Local-global principles

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R = comm. ring
$I \trianglelefteq R$ $\mu(I)$ = minimal number of generators
$R = \mathrm{PID} \rightsquigarrow \mu(I) \leq 1$
$R - \mathrm{D.D} \rightsquigarrow \mu(I) \leq 2$
Hilbert: $R = \mathbb{Z}[x_1, \dots, x_n] \rightsquigarrow \mu(I) < \infty$

<u>Q</u>: Can one say that $\mu(I)$ is bounded by a function of $\dim R$
(Krull)

<u>Answer</u> ① (Cohen): No: $\exists$ Noetherian rings of $\dim \geq 1$ for which $\mu(I)$ is not bounded
② Sally-Vasconser
$I = \mathfrak{p}$ (prime, $\dim R \leq 2$
$\rightsquigarrow \exists$ rings for which $\{\mu(\mathfrak{p}) \mid \mathfrak{p} \in \operatorname{Spec} R\}$ not bounded.
<u>But</u> for rings like $k[t_1, t_2]$
$\{\mu(\mathfrak{p})\}$ is bounded
③ Macaulay: $\mathbb{C}[t_1, t_2, t_3]$, $\exists$ ht 2 prime ideals for which ~~$\{\mu(\mathfrak{p}) \mid \mathfrak{p} \in \operatorname{Spec} R$~~ one requires arbitrary no of generators

<u>Next step</u>: prime ideals of $k[x_1, \dots, x_n]$ s.t.
$k[x_1, \dots, x_n]/\mathfrak{p}$ is regular
Forster: $\underline{n \leq \mu(\mathfrak{p})} \leq n+1 \Longleftrightarrow \mu(\mathfrak{p}/\mathfrak{p}^2) = n$
conj: $\mu(\mathfrak{p}) = n$ Forster-Swan
Forster's conjecture: $\mu(\mathfrak{p}/\mathfrak{p}^2) = n \Rightarrow \mu(\mathfrak{p}) = n$
classical results:
① $\mathfrak{m} \in \operatorname{Max}(k[x_1, \dots, x_n]) \Rightarrow \mu(\mathfrak{m}) = n$
② $\operatorname{ht} \mathfrak{p} = 1 \Rightarrow \mu(\mathfrak{p}) = 1$

$n = 1 \rightsquigarrow \mathrm{PID}$
$n = 2 \rightsquigarrow \operatorname{ht} \mathfrak{p} = 1$ or 2
First interesting case: $n = 3$, $\operatorname{ht} = 2$

50's Serre worked with ht 2 primes $/ k[t_1, t_2]$
under certain conditions
$\exists$ an exact seq. $R \longrightarrow P \longrightarrow I \longrightarrow 0$ — projective, $\operatorname{rk} P = 2$

$$R^n \longrightarrow I \longrightarrow 0$$

Q: can we say that P is free? $\rightsquigarrow R^2 \to I \to 0$
$\Rightarrow \mu(I) = 2$

1955:
Serre's conjecture
Can one say a f.g. projective module over a polynomial ring over a field is free?
Hilbert–Serre: Such modules are stably free:
$P \oplus R^m = R^n$

$$\rightsquigarrow \quad 0 \longrightarrow R^{\bullet} \longrightarrow R^n \longrightarrow R^n \longrightarrow 0$$

right-invertible matrix

$AB = I_m$
$\rightsquigarrow$ unimodular row –

Study of f.g. proj. modules = study of right-invertible matrices

P – projective over R
$m \in P \qquad O_P(m) = \{ f(m) : f \in P^* \}$
order ideal
m – unimodular $\Leftrightarrow O_P(m) = R$

Cowsik–Nori: see Book by Rao–Ischebeck "Ideals & Reality"
Forster's Conjecture: see Google: Rabeya Basu (Homepage) M.Phil. (pdf)
– proved by
Sathey – Mohomkumar
– generalized by Satya Mandal

Quillen–Suslin: P = f.g. proj. $R[x]$-module
If for some monic polynomial $f \in R[x]$
P_f is free over $R[x]_f$, then P is free over $R[x]$
Serre's conjecture follows: $R = k[x_1, \ldots, x_n]$
$P \in \mathcal{P}\mathrm{proj}(R)$
$P \otimes k(x_1, \ldots, x_n)$, is $k(x_1, \ldots, x_n)$-free
$\Rightarrow \exists f(x_1, \ldots, x_n) \in k[x_1, \ldots, x_n]$ s.t. P_f is free
After Nagata transformation
$t_n \longmapsto t_n$
$t_i \longmapsto t_i + t_n^{r_i}$ – invertible change of variables

$\rightsquigarrow f$ monic in t_n with coeff. from $k[t_1,\dots,t_{n-1}] = R'$

$\to P$ is $R'[t_n]$-free by Quillen–Suslin

– due to Horrocks for local rings
then Quillen–Suslin introduced localization!

P_M is extended from $R_M[x]$ $\forall M \in \mathrm{Max}(R)$

$\Rightarrow P$ is extended from $R[x]$?

– see Rao's book

or

Google R. Basu $\rightsquigarrow$ Amit Ray (Home) $\to$ notes on Serre's conj.

$GL_n(R)$, $Sp_{2m}(R)$, $O_{2m}(R)$, $GQ_{2m}(R)$, $GH_{2m}(R)$, $2 \in R^*$

unitary, hermitian

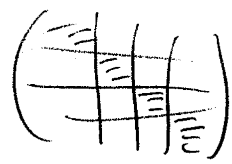
$\rightsquigarrow$ transvections subgroups

Symplectic & orthogonal groups:

considers the permutation $\sigma(2i) = 2i-1$, $\sigma(2i-1) = 2i$

Alternating forms: $\psi_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\psi_n = \psi_{n-1} \perp \psi_1$

Symmetric forms: $\tilde{\psi}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\tilde{\psi}_n = \tilde{\psi}_{n-1} \perp \tilde{\psi}_1$



$Sp_{2m}(R) = \{\alpha \in GL_{2m} \mid \alpha^t \psi_n \alpha = \psi_n\}$

$O_{2m}(R) = \{\alpha \dots \tilde{\psi}_n \dots = \tilde{\psi}_n\}$

$GL_n(R)$: elementary matrices $\begin{pmatrix} 1 & & \lambda \\ & \ddots & \\ & & 1 \end{pmatrix}$

$E_n(R) = \langle e_{ij}(\lambda) \mid \lambda \in R, i \neq j \rangle$

$ESp_{2m}(R) = \langle s\, e_{ij}(\lambda) \mid \lambda \in R \rangle$

form parameter is maximal,

$s\, e_{ij}(\lambda) = I_n + e_{ij}(\lambda) - (-1)^{i+j} e_{\sigma(j)\sigma(i)}$ if $\sigma(j) \neq i$
$= I_n + e_{ij}(\lambda)$ if $\sigma(j) = i$

for EO it is minimal

$EO_{2m}(R) = \langle o\, e_{ij}(\lambda) \mid \lambda \in R \rangle$

$o\, e_{ij}(\lambda) = I_n + e_{ij}(\lambda) - e_{\sigma(j)\sigma(i)}(\lambda)$

$G(n,R) = \underset{\geq 3}{GL_n(R)}$ or $\underbrace{Sp_{2m}(R) \text{ or } O_{2m}(R)}_{m \geq 3}$ $2m = n$

$E(n,R)$ = corresp. elem. subgroup

$e_1 E_n(R) = e_1 ESp_n(R)$

$S(n,R) = SL_n(R) / Sp_{2m}(R) / SO_{2m}(R)$

assumption: $n \geq 3, m \geq 3, 2 \in R^*$

Suslin's local-global principle. let $\alpha(x) \in G(n, R[x])$, $\alpha(0) = I_n$. If $\alpha_{\mathfrak{m}}(x) \in E(n, R_{\mathfrak{m}}[x])$ $\forall \mathfrak{m} \in Max(R)$, then $\alpha(x) \in E(n, R[x])$.

// Basu, Rao, Khanna

Fact: $E(n,R) \trianglelefteq G(n,R)$ | not true for $n=2$

$$\begin{pmatrix} 1-xy & -xy \\ xy & 1+xy \end{pmatrix}$$ P.M. Cohn

- see (unpublished) book by Gupta:

Normality $\equiv$ L-G. Principle. (Basu, Rao, Khanna)

$$\underset{\underset{E(n,R)}{\uparrow}}{\varepsilon}\, g e_{pq}(x^2y) \varepsilon^{-1} = \prod g e_{p_t q_t}(xY)$$

We define a map $M: R^n \times R^n \longrightarrow M(n,R)$

inner product $\langle\, ;\, \rangle$:

$$M(v,w) = \begin{cases} v w^t & \text{- linear case} \\ v\widehat{w} - \widehat{w} v & \text{- symplectic} \\ v\widehat{w} + \widehat{w} v & \text{- orthogonal} \end{cases}$$

v - column

$\hat{v} = v^t \psi_n$

$$\langle v, w \rangle = \begin{cases} v^t \cdot w & \text{- linear case} \\ \hat{v} \cdot w & \text{- otherwise} \end{cases}$$

(6) $I_n + M(v,w) \in E(n,R)$ | - see Vasenstein's proof for linear case

If $v \in E(n,R) e_1$

and $\langle v, w \rangle = 0$

- true for any associative R

the following are equivalent: // R is finite over $C(R)$
i.e. almost commutative

1. $E(n,R) \trianglelefteq G(n,R)$
2. $I_n + M(v,w) \in E(n,R)$ if $v \in Um(R)$, $\langle v,w\rangle = 0$
3. L.-G. Principle
4. Dilation Principle: If $\alpha(x) \in G(n, R[x])$, $\alpha(0) = I_n$
   $\alpha_s(x) \in E(n, R_s[x])$ for some $s \in C(R)$
   Then $\alpha(bx) \in E(n,R)$ for some $b \in (s^\ell)\, C(R)$ non-nilpotent (not a zero divisor?)
   $\ell \gg 0$
5. $\alpha(x) = I_n + X^d M(v,w)$, $v \in E(n,R)e_1$, ~~$C(R)$ is Noetherian~~
   $\langle v,w\rangle = 0 \implies \alpha(x) \in E(n, R[x])$
   and is a product of matrices of the form
   $\prod g e_{ij}(x h(x))$
   $h(x) \in R[x]$ for $d \gg 0$
6. $I_n + M(v,w) \in E(n,R)$ if $v \in E(n,R)e_1$ & $\langle v,w\rangle = 0$
7. $I_n + M(v,w) \in E(n,R)$ if $v \in G(n,R)e_1$ & $\langle v,w\rangle = 0$

$$7 \Rightarrow 6 \Rightarrow 5 \Rightarrow 4 \underset{\substack{\uparrow \\ \text{uses} \\ \text{Bak's patching}}}{\Rightarrow} 3 \underset{\substack{\uparrow \\ \text{we need almost comm.}}}{\Rightarrow} 2 \Rightarrow 1 \Rightarrow 7$$

$u = \varepsilon e_1$

$$I_n + M(u,w) = \varepsilon(I_n + M(e_1, w_1))\varepsilon^{-1}$$

$$w_1 = \begin{cases} \varepsilon^t w, & \text{linear} \\ \varepsilon^{-1} w, & \text{otherwise} \end{cases}$$

$$\langle e_1, w_1\rangle = \langle v,w\rangle = 0 \implies w_1 = (0, w_{12}, w_{13}, \ldots)$$

$\boxed{\text{Lemma}}$ If $\varepsilon = \varepsilon_1 \varepsilon_2 \ldots \varepsilon_r$, ε_i are elementary generators,
then $\varepsilon\, g e_{pq}(X^{2^r m} Y)\,\varepsilon^{-1} = \prod g e_{p_t q_t}(x h_t(x,y))$

$(5) \Rightarrow (4)$: $X \mapsto X^d$

$(3) \Rightarrow (2)$: over semilocal every row is completable