19.10.2011
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Applications of Local - Global Principle & injective stability for K_1

① Applications of LG

a) ~~$NK_1(R)$~~ $SK_1(R[x])$ has no k-torsion if $1/k \in R$, R = local

b) a LG Principle for commutator subgroup

② Injective stability – Improved stability over affine algebras

$$NK_1(R) := \mathrm{Ker}\left(K_1(R[x]) \xrightarrow{x \mapsto 0} K_1(R)\right)$$

<u>Th.</u> R = assoc. ring, $1/k \in R$; Then $NK_1(R)$ has no k-torsion

Observation: R = local $\Rightarrow SL_n(R) = E_n(R)$

Claim: $SK_1(R[x]) = NK_1(R)$ if R = comm. local

$\subseteq$ – clear

Let $\alpha(x) \in NK_1(R) \Rightarrow [\alpha(0)] = [1]$

we can replace $\alpha(x)$ by $\alpha(x)\alpha(0)^{-1} \to \det(\ldots) = 1$

$\rightsquigarrow \in SK_1(R[x])$

Two matrices $\alpha \in M_r(R)$, $\beta \in M_s(R)$ are stably equivalent if $\exists\, \varepsilon_1, \varepsilon_2 \in E_t(R)$ s.t.

$$\varepsilon_1 (\alpha \perp I_{t-r}) \varepsilon_2 = \beta \perp I_{t-s}$$

<u>Higman Lemma</u> // should be generalised to unitary groups

$$\alpha(x) \approx \begin{pmatrix} I_r & -a_n x \\ 0 & I_r \end{pmatrix} \begin{pmatrix} a_0 + a_1 x + \cdots + a_n x^n & 0 \\ 0 & I_r \end{pmatrix} \begin{pmatrix} I_r & 0 \\ x^{n-1} & I_r \end{pmatrix}$$

$$\downarrow$$

$$I + NX$$

$\alpha(x) \in GL_n(R[x])$, $\alpha(0) = I \rightsquigarrow N$ – nilpotent

$$R_t = R[X] / (X^{t+1})$$

$\boxed{\text{Lemma 1}}$ Let $P(x) \in R[x]$. Then in R_t we get

$$(1 + x^r P(x)) = (1 + x^r P(0))(1 + x^{r+1} Q(x))$$

with $r > 0$, $\deg q(x) < t - r$.

$\boxed{\text{Lemma 2}}$

$$(1+x^r P(x))^{k^n} = 1$$
$$\Rightarrow (1+x^r P(x)) = (1+x^{n+1} Q(x))$$

$$[\alpha(x)] \xrightarrow{\text{Higman}} [I+NX]$$

$[I+Nx]^k = 1 \Rightarrow [I+kNx] = [I]$

$x \longmapsto \frac{1}{k} x \quad \rightsquigarrow [I+Nx] = [1]$

// It is a necessary condition: see Gubeladze's examples.

$\cancel{\alpha \in} \begin{pmatrix} 1+xy, & x^2 \\ -y^2, & 1+xy \end{pmatrix} \qquad R[x^2, yx, y^2]$

$R = \mathbb{Z}_2 \rightsquigarrow \alpha^2 \in E_3(\mathbb{Z}_2[x^2, yx, y^2])$

Graded version:

$R = R_0 \oplus R_1 \oplus \dots$

$\cancel{N = N_0 \oplus N_1 \oplus} \dots$

$\frac{1}{k} \in R_0, \quad \alpha(x) \in GL_n(R[x]) \quad \alpha(0) = I_n$

If $\alpha_m(x) \in E_n(R_m[x]) \quad \forall m \in Max(R_0)$

$\Rightarrow \alpha(x) \in E_n(R[x])$

$\theta : R \longrightarrow R[x]$

$(a_0, a_1, \dots) \mapsto a_0 + a_1 x + a_2 x^2 + \dots$

Let $N = N_0 \oplus N_1 \oplus \dots$ – a graded R-~~matrix~~, $\quad 1/k \in R_0$

If $[(I+N)]^k = [I]$ in $SK_1(R)$

$\rightarrow [(I+N)] = [I+N_0]$.

In particular, if R_0 is reduced / local ring, then $SK_1(R)$ has no k-torsion

$\left[\begin{array}{l} R_0[x]^* = R_0^* \\ N_0 \text{ – nilpotent; } \det(I+N_0 x) = 1 \rightsquigarrow I+N_0 \in SL_r = E_r \rightsquigarrow [I+N_0] = [I] \end{array} \right.$

Anderson's conjecture:

is $P \in \mathcal{P}(R[M])$ free?

$\uparrow$ monoid ring (commutative)

Answer: Gubeladze

$R = PID \rightsquigarrow$ YES

Swan gave an algebraic proof:

True for R = Dedekind Domain

Proof of ① via LG

Claim: $[I_s + Nx]^k = [I] \Rightarrow [(I+NX)]^k = [I + kNX]$

(N - nilpotent), $N^t = 0$ $\qquad$ induction on (t,k)

First we show $[(I_s + NX)^{-1}] = [(I_s - NX)]$

suppose $t = 1, k > 0$;

$t = 2 \Rightarrow$ it is correct

Define $\delta(x) = (I_s + NX)(I_s - NX) = I_s - N^2 X^2$

$\delta_1(x) = I_s - N^2 x$

$\rightsquigarrow [\delta_1(x^2)]$ has k-torsion

$\boxed{\text{Lemma}}$: N = nilpotent

$I_r + NX^2 \in E_r(R[x]) \Rightarrow (I_r + NX)^2 \in E_{2r}(R[x])$

Proof apply transfer homo (see Milnor, "K-Theory")

$$Tr : K_1(R[x]) \longrightarrow K_1(R[x^2])$$

$$I_r + N x^2 \longmapsto (I_r + N x^2)^2$$

$$\in E_r(R[x]) \Longrightarrow \in E_{2r}(R[x^2])$$

Now apply $x^2 \mapsto x \rightsquigarrow (I_r + Nx)^2 \in E_{2r}(R[x])$

$\Rightarrow [\delta_1(x)]^{2k} = [I]$

$1/2 \in R$, By induction: $[\delta_1(x)]^{2k} = (I_s - N^2 x)^{2k}$

$\Rightarrow [\delta_1(x^2)]^k = [I]$

by induction $\Rightarrow [\delta_1(x^2)] = [I] \xRightarrow{x^2 \mapsto x} [\delta_1(x)] = [I]$

Define $\beta(X,T) = (I_s + N(X+T))(I_s + N(X-T))$

$= (I_s + NX)^2 (I_s - (I_s + NX)^{-2} N^2 T^2)$

it has k-torsion $\Rightarrow$

$[\beta(X,T)] = [I] \rightsquigarrow \beta(X,T)_m \in \boxed{(I_s - (I_s+NX)^{-2} N^2 T^2)} \cdot E_n(R[X]_m[T])$

$\forall m \in \mathrm{Max}(R[X])$

$\gamma(T) = (I_s - (I_s + NX)^{-2} N^2 T)_m$

Therefore, $[\gamma(T^2)]^k = [\beta_m(X,T)]^k = [I]$ in $K_1(R[X]_m[T])$

$\cancel{\Rightarrow \beta_m(X,T)}$ $\Rightarrow [\gamma(T)]^{2k} = [I]$

$\Rightarrow [\gamma(T)] = [I]$ as $((I_s + NX^2)^{-2} N^2)^t = 0$

$\xrightarrow{T \mapsto T^2} [\gamma(T^2)] = [I]$

$[\beta_m(X,T)] = [I]$, i.e. $\beta_m(X,T) \in E_r(R[X]_m[T])$

by LG $\rightsquigarrow [\beta(X,T)] = [\beta(X,0)] =$

$= [(I_s + NX)^2]$

Put $T = (k-1)X$

$\rightsquigarrow [(I_s + NX)^2] = [(I_s + kNX)(I_s - (k-2)NX)]$

Apply $X \longmapsto -X$

$[I_s + kNX] = [(I_s + NX)^2] \cdot [I_s + (k-2)NX]$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \| $

$\qquad\qquad\qquad\qquad\qquad\qquad [I_s + NX]^{k-2}$

$\rightsquigarrow [I_s + NX]^k$

$\rightsquigarrow [I_s + kNX] = [I] \rightsquigarrow X \longmapsto \frac{1}{k} X.$

L-G for commutator subgroups

Let $\alpha(x) \in GL_n(R[x])$, $\alpha(0) = Id$

If $\alpha_m(x) \in [SL_n(R_m[x]), SL_n(R_m[x])]$ $\forall m \in Max(R)$

then $\alpha(x) \in [SL_n(R[x]), SL_n(R[x])]$

- true also for $Sp_{2n}(R)$

Let $Q = P \oplus R$ // $Q = P \oplus R^2$ for Sp

Pratyusha Chatterjee, Ravi Rao: $\approx$ K-Theory?

$\alpha_m(x) \in E_n(R_m[x], I[x]) \rightsquigarrow \alpha(x) \in E_n(R[x], I[x])$

if $\begin{cases} \alpha(x) \equiv \alpha(0) \mod I \\ \alpha(0) = 1 \mod I \end{cases}$ or ?

L-G for transvections:

If $\alpha(x) \in Aut(Q[x])$, $\alpha(0) = Id$,

$\alpha_m(x) \in E_r(R_m[x])$ for every $m \in Max(R)$

Then $\alpha(x) \in T(Q)$.

Injective stability:

R = affine algebra over perfect C_1-field // alg. closed, finite, ...

~~then~~ $\dfrac{SL_n(R)}{E_n(R)} \xrightarrow{\text{stability map}} \dfrac{SL_{n+1}(R)}{E_{n+1}(R)}$

$f(x_1, \dots, x_n) = 0$, $n > d$ (d = deg of f)

it has a non-triv. solution

$\boxed{\text{24 hrs local cohomology BOOK!}}$

If γ is stably elementary

$\exists$ a homotopy between γ & I_n

Moreover if R = regular. THEN

the map is bijective for $n \geq d+1$

we can replace R by proj mod

SL_n by Sp_{2n} $\rightsquigarrow$ map is bijective

for $n \geq d+1$, if n = odd

d, if n = even.

Orthogonal case

$n \geq 2d+4$ (Vaserstein)

- cannot be improved in general

if so then that will imply

every vector in $Um_{d+1}(A[x])$ is completable

$A = \dfrac{k[x, y, z]}{(z^7 - x^2 - y^3)}$, $k = \mathbb{C}$

$Um_3(A[T, T^{-1}][\cdots], (\cdots))$

$\neq e_1(E_3(\dots))$