APPLICATIONS OF DE JONG'S THEOREM ON ALTERATIONS

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1. Introduction

The purpose of this article is to give a survey of some of the applications of de Jong's theorem on alterations [dJ]. Most of the applications fall into one of the following two categories:

The first type of application deals with contravariant functors $\mathcal{F}$ from some subcategory of the category of schemes to the category of $\mathbb{Q}$-vector spaces with extra structure (e.g. Galois action), which are equipped with a trace map for finite étale morphisms. In a situation like this, for a given scheme X, $\mathcal{F}(X)$ will be a direct summand of $\mathcal{F}(X')$ for an alteration X' of X. This allows to deduce properties of $\mathcal{F}(X)$ for general X if one only knows the same property for smooth schemes. The following are examples of this kind of application: The independence of l in Grothendieck's monodromy theorem, the p-adic monodromy theorem, finiteness of rigid cohomology, and a (conditional) vanishing theorem for motivic cohomology. All but the last application were already discussed by Berthelot in his Bourbaki talk, and we follow his exposition.

The second type of application is more direct. There are certain Grothendieck topologies which admit proper surjective maps as coverings. For such a topology, de Jong's theorem tells us that any variety is locally smooth. The two main examples we are giving here are Deligne's topology of universal cohomological descent, and the h-topology of Suslin and Voevodsky. In the first case, we get a generalization to characteristic p of Deligne's theorem that any scheme admits a proper hypercovering. In the second case, a theorem of Suslin and Voevodsky comparing their singular cohomology of varieties to étale cohomology, and a theorem of Suslin comparing Bloch's higher Chow groups to étale cohomology, generalize to characteristic p.

We must point out that none of the work presented here is original, and that our exposition follows other papers closely in parts.

2. The Theorem, Serre's conjecture

In this section, we explain notation, give de Jong's theorem, and give as a first application the proof of Serre's conjecture on intersection multiplicities.

Let X be an integral Noetherian scheme. An *alteration* X' of X is an integral scheme X' together with a proper dominant morphism $\varphi : X' \to X$ which is finite over a non-empty open subset of X.

There are two versions of de Jong's theorem; the first one deals with varieties over fields k, and the second one with varieties over complete discrete valuation rings. We denote by k-variety an integral scheme which is separated and of finite type over k.

Theorem 2.1. [dJ, Theorem 4.1] *Let X be a variety over a field k and $Z \subseteq X$ a proper closed subset. Then there exists an alteration $\varphi : X' \to X$ and an open immersion of X' into a regular, projective k-variety $\bar{X}'$, such that the closed subset $\varphi^{-1}(Z) \cup (\bar{X}' - X')$ is the support of a strict normal crossing divisor on $\bar{X}'$.*

There is a finite extension k' of k such that the structure morphism $\bar{X}' \to k$ factors through k' and that $\bar{X}'$ is geometrically irreducible and smooth over k'. If k is perfect, then $\bar{X}'$ is smooth over k, and φ can be chosen generically étale.

Let $S = \operatorname{Spec} A$ be the spectrum of a complete discrete valuation ring A with generic point η and closed point s. An *S-variety* is an integral scheme X, separated, flat and of finite type over S. Let X be an S-variety whose closed fiber X_s has irreducible components X_i, $i \in I$. For $J \subseteq I$, let $X_J = \cap_{j \in J} X_j$. Then X is *strictly semi-stable* over S, if

- X_η is smooth over $k(\eta)$
- X_s is reduced
- each X_i is a divisor on X
- for each $J \subseteq I$, X_J is smooth over $k(s)$ of codimension $\#J$ in X.

Let (X, Z) be a pair consisting of an S-variety X together with a closed subset $Z \subseteq X$, viewed as a reduced closed subscheme. Write $Z = Z_f \cup Z'$, where $Z_f \to S$ is flat and $Z' \subseteq X_s$. Let Z_i be the irreducible components of Z_f, and $Z_J = \cap_{j \in J} Z_j$. The pair (X, Z) is a *strictly semi-stable pair* if

- X is strictly semi-stable over S
- Z is divisor with normal crossings on X
- for each $J \subseteq I$, Z_J is a union of strictly semi-stable S-varieties.

Theorem 2.2. [dJ, Theorem 6.5] *Let X be an S-variety and $Z \subseteq X$ a proper closed subset containing the closed fiber. Then there exists a discrete valuation ring A', finite over A, a variety X' over $S' = \operatorname{Spec} A'$, an alteration $\varphi : X' \to X$ over S, and an open immersion $j : X' \to \bar{X}'$ of S'-varieties, such that $\bar{X}'$ is projective over S' with geometrically irreducible generic fiber, and $(\bar{X}', \varphi^{-1}(Z)_{\mathrm{red}} \cup (\bar{X}' - X'))$ is a strictly semi-stable pair.*

2.1. **Serre's conjecture.** The first application we give concerns intersection multiplicities, see [R2] for an overview. We give it here because it does not fit into one of the other categories mentioned in the introduction.

Let A be a regular local ring of finite Krull dimension with maximal ideal $\mathfrak{m}$ and residue field k. Let M and N be two finitely generated A-modules such that $M \otimes_A N$ is of finite length. This implies $\dim_A M + \dim_A N \leq \dim A$, where $\dim_A M$ is the Krull dimension of the ring $A/\operatorname{Ann}_A(M)$. Geometrically, if M and N are ideals of A defining subvarieties, one would like to define the multiplicity of intersection of these subvarieties in the point given by the maximal ideal of A. Of course, one wants this multiplicity to be non-negative, and to be zero if the subvarieties do not meet. Serre [Se] proposed to define the intersection multiplicity of M and N as

$$\chi_A(M,N) = \sum_{i\geq 0}(-1)^i \lg_A \operatorname{Tor}_i^A(M,N),$$

and conjectured the following properties:

Theorem 2.3. *Under the above assumptions, we have*

Positivity: $\chi_A(M,N) \geq 0$
Annihilation: $\chi_A(M,N) = 0$ *if* $\dim_A M + \dim_A N < \dim A$

Serre proved this for A of equal characteristic, and for A of unequal characteristic and non-ramified. The annihilation conjecture was proved by Gillet-Soulé [GS] and Roberts [R1]. Finally, using de Jong's theorem, Gabber [GA] proved the positivity conjecture, see [B1] for more details.

3. Grothendieck topologies for which alterations are coverings

In this section, we give some applications which use Grothendieck topologies admitting proper surjective maps as coverings. De Jong's theorem implies that every variety is locally smooth for such a topology.

3.1. **Proper hypercoverings.** (see Deligne [D, Section 5]) Let Δ be the category with objects finite ordered sets $[n] := \{0,\dots,n\}$ and morphisms maps respecting the ordering; let Δ_t be the full subcategory of sets $[n]$ with $n \leq t$. Recall that a simplicial object (respectively a t-truncated simplicial object) in the category $\mathcal{C}$ is a contravariant functor $U_{\bullet} : \Delta \longrightarrow \mathcal{C}$ (respectively $U_{\bullet} : \Delta_t \longrightarrow \mathcal{C}$). One usually writes X_n for $X_{\bullet}([n])$. The restriction functor sk_t (t-skeleton) from simplicial objects to t-truncated simplicial objects has a right adjoint functor cosk_t (t-coskeleton) such that $\mathrm{sk}_t = \mathrm{sk}_t \mathrm{cosk}_t \mathrm{sk}_t$. The notion of a simplicial scheme generalizes the notion of a scheme by taking $X_n = X$ for all n, and all simplicial maps the identity.

A sheaf $\mathcal{F}^{\bullet}$ on a simplicial topological space $X_{\bullet}$ is a family of sheaves $\mathcal{F}^n$ on X_n together with morphisms of sheaves on X_m, $f^*\mathcal{F}^n \longrightarrow \mathcal{F}^m$, for each map $f : [n] \longrightarrow [m]$ satisfying obvious compatibilities. A sheaf on $X_{\bullet}$ can be viewed as a functor on pairs (n, U) with $U \subseteq X_n$, satisfying certain compatibility conditions. In particular, the sheaves on $X_{\bullet}$ can be viewed as the category of sheaves on a site. The global sections of the sheaf $\mathcal{F}^{\bullet}$ are

$$\Gamma(X_{\bullet}, \mathcal{F}^{\bullet}) = \ker\left(\Gamma(X_0, \mathcal{F}^0) \longrightarrow \Gamma(X_1, \mathcal{F}^1)\right),$$

where the map is the difference of the maps induced by the two maps ∂_0, ∂_1 from $[0]$ to $[1]$. Let $H^i(X_{\bullet}, \mathcal{F}^{\bullet})$ be the ith derived functor of the global section functor. Looking at an acyclic resolution (for example the Godement resolution), one sees that there is a spectral sequence

$$E_1^{pq} = H^q(X_p, \mathcal{F}^p) \Rightarrow H^{p+q}(X_{\bullet}, \mathcal{F}^{\bullet}).$$

Let $a : X_{\bullet} \to S$ be an augmented simplicial scheme, i.e. a simplicial scheme together with a map $X_0 \to S$. This induces a (unique) map $a_n : X_n \to S$ for each n, and a functor a^* from sheaves on S to sheaves on $X_{\bullet}$, sending $\mathcal{F}$ to the sheaf $a_n^*\mathcal{F}$ on X_n. The functor a^* has a left adjoint a_*, explicitly,

$$a_*\mathcal{F}^{\bullet} = \ker\left(a_{0*}\mathcal{F}^0 \xrightarrow{\partial_0^*-\partial_1^*} a_{1*}\mathcal{F}^1\right).$$

This can be derived to give a functor

$$Ra_* : D^+(X_{\bullet}) \to D^+(S)$$

from the derived category of bounded above complexes of sheaves of abelian groups on $X_{\bullet}$ to the corresponding category on S. Let $\varphi : \mathrm{id} \to Ra_*a^*$ be the associated adjunction morphism. Then a is said to be of *cohomological descent* if φ is an isomorphism. Since $\Gamma(X_{\bullet}, \mathcal{F}^{\bullet}) = \Gamma(S, Ra_*\mathcal{F}^{\bullet})$ for any sheaf on $X_{\bullet}$, cohomological descent implies

$$H^i(S, \mathcal{F}) \cong H^i(S, Ra_*a^*\mathcal{F}) \cong H^i(X_{\bullet}, a^*\mathcal{F}^{\cdot}),$$

and the spectral sequence above reads

$$E_1^{p,q} = H^q(X_p, a_p^*\mathcal{F}) \Rightarrow H^{p+q}(S, \mathcal{F}).$$

If we can find for a given S an $a : X_{\bullet} \to S$ of cohomological descent such that all the schemes X_n are smooth, then this formalism allows us to study the cohomology groups of singular schemes in terms of the cohomology groups of smooth schemes.

We have the following basic example of morphisms of cohomological descent [SGA 4, V bis]: A t-truncated simplicial scheme $X_{\bullet}$ is called a t-truncated proper hypercovering if the adjoint maps

$$\varphi_{n+1} : X_{n+1} \to (\operatorname{cosk}_n \operatorname{sk}_n X_{\bullet})_{n+1} \tag{1}$$

are proper and surjective for all $n \leq t-1$. In this case, the map $\operatorname{cosk}_t X_{\bullet} \to S$ is of cohomological descent. This construction is used to prove the following theorem:

Theorem 3.1. *Let S be a variety over a perfect field k. Then there exists a simplicial scheme $\bar{X}_{\bullet}$, projective and smooth over k, a strict normal crossing divisor $D_{\bullet}$ in $\bar{X}_{\bullet}$ with open complement $X_{\bullet} = \bar{X}_{\bullet} - D_{\bullet}$, and an augmentation $a : X_{\bullet} \to S$ which is a proper hypercovering of S.*

For the proof one constructs inductively, using de Jong's theorem, t-truncated simplicial schemes ${}_tX_{\bullet}$ over S with compactification ${}_t\bar{X}_{\bullet}$, such that ${}_tX_{\bullet}$ satisfies the condition (1). The limit of these t-truncated schemes then satisfies the statement of the theorem, see Deligne's [D, 6.2.5].

3.2. **Singular cohomology of varieties.** Suslin and Voevodsky define in [SV] singular homology $H_*^{\mathrm{sing}}(X, A)$ and cohomology groups $H^*_{\mathrm{sing}}(X, A)$ for any scheme X of finite type over a field k, and any abelian group A. For $k = \mathbb{C}$, and $A = \mathbb{Z}/n$, these groups generalize the usual singular homology groups. We give a short outline of the construction, see Levine [L2] for another survey.

Let $\mathcal{F}$ be a presheaf of abelian groups on Sch/k, the category of schemes of finite type over a field k of exponential characteristic p. We define presheaves $\mathcal{F}_q$ by

$$\mathcal{F}_q(X) := \mathcal{F}(X \times \Delta_q). \tag{2}$$

Here

$$\Delta_q = \operatorname{Spec} k[t_0, \dots , t_q]/(\sum t_i = 1)$$

is the algebraic q-simplex. As in topology, $\Delta_\bullet$ is a cosimplicial scheme, hence every presheaf $\mathcal{F}$ on Sch/k gives rise to a simplicial presheaf $\mathcal{F}_\bullet$ on Sch/k via (2). By the Dold-Kan equivalence, this corresponds to a complex of presheaves $\mathcal{F}_*$ on Sch/k. We let

$$C_*(\mathcal{F}) = \mathcal{F}_*(k)$$

be the global sections over k of this complex of presheaves. Note that in order to define $C_*(\mathcal{F})$, we only need to know the values of $\mathcal{F}$ on the algebraic q-simplices, for example it suffices for $\mathcal{F}$ to be defined on smooth schemes over k.

Let $c_0(X)$ and $z_0(X)$ be the presheaf which associates to every smooth connected k-scheme S the free abelian group generated by the closed integral subschemes $Z \subseteq X \times S$ which are finite and surjective over S and quasi-finite over S, respectively. Note that if X is proper, then $c_0(X) = z_0(X)$. For an abelian group A one defines

$$H_*^{\mathrm{sing}}(X, A) = \mathrm{Tor}_*^{\mathrm{Ab}}(C_*(c_0(X)), A),$$
$$H^*_{\mathrm{sing}}(X, A) = \mathrm{Ext}^*_{\mathrm{Ab}}(C_*(c_0(X)), A).$$

This generalizes singular (co)homology; for X a scheme of finite type over $\mathbb{C}$, one has the following natural isomorphisms [SV, Theorem 8.3]:

$$H_*^{\mathrm{sing}}(X, \mathbb{Z}/m) \xrightarrow{\sim} H_*(X(\mathbb{C}), \mathbb{Z}/m)$$
$$H^*_{\mathrm{sing}}(X, \mathbb{Z}/m) \xleftarrow{\sim} H^*(X(\mathbb{C}), \mathbb{Z}/m).$$

The right hand side is the ordinary (co)homology of the $\mathbb{C}$-valued points of X.

Suslin and Voevodsky also show that for X separated of finite type over an algebraically closed field of characteristic 0, their singular cohomology groups agree with étale cohomology groups. Using de Jong's theorem on alterations, one can show that the last hypothesis is spurious:

Theorem 3.2. [SV, Corollary 7.8] *Let X be a separated scheme of finite type over an algebraically closed field k, and let m be prime to the characteristic of k. Then*

$$H^*_{\mathrm{sing}}(X, \mathbb{Z}/m) \cong H^*_{\text{ét}}(X, \mathbb{Z}/m).$$

The rest of this subsection is devoted to give a sketch of this theorem. We introduce a Grothendieck topology on Sch/k, the h-topology. An h-cover of a scheme X is a finite family of morphisms of finite type $\{p_i : X_i \to X\}$ such that the map $\coprod X_i \to X$ is a universal topological epimorphism. The h-topology is the Grothendieck topology generated by all h-coverings. The h-topology is finer than the étale topology. In our context it is important to note that an alteration is an h-covering; in particular every separated and integral scheme of finite type over k is locally smooth for the h-topology by theorem 2.1. For any presheaf $\mathcal{F}$ on Sch/k we denote the associated sheaf for the h-topology by $\mathcal{F}_{\mathrm{h}}$.

We use a collection of theorems in [SV] to express étale cohomology in terms of the h-topology: The presheaf $c_0(X)$ can be extended to a presheaf on normal integral schemes with the same definition on objects; however one has to invert the characteristic of k for functoriality [SV, Section 5]. This presheaf can be further extended to a presheaf on all schemes of finite type over k [SV, Section 6], which we will again denote by $c_0(X)$. The analogous statements for $z_0(X)$ hold.

By [SV, Theorem 6.7], after inverting the characteristic of k, the h-sheaf $c_0(X)_{\mathrm{h}}$ is isomorphic to the free sheaf $\mathbb{Z}(X)_{\mathrm{h}}$ generated by X, i.e. the sheaf associated to the presheaf which sends U to the free abelian group generated by $\mathrm{Hom}(U, X)$. This implies the following

Lemma 3.3. *Let X be a separated scheme over k and m prime to the characteristic of k. Then we have isomorphisms*

$$\mathrm{Ext}^*_{\mathrm{h}}(c_0(X)_{\mathrm{h}}, \mathbb{Z}/m) \cong H^*_{\text{ét}}(X, \mathbb{Z}/m)$$
$$\mathrm{Ext}^*_{\mathrm{h}}(z_0(X)_{\mathrm{h}}, \mathbb{Z}/m) \cong H^*_{\text{ét},c}(X, \mathbb{Z}/m).$$

Proof. For an étale sheaf $\mathcal{F}$, we get by comparing to an intermediate Grothendieck topology, the qfh-topology, the isomorphisms [SV, Corollary 10.10]

$$\mathrm{Ext}^*_{\text{ét}}(\mathcal{F}, \mathbb{Z}/m) \cong \mathrm{Ext}^*_{\mathrm{qfh}}(\mathcal{F}_{\mathrm{qfh}}, \mathbb{Z}/m) \cong \mathrm{Ext}^*_{\mathrm{h}}(\mathcal{F}_{\mathrm{h}}, \mathbb{Z}/m).$$

Hence we have

$$\mathrm{Ext}^*_{\mathrm{h}}(c_0(X)_{\mathrm{h}}, \mathbb{Z}/m) \cong \mathrm{Ext}^*_{\mathrm{h}}(\mathbb{Z}(X)_{\mathrm{h}}, \mathbb{Z}/m) \cong \mathrm{Ext}^*_{\text{ét}}(\mathbb{Z}(X), \mathbb{Z}/m) \cong H^*_{\text{ét}}(X, \mathbb{Z}/m).$$

To prove the second isomorphism, we choose an open embedding $j : X \to \bar{X}$ into a complete separated scheme $\bar{X}$. Let $i : Y \to \bar{X}$ be the closed embedding of the complement. There is an exact sequence of h-sheaves [Su]:

$$0 \to z_0(Y)_{\mathrm{h}} \xrightarrow{i_*} z_0(X)_{\mathrm{h}} \xrightarrow{j^*} z_0(U)_{\mathrm{h}} \to 0. \tag{3}$$

Comparing the associated long exact $\mathrm{Ext}^*_{\mathrm{h}}(-, \mathbb{Z}/m)$-sequence to the long exact Gysin sequence for étale cohomology with compact supports, the second statement follows from the first. ⥀

To apply the following theorem, we need another definition. A presheaf $\mathcal{F}$ is a *homotopy invariant presheaf with transfers* if the projection induces an isomorphism $\mathcal{F}(X) \xrightarrow{\sim} \mathcal{F}(X \times \mathbb{A}^1)$, and if every element of $c_0(X)(Y)$ induces a map $\mathcal{F}(X) \to \mathcal{F}(Y)$. As an example, any sheaf for the qfh-topology can be equipped with transfers [SV, Section 6]. On the other hand, taking the homology groups of the complex $\mathcal{F}_*$ is a functorial way of making $\mathcal{F}$ homotopy invariant [SV, Corollary 7.5]. In particular, for any qfh-sheaf the homology presheaves $\mathcal{H}_q(\mathcal{F}_*)$ are homotopy invariant presheaves with transfers.

Theorem 3.4. *(Rigidity Theorem* [SV, Theorem 4.5]*) Let k be an algebraically closed field, $\mathcal{F}$ a homotopy invariant presheaf with transfers on Sch/k, and let m be prime to the characteristic of k. Then there are canonical isomorphisms*

$$\mathrm{Ext}^*_{\mathrm{h}}(\mathcal{F}_{\mathrm{h}}, \mathbb{Z}/m) \cong \mathrm{Ext}^*_{\mathrm{Ab}}(\mathcal{F}(k), \mathbb{Z}/m).$$

Sketch of proof: Let $\mathcal{F}_0$ be the constant presheaf $\mathcal{F}(\mathrm{Spec}\, k)$. Let $\mathcal{F}'$ be the cokernel of the natural map $\mathcal{F}_0 \longrightarrow \mathcal{F}$, which is an inclusion because k is algebraically closed, hence every scheme of finite type has a k-rational point.

An explicit calculation [SV, Theorem 4.4] shows that for a homotopy invariant m-torsion presheaf with transfer $\mathcal{G}$, and X^h_x the henselization of the smooth scheme X at a closed point x, $\mathcal{G}(X^h_x) \cong \mathcal{G}(k)$. Since alterations and étale covers are h-covers, every scheme is locally smooth for the h-topology by theorem 2.1.

Applying this to the presheaves $\mathcal{F}'/m$ and ${}_m\mathcal{F}'$ (cokernel and kernel of multiplication by m of $\mathcal{F}'$), which are again homotopy invariant presheaves with transfers, we see that $\mathcal{F}'_{\mathrm{h}}$ is uniquely m-divisible, hence the natural map $(\mathcal{F}_0)_{\mathrm{h}} \longrightarrow \mathcal{F}_{\mathrm{h}}$ induces the isomorphism of the theorem, noting

$$\mathrm{Ext}^*_h(((\mathcal{F}_0)_{\mathrm{h}}, \mathbb{Z}/m) \cong \mathrm{Ext}^*_{\mathrm{Ab}}(\mathcal{F}(k), \mathbb{Z}/m).$$

$\circlearrowleft$

Corollary 3.5. *For any homotopy invariant presheaf with transfers $\mathcal{F}$ we have*

$$\mathrm{Ext}^*_{\mathrm{h}}(\mathcal{F}_{\mathrm{h}}, \mathbb{Z}/m) \cong \mathrm{Ext}^*_{\mathrm{h}}((\mathcal{F}_*)_{\mathrm{h}}, \mathbb{Z}/m) \cong \mathrm{Ext}^*_{\mathrm{Ab}}(C_*(\mathcal{F}), \mathbb{Z}/m).$$

Sketch of proof: To prove the first isomorphism, one uses the first hypercohomology spectral sequence

$$E_1^{pq} = \mathrm{Ext}^q_{\mathrm{h}}((\mathcal{F}_p)_{\mathrm{h}}, \mathbb{Z}/m) \Rightarrow \mathrm{Ext}^{p+q}_{\mathrm{h}}((\mathcal{F}_*)_{\mathrm{h}}, \mathbb{Z}/m),$$

where $(\mathcal{F}_*)_{\mathrm{h}}$ is the sheafification of the simplicial presheaf $\mathcal{F}_*$ on Sch/k. This spectral sequence collapses at E_2 to the isomorphism $\mathrm{Ext}^q_{\mathrm{h}}(\mathcal{F}_{\mathrm{h}}, \mathbb{Z}/m) \cong \mathrm{Ext}^q_{\mathrm{h}}((\mathcal{F}_*)_{\mathrm{h}}, \mathbb{Z}/m)$, [SV, Corollary 7.3].

To prove the second isomorphism, Suslin and Voevodsky employ the second hypercohomology spectral sequence

$$E_2^{pq} = \mathrm{Ext}^p_{\mathrm{h}}(\mathcal{H}_q((\mathcal{F}_*)_{\mathrm{h}}), \mathbb{Z}/m) \Rightarrow \mathrm{Ext}^{p+q}_{\mathrm{h}}((\mathcal{F}_*)_{\mathrm{h}}, \mathbb{Z}/m).$$

Since sheafification is exact, $\mathcal{H}_q((\mathcal{F}_*)_{\mathrm{h}}) \cong \mathcal{H}_q(\mathcal{F}_*)_{\mathrm{h}}$, and by definition $\mathcal{H}_q(\mathcal{F}_*)(k) = H_q(\mathcal{F}_*(k)) = H_q(C_*(\mathcal{F}))$. The rigidity theorem now shows that

$$\begin{aligned}\mathrm{Ext}^*_{\mathrm{h}}(\mathcal{H}_q((\mathcal{F}_*)_{\mathrm{h}}), \mathbb{Z}/m) &\cong \mathrm{Ext}^*_{\mathrm{h}}(\mathcal{H}_q(\mathcal{F}_*)_{\mathrm{h}}, \mathbb{Z}/m) \\ &\cong \mathrm{Ext}^*_{\mathrm{Ab}}(\mathcal{H}_q(\mathcal{F}_*)(k), \mathbb{Z}/m) \cong \mathrm{Ext}^*_{\mathrm{Ab}}(H_q(C_*(\mathcal{F})), \mathbb{Z}/m).\end{aligned}$$

Hence the natural map induced by taking the associated constant sheaf from the spectral sequence

$$E_2^{pq} = \mathrm{Ext}^p_{\mathrm{Ab}}(H_q(C_*(\mathcal{F})), \mathbb{Z}/m) \Rightarrow \mathrm{Ext}^{p+q}_{\mathrm{Ab}}(C_*(\mathcal{F}), \mathbb{Z}/m)$$

is an isomorphism on E_2-terms, and gives an isomorphism of the abutments. $\circlearrowleft$

To finish the proof of Theorem 3.2, we apply the Corollary to $\mathcal{F} = c_0(X)$, noting that the presheaf $c_0(X)$ is actually a qfh-sheaf, hence admits transfers. We get

$$H^*_{\mathrm{sing}}(X, \mathbb{Z}/m) = \mathrm{Ext}^*_{\mathrm{Ab}}(C_*(c_0(X)), \mathbb{Z}/m) \cong \mathrm{Ext}^*_{\mathrm{h}}(c_0(X)_{\mathrm{h}}, \mathbb{Z}/m),$$

and conclude with Lemma 3.3.

3.3. **Higher Chow groups and étale cohomology.** Let $z^i(X, -)$ be Bloch's cycle complex, i.e. $z^i(X, n)$ is the free abelian group generated by the closed irreducible subschemes of codimension i of $X \times \Delta^n_k$ which intersect all faces properly; see [Bl] for the basic properties. Then for an abelian group A, higher Chow groups with A-coefficients are defined as

$$\mathrm{CH}^i(X, n, A) = H_n(z^i(X, -) \otimes A). \tag{4}$$

Suslin proves in [Su] that for an equidimensional scheme over an algebraically closed field k of characteristic 0, higher Chow groups are dual to étale cohomology with compact support. Again, the hypothesis that the base field has characteristic 0 is spurious:

Theorem 3.6. *Let X be an equidimensional quasi-projective scheme over an algebraically closed field k, and let $i \geq d = \dim X$. Then for any m prime to the characteristic of k,*

$$\mathrm{CH}^i(X, n, \mathbb{Z}/m) \cong H^{2(d-i)+n}_{\text{ét},c}(X, \mathbb{Z}/m(d-i))^\vee.$$

In particular, if X is smooth, we have

$$\mathrm{CH}^i(X, n, \mathbb{Z}/m) \cong H^{2i-n}_{\text{ét}}(X, \mathbb{Z}/m(i)).$$

Proof. Suslin shows that for X an *affine* equidimensional scheme of dimension d, the canonical injection of complexes

$$C_*(z_0(X)) \longrightarrow z^d(X, -)$$

is a quasi-isomorphism [Su, Theorem 2.1]. We show how one can get the general statement from this. We proceed by induction on the dimension of X. For a quasi-projective scheme X, one can find an effective Cartier divisor $Y \subset X$ such that the open complement U is affine.

It follows from Corollary 3.5 that we have for a complex of presheaves $\mathcal{F}^\cdot$

$$\begin{aligned} H_*(C_*(\mathcal{F}^\cdot) \otimes^{\mathbb{L}} \mathbb{Z}/m)^\vee &\cong H^*(\mathbb{R}\operatorname{Hom}(C_*(\mathcal{F}^\cdot), \mathbb{Z}/m)) \\ &= \mathrm{Ext}^*_{\mathrm{Ab}}(C_*(\mathcal{F}^\cdot), \mathbb{Z}/m) \cong \mathrm{Ext}^*_{\mathrm{h}}(\mathcal{F}^\cdot_{\mathrm{h}}, \mathbb{Z}/m). \end{aligned}$$

In particular, the complex $C_*(\mathcal{F}^\cdot) \otimes^{\mathbb{L}} \mathbb{Z}/m$ is acyclic if the complex of sheaves $\mathcal{F}^\cdot_{\mathrm{h}}$ is exact. Applying this to the exact sequence (3), we get the upper short exact sequence in the following commutative diagram of complexes of abelian groups

$$\begin{array}{ccccc} C_*(z_0(Y)) \otimes^{\mathbb{L}} \mathbb{Z}/m & \longrightarrow & C_*(z_0(X)) \otimes^{\mathbb{L}} \mathbb{Z}/m & \longrightarrow & C_*(z_0(U)) \otimes^{\mathbb{L}} \mathbb{Z}/m \\ \downarrow & & \downarrow & & \downarrow \\ z^{d-1}(Y, -) \otimes^{\mathbb{L}} \mathbb{Z}/m & \longrightarrow & z^d(X, -) \otimes^{\mathbb{L}} \mathbb{Z}/m & \longrightarrow & z^d(U, -) \otimes^{\mathbb{L}} \mathbb{Z}/m. \end{array}$$

The lower row is an exact triangle in the derived category by [Bl]. Since the outer vertical maps are quasi-isomorphisms by induction and the affine case, the same holds for the middle vertical map.

The theorem follows now for $i = d$ because by Lemma 3.3 and Corollary 3.5

$$\begin{aligned}\mathrm{CH}^d(X, n, \mathbb{Z}/m) &= H_n(z^d(X, -) \otimes^{\mathbb{L}} \mathbb{Z}/m) \cong H_n(C_*(z_0(X)) \otimes^{\mathbb{L}} \mathbb{Z}/m) \\ &\cong \mathrm{Ext}^n_{\mathrm{Ab}}(C_*(z_0(X)), \mathbb{Z}/m)^\vee \cong \mathrm{Ext}^n_{\mathrm{h}}(z_0(X)_{\mathrm{h}}, \mathbb{Z}/m)^\vee \cong H^n_{\text{ét},c}(X, \mathbb{Z}/m)^\vee.\end{aligned}$$

The general case can be derived by applying this to $X \times \mathbb{A}^{i-d}$, and using homotopy invariance. ⥀

4. Applications using trace maps

There are two main mechanisms how de Jong's theorem is used to prove properties of cohomology groups. Since the mechanism of the proof is the most important point and has to be adapted to the specific situation, we are somewhat vague in the formulation:

Lemma 4.1. *Let $\mathcal{P}$ be a property of cohomology groups of varieties over finite extensions of a perfect field K. Suppose that $\mathcal{P}$ is*

1. *preserved by extensions*
2. *holds for the cohomology of smooth projective varieties*

Assume that the cohomology theory

1. *has a long exact localization sequence*
2. *has trace maps for finite étale maps*

Then $\mathcal{P}$ holds for the cohomology groups of all varieties.

Proof. Using the localization sequence and induction, one sees that it is equivalent to prove property $\mathcal{P}$ for the cohomology of a scheme X or of some open subscheme U of X. On the other hand, we can use Theorem 2.1 to show that for a given X, there is an X', an alteration $\varphi : X' \to X$ and an open embedding of X' into a smooth projective scheme $\bar{X}'$. Let U be a sufficiently small smooth open subscheme of X, then the morphism $U' = U \times_X X' \to U$ is a finite map between smooth schemes. Using the trace map, we see that the cohomology of U is a direct summand of the cohomology of U'. Since $\mathcal{P}$ holds for the cohomology of $\bar{X}'$, using the localization sequence it holds for the cohomology of U', hence for U and finally for X. ⥀

Lemma 4.2. *Let $\mathcal{P}$ be a property of cohomology groups of varieties over finite extensions of the field of fractions K of a Henselian discrete valuation ring A. Suppose that property $\mathcal{P}$*

1. *holds for the cohomology groups of the generic fibers of semi-stable schemes over A*
2. *is inherited by direct summands of cohomology groups*

Suppose that the cohomology theory admits a trace map for finite étale maps. Then property $\mathcal{P}$ holds for all cohomology groups of smooth and proper varieties over K.

Proof. Let X be a scheme which is smooth and proper over K. By Theorem 2.2 we can find a finite extension K' of K, a strictly semi-stable scheme $\mathcal{X}'$ over the ring of integers of K' with generic fiber X', and a K-alteration $\varphi : X' \to X$. Property $\mathcal{P}$ holds for X' by hypothesis, and the cohomology group of X is a direct summand of the cohomology group of X' using the trace map. ◌

The following examples use these two methods with only minor modifications. The first two examples were discussed by Berthelot in [B1].

4.1. **Monodromy, $l \neq p$.** Let A be a Henselian discrete valuation ring with field of fractions K, and X be a K-scheme of finite type. Let $\bar{K}$ be the algebraic closure of K, $X_{\bar{K}} = X \times_K \bar{K}$, and denote by $I \subseteq G = \mathrm{Gal}(\bar{K}/K)$ the inertia subgroup of the Galois group. We fix a prime l different from the characteristic of K. Then the étale cohomology groups $H^i_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_l)$ and the étale cohomology groups with compact support $H^i_{\text{ét},c}(X_{\bar{K}}, \mathbb{Q}_l)$ are equipped with an action of the Galois group G, giving an l-adic representation of I.

If l is also different from the residue characteristic of A, then the monodromy theorem of Grothendieck [SGA 7, Th. 2.2] states that the l-adic representation $H^i_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_l)$ of I is quasi-unipotent, i.e. there is a subgroup of finite index $I' \subseteq I$ such that $g - \mathrm{id}$ acts nilpotently for each $g \in I'$. It has been observed by Deligne that de Jong's theorem implies that such an I' can be chosen independently of l:

Theorem 4.3. *There exists a subgroup $I' \subseteq I$ of finite index such that the action of I' on $H^i_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_l)$ and on $H^i_{\text{ét},c}(X_{\bar{K}}, \mathbb{Q}_l)$ is unipotent for each $l \neq p$.*

Sketch of proof:(see Berthelot [B1]) If X is the generic fiber of a semi-stable scheme, then the action of I is seen to be unipotent by using the vanishing cycle spectral sequence. Using the method of Lemma 4.2, we see that, after a finite extension of K (which amounts to replacing I by a subgroup of finite index), the theorem holds for all smooth and proper schemes over K.

If X is separated and of finite type over K, one can use the method of Lemma 4.1 to show that the cohomology with compact support $H^i_{\text{ét},c}(X_{\bar{K}}, \mathbb{Q}_l)$ has the property of the theorem. For X smooth and separated over K, the statement about $H^i_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_l)$ follows from the previous case by Poincaré duality.

In the general case one uses the spectral sequence for a Cech-covering to reduce to the case X affine, and then reduces to the case X integral. We can apply Theorem 3.1 to construct a proper hypercovering $X'_{\bullet} \xrightarrow{\varphi} X$ such that all X_i are smooth and such that the adjunction map $\mathbb{Q}_{l,X} \to R\varphi_*\mathbb{Q}_{l,X'_{\bullet}}$ is an isomorphism. The hypercohomology spectral sequence and the result for $H^t_{\text{ét}}(X'_{s,\bar{K}}, \mathbb{Q}_l)$ proves the result for $H^i_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_l)$. ◌

4.2. **Monodromy, $l = p$.** The above techniques can also be used to study the p-adic representations $H^*_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_p)$ if K is of characteristic 0 with residue characteristic p. For simplicity we assume that K is a finite extension of $\mathbb{Q}_p$. Let K_0 be the maximal unramified subextension of K, with Frobenius endomorphism σ, and let G be the absolute Galois group of K.

Let us recall some basic properties of Fontaine's rings [Fo]

$$B_{\mathrm{crys}} \subseteq B_{\mathrm{st}} \subseteq B_{dR}; \quad B_{HT}.$$

These rings carry a structure of a G-module, and for the invariants one has

$$B_{HT}^G \cong B_{dR}^G \cong K;$$
$$B_{\mathrm{crys}}^G \cong B_{\mathrm{st}}^G \cong K_0.$$

The ring B_{dR} is a complete discrete valuation field with residue field $\bar{K}^\wedge$, the completion of the algebraic closure of K. The algebra B_{HT} is the graded algebra associated to the filtration given by the valuation of B_{dR}, and

$$B_{HT} = \operatorname{gr} B_{dR} = \bigoplus_{i \in \mathbb{Z}} \bar{K}^\wedge(i).$$

The K_0-algebra B_{crys} is equipped with a σ-semilinear automorphism φ, and a G-equivariant injective homomorphism $B_{\mathrm{crys}} \otimes_{K_0} K \to B_{dR}$ which induces a filtration on $B_{\mathrm{crys}} \otimes_{K_0} K$. The associated map of graded algebras is an isomorphism.

Finally, B_{st} is a G-invariant polynomial extension in one variable u of B_{crys} inside B_{dR}; we extend φ to B_{st} by setting $\varphi(u) = pu$. The monodromy operator $N : B_{\mathrm{st}} \to B_{\mathrm{st}}$ is the unique B_{crys}-derivation such that $Nu = 1$; it follows that $N\varphi = p\varphi N$ and we can recover B_{crys} as the kernel of N. The natural injection $B_{\mathrm{st}} \otimes_{K_0} K \to B_{dR}$ induces a filtration on $B_{\mathrm{st}} \otimes_{K_0} K$, and $u \in \mathrm{Fil}^1$.

For a p-adic representation E of $G = \mathrm{Gal}(\bar{K}/K)$ and $*$ one of the symbols crys, st, dR and HT, Fontaine defines

$$D_*(E) = (B_* \otimes_{\mathbb{Q}_p} E)^G.$$

Then E is said to be crystalline, semi-stable, de Rham or Hodge-Tate, if the canonical injection

$$\alpha_* : B_* \otimes_{B_*^G} D_*(E) \to B_* \otimes_{\mathbb{Q}_p} E \tag{5}$$

is an isomorphism, for $*$ the corresponding symbol. It is easy to see the following implications:

$$\text{crystalline} \Rightarrow \text{semi-stable} \Rightarrow \text{de Rham} \Rightarrow \text{Hodge-Tate}.$$

Of special interest is the case of the representation $H^*_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_p)$, in this case $D_*(H^*_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_p))$ can sometimes be identified with other cohomology theories, so that by (5) this cohomology theory and étale cohomology determine each other.

The following conjectures of Fontaine have been proved by Faltings [Fa] and Tsuji [T1] based on the work of a number of people (Fontaine, Hyodo, Kato, Messing...).

1. (Faltings) Let X be smooth and proper over K. Let $H^*_{dR}(X/K)$ be the de Rham cohomology of X, equipped with its Hodge filtration. Then the representation $H^*_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_p)$ is de Rham, and

$$B_{dR} \otimes_K H^*_{DR}(X/K) \cong B_{dR} \otimes_{\mathbb{Q}_p} H^*_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_p),$$

as filtered Galois-modules.

2. (Tsuji) Let X be the generic fiber of a proper, semi-stable scheme over A. Let $H^*_{\mathrm{st}}(X/W(k))$ be the logarithmic crystalline cohomology of Hyodo and Kato, equipped with a σ-linear endomorphism φ and a monodromy operator N satisfying $N\varphi = p\varphi N$. After extending scalars to K, it is isomorphic to de Rham cohomology, hence inherits the Hodge filtration. Then $H^*_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_p)$ is semi-stable and
$$B_{\mathrm{st}} \otimes_{W(k)} H^*_{\mathrm{st}}(X/W(k)) \cong B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} H^*_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_p),$$
compatible with Galois action, σ-semilinear endomorphism, monodromy operator, and filtration after extension of scalars to K.

Note that if X is the generic fiber of a smooth and proper scheme $\mathcal{X}$ over A, then logarithmic crystalline cohomology agrees with the usual crystalline cohomology, and the monodromy operator N is zero. In this situation, the above result has been proved by Faltings, and yields that $H^*_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_p)$ is crystalline, and
$$B_{\mathrm{crys}} \otimes_{W(k)} H^*_{\mathrm{crys}}(\mathcal{X}/W(k)) \cong B_{\mathrm{crys}} \otimes_{\mathbb{Q}_p} H^*_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_p),$$
as Galois modules with σ-semilinear endomorphism and filtration after extending scalars to K.

A representation E is called *potentially semi-stable*, if its restriction to an open subgroup of finite index of the Galois group is semi-stable. This is the closest analogy to the monodromy theorem of Grothendieck in the p-adic situation. Obviously, semi-stable representations are potentially semi-stable, and one can show that potentially semi-stable representations are de Rham. Using de Jong's and Tsuji's theorem, we get the following strengthening and alternate proof of (1):

Proposition 4.4. *For a smooth and proper scheme X over K, $H^*_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_p)$ is potentially semi-stable.*

Sketch of proof: By Tsuji's theorem, the theorem holds for the generic fiber of a proper semi-stable scheme. Since a subrepresentation of a semi-stable representation is again semi-stable, we see using the method of Lemma 4.2 that $H^*_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_p)$ is semi-stable as a $\mathrm{Gal}(\bar{K}/K')$-module, hence potentially semi-stable as a $\mathrm{Gal}(\bar{K}/K)$-module. ↺

Note that extensions of semi-stable representations need not be semi-stable, so that Lemma 4.1 does not apply to prove Proposition 4.4 for all K-varieties X. However, in a recent paper Tsuji uses proper hypercoverings to extend his method to prove the following generalization:

Theorem 4.5. [T2, Corollary 2.2.3] *Let X be a proper scheme over K. Then $H^*_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_p)$ is potentially semi-stable.*

4.3. **Finiteness of rigid cohomology.** Let k be a field of characteristic $p > 0$, W a Cohen-ring for k, and K its field of fractions. For a smooth, affine scheme X over k, Monsky and Washnitzer defined the cohomology $H^*_{MW}(X/K)$, which is a K-vector space. Not much is known about these groups. On the other hand, for X smooth and proper over k, Grothendieck defined crystalline cohomology groups $H^*_{\mathrm{crys}}(X/W)$, which are finitely generated W-modules. De Jong's theorem allows to prove finite generation of $H^*_{MW}(X/K)$, starting from the corresponding statement for crystalline cohomology.

The bridge between the two theories is Berthelot's rigid cohomology. We refer the reader to his paper [B2] for a proper definition of $H^*_{\mathrm{rig}}(X/K)$ for X a separated scheme of finite type over k, and rigid cohomology with support $H^*_{Z,\mathrm{rig}}(X/K)$ for Z a closed subscheme of X. We will only need the following properties:

Trace: If X' is étale over X, then there exists a trace map $H^*_{MW}(X'/K) \to H^*_{MW}(X/K)$.

Proper: If X is proper and smooth over k, then there is an isomorphism

$$H^*_{\mathrm{rig}}(X/K) \xrightarrow{\sim} H^*_{\mathrm{crys}}(X/W) \otimes_W K.$$

Affine: If X is affine and smooth over k, then there is an isomorphism

$$H^*_{\mathrm{rig}}(X/K) \xrightarrow{\sim} H^*_{MW}(X/K).$$

Gysin: Let $Y \to X$ be a closed immersion of codimension r between two smooth schemes over k which can be lifted to characteristic 0. If Z a closed subscheme of Y then there exists a Gysin isomorphism

$$H^*_{Z,\mathrm{rig}}(Y/K) \xrightarrow{\sim} H^{*+2r}_{Z,\mathrm{rig}}(X/K).$$

Excision: For $T \subseteq Z \subseteq X$, there is a long exact sequence

$$\ldots \to H^i_{T,\mathrm{rig}}(X/K) \to H^i_{Z,\mathrm{rig}}(X/K) \to H^i_{Z-T,\mathrm{rig}}(X-T/K) \to \ldots.$$

Theorem 4.6. *(Berthelot,* [B2, Théorème 3.1]*) Let X be a smooth, separated scheme over k and $Z \subseteq X$ a closed subscheme. Then the groups $H^*_{Z,\mathrm{rig}}(X/K)$ are finite dimensional K-vector spaces. In particular, if X is a smooth affine scheme over k, the groups $H^*_{MW}(X/K)$ are finite dimensional K-vector spaces.*

Sketch of proof:(see Berthelot [B1]) The proof is an induction over n on the following two assertions for each field k of characteristic p and all smooth and separated schemes X over k:

$(a)_n$: $H^*_{\mathrm{rig}}(X/K)$ is finite dimensional for X of dimension at most n.

$(b)_n$: $H^*_{Z,\mathrm{rig}}(X/K)$ is finite dimensional for each closed subscheme Z of dimension at most n.

To prove $(a)_n$ from (b_{n-1}) one applies the method of Lemma 4.1. The statement follows for smooth and proper schemes by comparison to crystalline cohomology. To get a trace map, one finds an alteration over the algebraic closure of the base field (which is then generically étale), and observes that this alteration is already defined over a finite extension of the base field. The proof of (b_n) from (a_n) does not require de Jong's theorem, so we omit it. $\circlearrowleft$

4.4. **Rational motivic cohomology in characteristic p.** We give an example of how de Jong's theorem can be used to study motivic cohomology of fields and varieties in characteristic p. For X a smooth variety over a field k, define motivic cohomology with coefficients in an abelian group A to be Bloch's higher Chow groups (4):

$$H^i(X, A(n)) = \mathrm{CH}^n(X, 2n-i, A).$$

By [L1], rationally motivic cohomology agrees with the weight n-part of the algebraic K-theory of X: $H^i(X, \mathbb{Q}(n)) \cong K_{2n-i}(X)^{(n)}_{\mathbb{Q}}$. A conjecture of Parshin states that if X is smooth and projective over a finite field, then $H^i(X, \mathbb{Q}(n)) = 0$ unless $i = 2n$. This is motivated by the idea that motivic cohomology should be the Ext-groups in a category of mixed motives, and that the category of mixed motives over a finite field should be semi-simple, hence the Ext-groups vanish. In order to convince algebraic geometers of the validity of Parshin's conjecture, we note that it is a consequence of Tate's conjecture on algebraic cycles and the conjecture that rational and numerical equivalence agrees for smooth, projective varieties over finite fields up to torsion [Ge, Theorem 3.3].

Using de Jong's theorem, we can show that Parshin's conjecture determines rational motivic cohomology of fields and of smooth varieties in characteristic p:

Theorem 4.7. *Let k be a field of characteristic $p > 0$ of transcendence degree e (possibly infinite) over $\mathbb{F}_p$. Assume Parshin's conjecture and let X be a variety of dimension d over k. Then*
i) $H^i(k, \mathbb{Q}(n)) = 0$ unless $i = n \leq e$.
ii) $H^i(X, \mathbb{Q}(n)) = 0$ unless $n \leq i \leq \min\{n+d, e+d\}$.

Proof. i) [Ge, Theorem 3.4] It follows from the definition that $H^i(k, \mathbb{Q}(n)) = 0$ for $i > n$. By induction, we can assume that if $n' < n$, then for any field k of characteristic p, $H^i(k, \mathbb{Q}(n')) = 0$ for $i \neq n'$ and for $n' > e$. Since motivic cohomology commutes with direct limits, we can assume that e is finite. By de Jong, we can find a smooth projective variety X over $\mathbb{F}_p$ such that the function field $k(X)$ of X is a finite extension of k. Since the composition of the inclusion and transfer map $H^i(k, \mathbb{Q}(n)) \to H^i(k(X), \mathbb{Q}(n)) \to H^i(k, \mathbb{Q}(n))$ is multiplication by the degree of the extension, we can assume $k = k(X)$. Now consider the coniveau spectral sequence for motivic cohomology [Bl]

$$E_1^{s,t} = \bigoplus_{x \in X^{(s)}} H^{t-s}(k(x), \mathbb{Q}(n-s)) \Rightarrow H^{s+t}(X, \mathbb{Q}(n)).$$

We have $H^i(k, \mathbb{Q}(n)) = E_1^{0,i}$, and the differentials leaving $E_r^{0,i}$ end in $E_r^{r,i-r+1}$, which is a subquotient of a sum of groups $H^{i+1-2r}(k(x), \mathbb{Q}(n-r))$ for various fields $k(x)$ of transcendence degree $e - r$. If $i < n$, then $i + 1 - 2r < n - r$, and if $n > e$, then $e - r > n - r$. Hence, by induction all differentials leaving $E_r^{0,i}$ are zero and $H^i(k, \mathbb{Q}(n)) = E_\infty^{0,i}$. This is a quotient of $H^i(X, \mathbb{Q}(n))$, which is trivial by Parshin's conjecture.

ii) [Ge, Corollary 3.5] The bound $i \leq n+d$ follows from the definition of higher Chow groups. For the other bounds, we use the coniveau spectral sequence and note that by (i) the E_1-terms vanish outside the specified bounds. ⥀

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