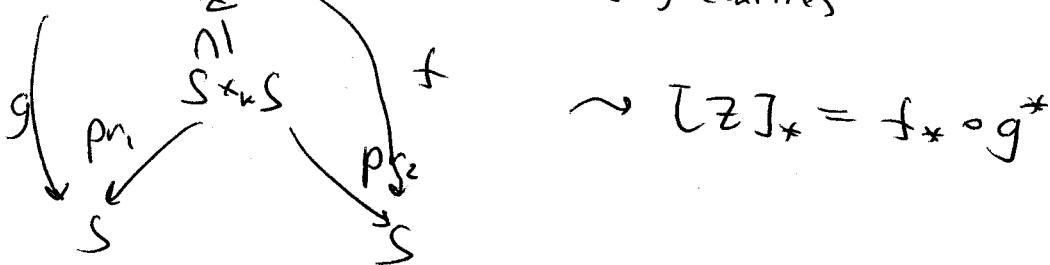
$$\downarrow H^1(G, \alpha_{L^*})$$

$$A_0(S) \xrightarrow{\phi_S} H^1(G, H_1(C_0(S)))$$

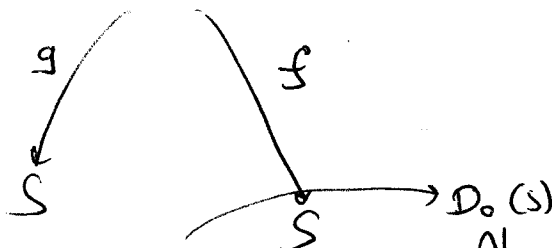
$$d = \sum [Z_i] \quad \begin{array}{l} Z_i \subset S \times S \\ \dim 2 \end{array}$$

→ we can assume $d = [Z]$

resolution of singularities



→ Y — smooth proj. $\dim=2$



$$k_2(k(S)) \longrightarrow \bigoplus_{x \in S^{(1)}} k(x)^* \xrightarrow{d_1^S} \bigoplus \mathbb{Z} \xrightarrow{\text{Deg}} \mathbb{Z}$$

$$\text{Im } d_1^S \subseteq D_0(S)$$

$$(n_x) \longmapsto \sum n_x [k(x):k]$$

Kernel Deg = $D_0(S)$

if S is rational then

$$\text{Im } d_1^S = D_0(S)$$

~~Deg is surjective~~

$L =$ splitting field of S ; S_L -rational motive of S_L is split

\leadsto exact sequence

$$G = \text{Gal}(L/k)$$

$$Z(S_L) \longrightarrow C_1(S_L) \longrightarrow D_0(S_L)$$

\parallel
Kernel $d_1^{S_L}$

\leadsto take Galois cohomology sequence:

Faddeev-Shapiro's lemma

$$\begin{array}{ccccc} C_1(S_L)^G & \longrightarrow & D_0(S_L)^G & \longrightarrow & H^1(G, Z(S_L)) \\ \parallel & & \parallel & & \\ C_1(S) & & D_0(S) & & \end{array}$$

\leadsto nat. isomorphism

$$A_0(S) \cong H^1(G, Z(S_L))$$

$$\leadsto Z(S_L) \longrightarrow H_2(C_0(S_L))$$

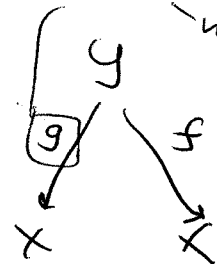
? = ϕ_S (Bloch map)

$$A_0(S) \cong H^1(G, Z(S_L)) \longrightarrow H^1(G, H_2(C_0(S_L)))$$

$$Z(S_L) \longrightarrow C_1(S_L) \longrightarrow D_0(S_L)$$

$$\begin{array}{ccccc} H_1(C_0(S_L)) & \downarrow f_* g_*^* & \downarrow f_* g_*^* & \downarrow f_* g_*^* & \\ \downarrow f_* g_*^* & Z(S_L) & \longrightarrow & C_1(S_L) & \longrightarrow & D_0(S_L) \\ H_1(C_0(S_L)) & & & & & \end{array}$$

if it is flat / S
no choices for σ .



$$(f_* g_*^*)^G = f_* g_*^*$$

Thm $k =$ perfect field

\leadsto RN is birational invariant in the cat. of all surfaces