Изоморфизмы в гомотопической категории

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$$\begin{tikzcd}
T = \mathbb{A}^1/\mathbb{G}_m \arrow[r, phantom, "\simeq"] \arrow[d, phantom, "\wr"] & \mathbb{G}_m \wedge S^1_s \arrow[d, phantom, "\shortparallel"] \\
(\mathbb{P}^1, \infty) & S^1_t
\end{tikzcd}$$

$(\mathbb{P}^1, \infty) \simeq \mathbb{P}^1/\mathbb{A}^1$:

$$\begin{tikzcd}
\mathbb{A}^1 \simeq pt \arrow[r] \arrow[d] & \mathbb{P}^1 \arrow[d] \\
pt \arrow[r] & \mathbb{P}^1/\mathbb{A}^1
\end{tikzcd}
\rightsquigarrow
\begin{tikzcd}
pt \arrow[r] \arrow[d] & \mathbb{P}^1 \arrow[d] \\
pt \arrow[r] & (\mathbb{P}^1, \infty)
\end{tikzcd}$$

$$\mathbb{P}^1/\mathbb{A}^1 \underset{\substack{\uparrow \\ \text{вырезание}}}{\simeq} \mathbb{A}^1/\mathbb{G}_m$$

$$\begin{tikzcd}
\underset{(\mathbb{G}_m, 1)}{\mathbb{G}_m} \arrow[r, "i"] & \underset{(\mathbb{A}^1, 1)}{\mathbb{A}^1}
\end{tikzcd}$$ — вложение. Нарисуем конус

$$\begin{tikzcd}
(\mathbb{G}_m, 1) \arrow[r, "i"] \arrow[d] & (\mathbb{A}^1, 1) \simeq pt \arrow[d] \arrow[dd, dashed, bend left] \\
\mathbb{G}_m \wedge \Delta^1_s \arrow[r] \arrow[d, "\wr"] \arrow[ur, phantom, "\text{стягивание}"] & \mathrm{Cone}(i) \arrow[d, "\wr"] \arrow[r, phantom, "\simeq"] & \mathbb{G}_m \wedge S^1_s \\
pt \arrow[r] & \mathbb{A}^1/\mathbb{G}_m
\end{tikzcd}$$

$$\begin{tikzcd}
(\mathbb{G}_m, 1) \arrow[r] \arrow[d] & pt \arrow[d] \\
\mathbb{G}_m \wedge \Delta^1_s \arrow[r] & (\mathbb{G}_m, 1) \wedge S^1_s
\end{tikzcd}$$

$\rightsquigarrow$

$$\begin{tikzcd}
(\mathbb{G}_m, 1) \wedge S^0_s \arrow[r] \arrow[d] & (\mathbb{G}_m, 1) \wedge pt \arrow[d] \\
\mathbb{G}_m \wedge \Delta^1_s \arrow[r] & (\mathbb{G}_m, 1) \wedge S^1_s
\end{tikzcd}$$

$\downarrow$

$$\begin{tikzcd}
S^0_s \arrow[r] \arrow[d] & pt \arrow[d] \\
\Delta^1_s \arrow[r] & S^1_s \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

$$(\mathbb{A}^2\setminus\{0\},(1,1)) \simeq S^{3,2} := T\wedge(\mathbb{G}_m,1) \cong S^1_s\wedge(\mathbb{G}_m,1)^{\wedge 2}$$

$$(\mathbb{A}^n\setminus\{0\},(1,1,\dots,1)) \simeq S^{2n-1,n} \qquad // (\mathbb{G}_m,1)\simeq S^{1,1}$$

$$\{\sum_{i=1}^n x_i y_i = 1\} = SL_n/SL_{n-1}: \qquad SL_n \curvearrowright \{x_i,y_i\}: \; x\mapsto gx,\; y^T\mapsto y^Tg^{-1}$$

$$SL_{n-1} = \mathrm{Stab}((1,0,\dots,0),(1,0,\dots,0))$$

$$\{\sum_{i=1}^n x_iy_i=1\} \xrightarrow{\mathbb{A}^{n-1}} \mathbb{A}^n-\{0\}$$

$$H^{i,j}(\mathbb{A}^n-\{0\}) = H^{i,j}(\sum_{i=1}^n x_iy_i=1)$$

$$H^{i,j}(\mathbb{A}^n-\{0\}) = H^{i,j}((\mathbb{A}^n-\{0\},1))\oplus H^{i,j}(pt)$$

$$H^{i,j}(X) = H^{i,j}((X_+,+))\oplus H^{i,j}(pt)$$

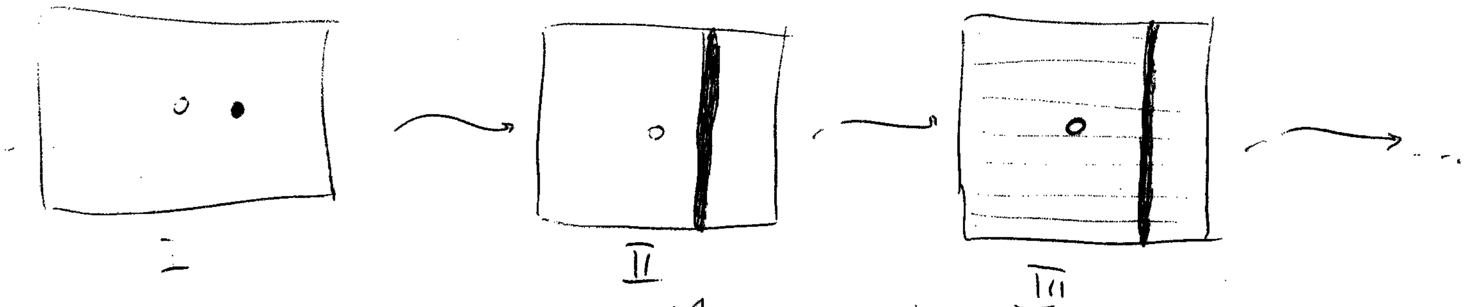
$$(pt_+,+) \rightleftarrows (X_+,+) \longrightarrow (X,*)$$

расщепление

⤳ есть расщепление на когомологиях

$$= H^{i,j}(S^{2n-1,n})\oplus H^{i,j}(pt_+)$$
$$\| $$
$$H^{i-2n+1,j-n}(pt_+)\oplus H^{i,j}(pt_+)$$

почему $(\mathbb{A}^2-\{0\},(1,\dots,1)) \simeq T\wedge(\mathbb{G}_m,1)$

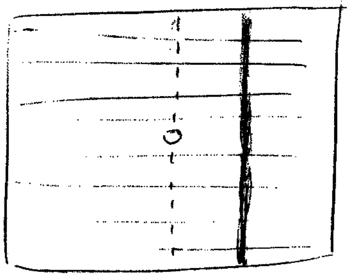


$$\begin{tikzcd}
 & & \mathbb{A}^1 \arrow[r] \arrow[d] & pt \arrow[d] \\
 & (1\times\mathbb{A}^1)\cup(\mathbb{A}^1\times\mathbb{G}_m) \arrow[ur, "\sim"] \arrow[r] \arrow[d] & \mathbb{A}^2-\{0\} \arrow[r] \arrow[d] & \mathrm{II} \\
 & pt \arrow[r] & \mathrm{III} &
\end{tikzcd}$$

(см. Morel–Voevodsky, Lemma 2.11)

$$\begin{tikzcd}
\mathbb{G}_m \arrow[r] \arrow[d] & \mathbb{A}^1\times\mathbb{G}_m \arrow[r, "\sim"] \arrow[d] & \mathbb{G}_m \arrow[d] \\
1\times\mathbb{A}^1 \arrow[r] & (1\times\mathbb{A}^1)\cup(\mathbb{A}^1\times\mathbb{G}_m) & \mathbb{A}^2
\end{tikzcd}$$

… вырезание →



$$= \mathbb{G}_m \wedge \underbrace{T}_{*}$$

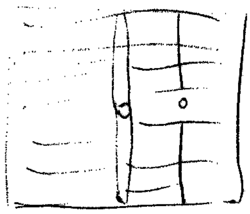
$$\| $$

$$(\mathbb{G}_m, 1) \wedge (\mathbb{A}^1/\mathbb{G}_m)$$

$H_Z(U)$

$\cong$ если $\overline{X} \cap Z = \emptyset$

$H_Z(U \setminus X)$



$$= Th(N_{\mathbb{G}_m \hookrightarrow \mathbb{A}^2 - \{0\}})$$

$$\cong$$

$$\mathbb{A}^2 - \{0\} \big/ \mathbb{A}^2 - \{0\} - \mathbb{G}_m$$