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ALTERATIONS CAN REMOVE SINGULARITIES

FRANS OORT

H. Hironaka, 1964:
> *In characteristic zero any variety can be modified into a nonsingular variety.*

A. J. de Jong, 1995:
> *Any variety can be altered into a nonsingular variety.*

ABSTRACT. This note is a review of: A. J. de Jong - *Smoothness, semi-stability and alterations.* Manuscript 1995, 39 pp.: Publications Mathématiques I.H.E.S. **83**, 1996, pp. 51-93.

INTRODUCTION

Resolution of singularities is a subject with a rich history and important applications. Often it is useful to "desingularize" varieties, because more methods are available for the study of nonsingular varieties than for varieties in general.

For many years partial results added to a growing amount of information, crowned by Hironaka's resolution of singularities in characteristic zero in 1964. This work of Hironaka had a great impact on many branches of mathematics. Below we mention only some of the many who contributed to this, and we give some references.

It seemed that the topic was getting very complicated. Hironaka's ingenuous proof had many applications. It was not easy to understand the fine details of his proof. Generalizing that method to varieties in positive characteristic has failed up to now: resolution of singularities *in positive characteristic* has been a topic to which many years of intensive research have been devoted. Up to now the status is not clear. For the general question of resolution of singularities in positive characteristic we seem to have neither a fully verified theorem nor a counterexample.

The algorithms involved in Hironaka's theory are difficult to understand in more complicated situations.

It seemed that we had reached a lull in development in this subject, until a totally new idea came about. In 1995 Johan de Jong approached the problem from a different angle. The idea of the proof is surprisingly easy. Moreover, for several applications this is sufficient. His approach is very geometric, and hence it works in many situations. For example it applies also in mixed characteristic and in positive characteristic. Without much effort it gives a new, easy proof of a weaker form of Hironaka's theorem. It is a radical new idea in a field which seemed to be at an impasse. Below we explain the difference in the methods by Hironaka and by de Jong, but for the moment let us note a striking aspect.

In the approach taken by Hironaka singularities of a variety are studied closely, invariants are defined, and methods are applied (blowing up, algorithms involved) in order to improve the situation, in the sense that the given invariants get "better". The algorithm then should terminate (and in fact, in characteristic zero it does), resulting in the construction of a regular variety. A big advantage of this process developed by Hironaka (and by many others) is the fact that usually it is very explicit, it is canonical in a certain sense, and once it works, the result is in its strongest form.

- The result by Hironaka is the strongest possible result for varieties *in characteristic zero*. It is not known whether it also applies in several other situations. Especially for varieties moving in a family (the "relative situation") it is not clear how to apply this method. - In this approach one focuses from the beginning on the singularities of one variety and tries to remove them stepwise by blowing up.

In the approach by Johan de Jong singularities in the beginning are completely ignored. The variety is fibered by curves, and certain operations are performed, creating possibly many more singularities, until this fibering is in a manageable form. Induction on the dimension allows us to assume the base space is regular, and only then, finally, attention is paid to the singularities. But these are like normal crossings singularities of an algebraic curve, and an easy, explicit blowing up finishes the job.

- The method as proposed by de Jong is more geometric; it can be used over an arbitrary field or in a relative situation. - In this approach at first a geometric procedure is applied (which has nothing to do with singularities; actually it even might create new singularities, and it might blow up in a regular point), until a nice situation is reached (a fibering with nodal curves over a regular variety). Only then the process of blowing up in singular points is carried out (which in this case is completely elementary). In the process certain choices have to be made, so it seems this method does not result in a "canonical" resolution. Also, during the process sometimes we have to extend the function field (i.e. the morphism transforming the variety into a non-singular one need not be birational); we shall explain where and why this is necessary.

This survey on the method of alterations as exposed in [14] is written for those who want to have a first impression about the general idea: we give a brief sketch of the proof of obtaining non-singular varieties by alterations. We hope and expect that this is accessible for everyone who has basic mathematical knowledge. For everyone in need of more precise statements and proofs, we refer to [14], [15], [3], [7]. Anyone

interested in a description of the proof or in certain applications of the method of alterations might turn to the nice and precise survey by Berthelot; see [5].

In this survey we mainly work with varieties over a field, avoiding the terminology of schemes where possible.

An apology: We have simplified statements, and we only sketch ideas of the proofs. This is a simplified survey, not a research paper.

Here is the basic idea of the construction by de Jong:

- For a given variety X of dimension d we produce a morphism $f : X \to Y$ with dim $Y = d - 1$, and all fibres of f are curves (here we might have to apply a modification to X).
- After an alteration on the base (also using induction, i.e. supposing that the theorem is already true for varieties of dimension $d - 1$) we arrive at a (new) morphism $f : X \to Y$ where Y is regular, and all fibres are curves with only ordinary nodes as singularities.
- Then an explicit (and easy) method of resolution of singularities finishes the job (repeated blowing up of X removes all singularities; this is easy for curves, not much more difficult for families of nodal curves over a regular base having certain properties, such as the fact that degeneration takes place only over a strict normal crossings divisor).

Suggestions to the reader:

(1) In case you want to have a first encounter with these new ideas, and if you want to see how alterations can be constructed, you can read the definitions (1.1) and (1.2) below, explaining what is meant by "modification" and "alteration", and follow the 5 steps in Section 2 leading to an insight into the proof of the Theorem 1.4, which says that any variety can be altered into a non-singular variety.

Also you can consult the notes [4].

(2) In case you have experience in algebraic geometry, e.g. you know enough about stable curves, you can read Definition 1.2, and construct a proof of Theorem 1.4 below. You will soon find out that the strategy is clear; you will have to fill in some technicalities.

(3) If you are interested in the fine points of the mechanism of alterations, e.g. the precise description of the theorem, the arithmetic case, alterations of families of curves, and applications, please read the original papers. They are written in a clear style, and they are easy to follow for those who have a basic understanding of the language of schemes.

We have discussed the differences between the results obtained by the methods developed by Hironaka (and many people using that approach) on the one hand and by the alteration method on the other hand. It might very well be that a combination of the two approaches will finally prove to give the strongest results and will provide the best approach.

1. Some definitions and the main results

This paper can be used to obtain a first impression of the subject. For more detailed definitions and a discussion on some concepts used we refer to the appendix.

The word *variety* will be used in the sense explained in [12], or in [23], or in [18], or in [26]. In particular it will be irreducible. Although we work in a more general situation, you might think of: a variety is a set $X \subset \mathbb{P}^n(\mathbb{C})$ defined by homogeneous

polynomial equations, which moreover is irreducible, i.e. X is not the union of two proper Zariski-closed subsets.

We shall use the terminology *non-singular* or *regular* in the usual sense: this means that all local rings of points on the variety are supposed to be regular local rings; see [12], page 32. We say that a variety X is *complete* (a notion stemming from Chevalley) if the projection $X \times V \to V$ on the second factor is a closed morphism (in the Zariski topology) for every variety V. Note that:

- a variety X over the complex numbers is complete if and only if the set of $\mathbb{C}$-rational points $X(\mathbb{C})$ is *compact* in the classical topology; see [21], Th. 2 on page 85.
- Any projective variety is complete (this is sometimes called the "main theorem of elimination theory").

Here are the two basic definitions:

Definition 1.1. Let X be a variety. A *modification* of X is a morphism of varieties $\varphi : X' \to X$ which is birational and proper.

This implies that the morphism φ gives an inclusion of function fields $K(X') \supset K(X)$ which is an equality $K(X') = K(X)$, that there exists a non-empty Zariski open set $U \subset X$ such that the restriction $\varphi|_{U'} \colon U' \xrightarrow{\sim} U$ is an *isomorphism*, and that for every point $x \in X$ the fibre X'_x is complete.

Sometimes a term like: "blowing up", "dilatation", "quadratic transformation", "Cremona transformation" is used to indicated this or a closely related concept.

Here are two typical examples of modifications.

- Let X be a variety, and let $\widetilde{X} \to X$ be its normalization (e.g. see [12], I.3.9A on page 20, and Exc. I.3.17 on page 23). This is a modification, and in this case the fibres of the modification are finite.
- Let X be a variety, $Z \subset X$ be a proper subvariety, and let $X' \to X$ be a blowing up of X along Z; see [12], pp. 28 - 30, page 163 and Proposition II.7.14.

Remark. In case $X = C$ is an algebraic curve, it follows that the normalization morphism $C^\sim \to C$ gives a regular curve and the morphism gives the resolution of singularities of C. For example the normalization process transforms a simple node into two regular disjoint branches, it transforms a cusp into one regular branch, etc.

However, an algebraic surface which is normal need not be regular. Here is an easy example: consider the surface $V \subset \mathbb{A}^3$ given as zeros of the polynomial $XZ - Y^2$; this surface is normal and singular. In this case there is no finite birational morphism resolving this singularity.

A generically bijective morphism between projective varieties in characteristic zero is a modification. Note that a bijective morphism in positive characteristic need not be birational (it can be purely inseparable without giving an isomorphism on the function fields); one might encounter such examples when trying to resolve singularities in positive characteristic.

Definition 1.2 (A. J. de Jong). Let X' and X be varieties. An *alteration* is a morphism $\varphi : X' \to X$ which is dominant, proper and generically finite.

"Dominant" means that the image $\varphi(X')$ is dense in X; equivalently: $\varphi(X')$ contains an open, dense subset of X. We see that dominant + proper implies

that φ is surjective (say, on geometric points). An alteration gives an inclusion of function fields $K(X') \supset K(X)$, which is a finite extension, and there is a non-empty Zariski open set $U \subset X$ such that the restriction $\varphi' : U' \to U$ is a *finite morphism.* For every point $x \in X$ the fibre X'_x is complete.

Suppose X' and X are projective varieties of the same dimension and $\varphi : X' \to X$ is a surjective morphism. Then it is an alteration.

A modification is an alteration, but not necessarily conversely. A finite covering which is not birational is an alteration, which is not a modification. A purely inseparable morphism which is not an isomorphism is an alteration, which is not a modification.

An alteration $\varphi : X' \to X$ can be factored as

$$X' \xrightarrow{\psi} X'' \xrightarrow{\rho} X,$$

such that ρ is a finite alteration, and ψ is a modification: if X' is normal, take for X'' the normalization of X in $K(X')$; even if X' is not normal, such a factorization exists (and it is called the "Stein factorization"; compare [12], III.11.5).

The terminology "alteration" was invented by Johan de Jong. In [pre-13] it appeared for the first time. A new word marked the start of a new idea!

Terminology. We speak of *resolution of singularities* of X if we consider a modification $\varphi : X' \to X$ with X' non-singular; for a morphism φ which is only known to be an alteration we shall not use the word "resolution".

1.3. Theorem. (Hironaka 1964; see [13], **the strong form of resolution of singularities.**) Let K be a field of characteristic zero. Let X be a variety over K. There exists a modification $\varphi : X' \to X$ such that for $X \supset U := X - \mathrm{Sing}(X)$ the induced morphism $\varphi' : U' \to U$ is an isomorphism, and X' is a non-singular variety.

Here $\mathrm{Sing}(X)$ is the (closed) set of singular points of X. We see that the strong form consists of blowing up a (possibly) singular variety in an ideal concentrated in the singular locus in such a way that the blown-up variety is non-singular.

We shall speak of *resolution of singularities in the "weak sense"* of X if $\varphi : X' \to X$ is a modification with X' regular (without requiring that the blowing up be centered only in $\mathrm{Sing}(X)$).

1.4. Theorem (de Jong 1995, [14], Theorem 4.1)**.** Let K be a field, and let X be a variety over K. There exists an alteration $\varphi : X' \to X$ where X' is a non-singular variety. If the field K is perfect, the alteration can be chosen to be separable.

Actually a much stronger theorem can be proved; see below for a precise formulation. Also note that there is no restriction on K (e.g. this can be a field of positive characteristic, it need not be algebraically closed, etc.).

In the construction of X' certain choices will be made; these will be arbitrary in a certain sense, and the degree of φ will depend on such choices.

Suppose you happen to start with a non-singular X; the construction given in [14] might produce a completely different X'. Also, when starting with a singular X, it might be that the morphism $\varphi : X' \to X$ thus constructed need not be finite above non-singular points of X.

1.5. Theorem. (Abramovich & de Jong 1996, [3]; Bogomolov & Pantev 1996, [7]; **a weak form of resolution of singularities.**) Let K be a field of characteristic

zero, and let X be a variety over K. There exists a modification $\varphi : X' \to X$ where X' is a non-singular variety.

Note that this is of course a corollary of the theorem by Hironaka recorded above. Moreover it proves less. However, the proof given by Abramovich and de Jong and by Bogomolov and Pantev is substantially easier than the proof by Hironaka.

2. A sketch of the construction of alterations giving a regular variety

In this section we give a strongly *simplified* form of the proof of Theorem 1.4, a simplification of the proof of [14], Theorem 4.1. We split up the proof in steps. A star attached to a step means that in that phase of the proof a finite extension of the function field might be involved: i.e. the alteration constructed need not be a modification. In steps without a star only modifications are used. In each new situation the new variety (dominating the previous one) again will be denoted by X. A much stronger version of the theorem is recorded in (3.1), and it is a rewarding exercise to follow the steps below and to fill in details in more refined situations.

We hope that this section gives you the flavor of the proof. In the steps below we do not provide precise references. In our description we will slip over several important technical details.

We start with a field K and a variety X over K.

2.1.   **Step 1.** We can assume $K = k$ is algebraically closed and X is projective and normal.

We intend to say: if we prove the theorem with this new additional data, then the theorem in the original, more general form follows. The main ingredient is Chow's Lemma (see [21], pp. 85-89; also see [12], Exercise II.4.10): *for a variety X over K, there exists a modification $X' \to X$, such that X' is quasi-projective.*

From now on, in each step, we shall replace X by a new variety X' over k which admits a modification or an alteration $X' \to X$, arriving finally at a regular variety and an alteration to the variety produced in Step 1.

2.2.   **Step 2.** Given X, construct a morphism $f : X \to Y$ of projective varieties such that for a non-empty Zariski open subset $U \subset Y$ and every geometric point $t \in U \subset Y$ the fibre X_t is an irreducible, complete, non-singular curve. Note: $\dim(Y) = \dim(X) - 1$.

Actually we need more. This step can be carried out in such a way that the smooth locus of f is dense in every irreducible component of every geometric fibre; i.e. there are no "multiple components" in geometric fibres.

This is a classical, geometric idea! Assume $\dim(X) = d$ and $X \subset \mathbb{P}^N$ (remember we work over k, an algebraically closed field); using Bertini's theorem, we see that we can find a linear subvariety $L \subset \mathbb{P}^N$ "in general position" with $\dim(L) = N - d$ such that the projection with center L gives a rational map $X \cdots \to \mathbb{P}^{d-1}$ where the generic fibre is a regular curve. (We indicate by "$\cdots \to$" a rational map, and use "$\to$" for a morphism, i.e. a rational map which is everywhere defined.) By blowing up X we can make this into a morphism. By counting constants the extra conditions can be achieved by choosing the center of the projection more carefully.

The strict transform. In the alteration-method of de Jong we need an operation called the "strict transform" (in [5], 815-12 the terminology "strict alteration" is used). Consider a morphism $X \to S$, and a base change $T \to S$. Assume T to be integral, and let $\eta \in T$ be its generic point. Then define $X' \subset T \times_S X$ as the closure of the fibre $(T \times_S X)_\eta$:

$$((T \times_S X)_\eta)^{Z-clos} \quad =: \quad X' \quad \subset \quad T \times_S X \quad \longrightarrow \quad X$$

Note that if the image of η is not in the image of $X \to S$ (i.e. if $T \times_S X \to T$ is not dominant), then the strict transform in the sense explained here is empty.

Remark. The notion given here is different from the usual notion of "strict transform" of a subvariety under a modification (compare with [12], II.7, the definition after 7.15).

Here is the main idea of the argument of de Jong: some of the fibres of f might have very bad singularities. We want to replace these by nodal curves (curves with only normal crossings as singularities). As we know that certain moduli spaces of curves carry such families, after an alteration of Y we can extend an open part of the parameterization $f : X \to Y$ to a nodal curve $\mathcal{C} \to Y$ actually in such a way that X and $\mathcal{C}$ are birational over Y. But that is not sufficient; we also need that the birational map $\beta : \mathcal{C} \cdots \to X$ thus obtained, which is a morphism β_0 over a dense open $U \subset Y$, actually is a morphism. The central idea is to equip $X \to Y$ with enough sections, these corresponding with sections making $\mathcal{C} \to Y$ into a *stable pointed curve*, which eventually will ensure that β_0 does not blow up points in $\mathcal{C}$, resulting in the conclusion that β is a morphism! In the next two steps and in the next definition this technical aspect of the proof is explained.

2.3. **Step 3*.** After applying alterations to X and to Y we arrive at a morphism $f : X \to Y$ as in Step 2 and mutually different sections $\sigma_1, \cdots, \sigma_n \in X(Y)$ such that every geometric component C' of every geometric fibre of f meets at least three of these sections in the smooth locus of f, i.e. in $C' \cap \mathrm{Sm}(f)$.

There is a "multi-section" in the situation of Step 2 having this property. After an alteration on Y and on X this becomes a union of sections.

Stable pointed curves (following Deligne & Mumford, Knudsen)**.** An algebraic curve is called *nodal* if it is complete, connected, and the singularities of C are not worse than ordinary double points. Its arithmetic genus is given by $g = \mathrm{dm}_k \, \mathrm{Hl}^1(C, \mathcal{O}_C)$.

Suppose C is a nodal curve of genus g over a field k, and let $P_1, \cdots, P_n \in C(k)$ with $2g - 2 + n > 0$; we write $\mathcal{P} = \{P_1, \cdots, P_n\}$; this is called a *stable n-pointed curve* if:

- the points are mutually different, $i < j \Longrightarrow P_i \neq P_j$,
- none of these marked points is singular, $P_i \notin \mathrm{Sing}(C)$,
- and $\mathrm{Aut}(C, \mathcal{P})$ is a finite group; under the previous conditions (and k algebraically closed) this amounts to the condition that for every regular rational irreducible component

$$\mathbb{P}_1 \cong C' \subset C, \quad \text{then} \quad \#(C' \cap (\mathcal{P} \cup \mathrm{Sing}(C)) \geq 3.$$

Families of curves are called stable n-pointed if all geometric fibres are stable n-pointed in the sense just defined, and where the markings are given by sections.

Historically, stable curves and stable pointed curves were introduced in order to construct in a natural way compactifications of moduli spaces (certainly the following names should be mentioned: Zariski, A. Mayer, Deligne, Mumford, Grothendieck, Knudsen, and many more). Among the vast amount of literature we mention only: [10], in this paper we find how stable curves can be used in order to compactify the moduli space of curves; see: [17], where stable pointed curves and families and moduli spaces of these are studied.

In [17] the terminology "pointed stable curves" is used; however, we think we should make a distinction between pointed curves which are stable (as we defined above, following Knudsen) and stable curves which moreover have some marked points. Hence we prefer the terminology "stable pointed curves". A technical remark: stable curves and stable pointed curves can be defined and used over arbitrary base schemes (only when level structures are involved do we want to exclude certain characteristics).

2.4.   **Step 4*.** Here we also use induction on the dimension of the base. After an alteration on the base we arrive at a regular variety Y, a family $g : \mathcal{C} \to Y$ of stable n-pointed curves, which is a morphism of projective varieties, which over a non-empty Zariski-open set $U \subset Y$ coincides with $f_{|U} : X'_U \to U$ with the sections σ_i. Moreover the birational morphism $\beta_0 : \mathcal{C}_U \to X'_U$ extends to a modification $\beta : \mathcal{C} \to X'$.

Here is the heart of the proof. We indicate some of the ideas.

Any family of curves which is stable over a non-empty set of the base can be extended to a stable curve after an alteration of the base. This follows by [10]; one could consult [9] (the precise statement we need follows from that paper). Also we can use [11], where a tautological family of nodal curves is constructed over a moduli space related with stable pointed curves with a level structure. The markings of the family $X' \to Y$ and of the stable n-pointed curve $\mathcal{C} \cdots \to Y$ correspond under the birational transformation thus defined. Now we want to show this extends to a morphism $\mathcal{C} \to Y$.

Then we apply induction on the dimension of the base: we suppose that the theorem we want to prove is valid for all varieties having dimension less than $\dim X$; after an alteration of the base we can suppose Y is regular and the strict transform of X has all the previous properties.

We take the closure $T \subset X \times_Y \mathcal{C}$ of the graph of $\beta_0 : \mathcal{C}_U \to X_U$. Then we apply the "flattening lemma" (see (5.3) below) thus arriving at X, T, and $\mathcal{C}$ flat over Y. All we have to show (modulo some technicalities) is that no point of a fibre of $\mathcal{C} \to Y$ is blown up to a component of a fibre of $X \to Y$. Using the markings and carefully studying the geometry, we show that indeed β_0 extends to a morphism β (as in the "three point lemma"; see 4.18 - 4.20 of [14]).

2.5.   **Step 5.** In all previous steps we can take into account degenerations, and we moreover can assume that the degeneration of the nodal curve $\mathcal{C} \to Y$ takes place over a strict normal crossings divisor. We blow up in codimension 2 components of $\mathrm{Sing}(X) \subset X$. After a finite number of such operations we arrive at a situation where X is regular in codimension 2. Analyzing singularities of X (which can be caused only by nodal singular points in fibres), one finds that an explicit blowing

up produces a regular X arriving at a regular X mapping via an alteration to the variety X we obtained in Step 1. Using the method of Step 1 one finishes (a sketch of) the proof of Theorem 1.4.

3. Stronger results achieved by alterations

Definition ([14], 2.4). A divisor $D \subset S$ on a variety S is called a **s**trict normal crossings divisor on S if:

i) the variety S is regular along D, and

ii) the divisor D is reduced and it is a union of irreducible components $D = \cup_{i \in I} \ D_i$, and

iii) for any subset $J \subset I$ the closed subscheme $D_J := \cap_{j \in J} \ D_i$ is a regular scheme in S of codimension $\#J$ in S.

In other words: every point of D is regular on S, and in such a point the (global) components of D intersect locally like coordinate hyperplanes.

3.1. Theorem ([14], Theorem 3.1). Let X be a variety over a field K, and let $Z \subset X$ be a proper closed subset. There exists an alteration

$$\varphi : X' \longrightarrow X$$

and an open immersion $j : X' \to X^c$ such that

i) X^c is a projective, regular variety, and

ii) the closed subset $D := j(\varphi^{-1}(Z)) \cup X^c - j(X')$ is a strict normal crossings divisor in X^c.

It is a rewarding exercise to go through the steps in Section 3 in this more general situation where the closed set $Z \subset X$ and the final resulting $D \subset X^c$ are taken into consideration (in a sense the proof becomes easier).

For certain applications it is convenient to have alterations respecting the action of a certain group:

3.2. Theorem. (See [14], Theorem 7.3 for a more extensive and more precise form.) Let k be an algebraically closed field, let X a projective variety over k, and suppose given a finite subgroup $G \subset \mathrm{Aut}(X)$. There exists an alteration $\varphi : X_1 \to X$ and a finite subgroup $G_1 \subset \mathrm{Aut}(X_1)$ such that:

i) the alteration φ is equivariant for a surjective homomorphism $G_1 \to G$, and for these actions,

ii) the variety X_1 is projective and non-singular, and

iii) the field extension

$$(k(X))^G \quad \subset \quad (k(X_1))^{G_1}$$

is purely inseparable.

3.3. Corollary (see [14], 7.4). Let k be an algebraically closed field, and X a projective variety over k. There exist a purely inseparable morphism $X_2 \to X$ and a modification $X_3 \to X_2$ such that X_3 has only quotient singularities.

We see that resolution of singularities in characteristic zero follows from resolution of quotient singularities. This is what is done in [3]: via the theorem of de Jong just quoted, using toroidal resolution of singularities, the weak form of resolution of singularities is achieved.

If we were able to resolve quotient singularities in positive characteristic (which up to now is not known in general), we would arrive at resolution of singularities

up to purely inseparable morphisms. It could be that the methods as developed by de Jong throw a new light on resolution of singularities in positive characteristic (I mean resolution via a modification, not only via an alteration).

3.4. **The arithmetic case.** Let R be a Dedekind domain, and let $S = \mathrm{Spec}(R)$. A morphism $S' \to S$ is called a finite extension of Dedekind schemes if there exists a finite extension $K \subset K'$ of the field of fractions K of R, and $S' = \mathrm{Spec}(R')$, where R' is the integral closure of R in K'.

3.5.   A simplified form of [14], Theorem 8.2 reads:

Theorem. Let $X \to S$ be an integral scheme over S, which is proper over S. There exists:

$$
\begin{array}{ccc}
X & \xrightarrow{\varphi} & X' \\
\downarrow & & \downarrow \\
S & \xrightarrow{\psi} & S',
\end{array}
$$

where ψ is a finite extension of Dedekind schemes, φ is an alteration, the morphism $X' \to S'$ is projective, and over a non-empty open set of S' it is smooth.

3.6. **Families of curves and alterations.** In [14], Theorem 6.8, later generalized in [15], we find a proof of the fact that any family of curves can be altered to a semi-stable family. A fact like this seemed more or less known; in these papers by de Jong we find the precise formulation.

4. Some references

The pioneering work of Zariski was the starting point of resolution of singularities in modern algebraic geometry (see [27], [28], [29]), resulting in resolution of singularities of varieties of dimension $d \leq 3$ in characteristic zero. For a historical account, see [18].

Resolution of singularities in characteristic zero for varieties of arbitrary dimension was first proved by Hironaka, [13]. There is a vast literature describing and expanding Hironaka's celebrated theorem; we mention: [25], [6].

Resolution of singularities of algebraic varieties in positive characteristic or of schemes in mixed characteristics seems unsolved in general. We mention [1], [2] solving the two dimensional cases. It might be (and we hope) that the method announced and partially carried out in [24] will give a final answer to the question of resolution of singularities in positive characteristic.

As already recorded, the method of de Jong (or a variant of it) proves (again) weak resolution of singularities in characteristic zero; see [3], [7].

For a more detailed history of the subject, and for various details involved in the proofs by Hironaka and by de Jong, see the volume *Resolutions of singularities*, proceedings to appear in connection with the Obergurgl (Austria) conference in September 1997.

Here are two questions which seem to be basic and which are unsolved:

- *Resolution of singularities in positive characteristic*: Given a variety X over a field of positive characteristic, does there exist a *modification* $X' \to X$ with X' non-singular?
- *Stable reduction theorem in arbitrary dimension*: Given a variety X over a field K with a discrete valuation v, with $\mathcal{O}_v \subset K$ the ring of v-integers, does there exist a semi-stable extension of X over an extension of $\mathcal{O}_v$? This is

solved for curves (Artin-Winters, Deligne-Mumford, Grothendieck). This is
solved in case $\text{char}(K) = 0$. The general case seems a difficult, open problem;
see [5], 1.3.

5. Appendix

In this appendix we discuss in more detail some concepts used above.

5.1. **Varieties.** Let K be a field. A variety *"defined over"* K in the sense of [26]
(in the language of schemes) is a separated scheme of finite type over K which is
irreducible and reduced, and hence integral, and which stays integral after extension
of the base field.

For a discussion about these concepts, see [12], II.4, and see [21], II.3-4. We will
adopt this definition in this paper.

However, one could define a variety over K as an integral separated scheme of
finite type over K (as we find in [14], 2.9).

A variety in this sense can become reducible or non-reduced after extending the
base field. E.g. an algebraic extension $K \subset L$ gives $X = \text{Spec}(L)$, an integral
scheme over K (a variety in the sense of [14]), which does not stay integral after
suitable base extension when $[L : K] > 1$; it is not "a variety defined over K" in
the sense of [26].

If S is a scheme, and $S' := S_{red}$, then the natural morphism $S' \to S$ is dominant,
proper and finite. We see that for the theory of alterations it does not make much
difference which definition is taken (either a "variety over K" in the sense of [14]
or a "variety defined over K" as used in more classical literature).

5.2. **Morphisms.** The terminology "smooth" will be used only in a relative situ-
ation: a morphism can be smooth. The terminology "regular", or "non-singular",
will be used in the absolute sense. A variety can be regular, which means that for
every point P in the variety the local ring at P is a regular local ring. If a morphism
$X \to \text{Spec}(K)$ is smooth, then X is regular (but please do not use the terminology
"a smooth variety"; that can be misleading and confusing).

A morphism between varieties is called *quasi-finite* if every fibre is finite. A
morphism is *proper* if it is universally closed. A morphism between projective
varieties is proper. Note that a proper, quasi-finite morphism between varieties
is finite: use [12], Exercise III.11.2. Let X be a variety of positive dimension, let
$\varphi : X \to Y$ be a morphism, and let $P \in X$ be a closed point; then the induced
morphism $X - \{P\} \to Y$ is not proper, and it is not finite. For example $X - \{P\} \hookrightarrow$
X is quasi-finite but not finite if $\dim X > 0$.

A morphism $\varphi : X \to Y$ between varieties is *dominating* if the image $\varphi(X)$
contains a dense open subset of Y. If so, it defines an inclusion $\varphi^* : K(Y) \hookrightarrow K(X)$
on the function fields. A dominating morphism between varieties of the same
dimension is generically finite, and it is quasi-finite above a dense open set in Y.

5.3. **The flattening lemma** (see [22], page 37, Th. 5.2.2; see [14], 2.19). Here
is the general question: consider a morphism $\varphi : X \to Y$, and try to choose
a modification $Y' \to Y$ in such a way that the strict transform $X' \to Y'$ (as
defined in Section 2) is flat. Raynaud and Gruson give a positive answer in quite a
general situation. In the proof above (in Step 4) we needed a relatively easy case
of this general procedure. Let $f : X \to Y$ be a projective morphism of varieties
(a relative curve is projective). Then there exists a dense open subset $U \subset Y$ over

which the morphism is flat. Using the theory of Hilbert schemes (as developed by Grothendieck), we can extend $X_U \to U$ to a flat $X_U \subset Q \to Z \supset U$. Closing the graph $U \subset Y' \subset Q \times Y$ (note that the projection $Y' \to Y$ is birational) and pulling back $X \to Y$ by strict transform, obtaining $X' \to Y'$, we have arrived at a flat morphism.

5.4. **Tautological families of nodal curves.** The moduli spaces of stable n-pointed *rational* curves (for $n \geq 3$) as defined by Knudsen are fine moduli spaces. However, for any $g > 0$, and for any $n \geq 0$ (if $g = 1$, then $n \geq 1$), there exists a stable n-pointed curve of genus g which has a non-trivial automorphism. This explains the fact that the moduli space $\overline{M_{g,n}}$ for $g > 0$ does not carry a "tautological family". Hence in order to extend families of stable curves to a complete base, we use curves with a level structure. By a generalization of a lemma of Serre (see [9], Lemma 3.5.1) we know that an automorphism of a stable curve inducing the identity on the m-torsion of its Jacobian (in characteristic not dividing m), and $m \geq 3$, is the identity. However, we do not know a good moduli functor for level structures on stable curves; see the discussion in [8], [20], [11]. But for $m \geq 3$ there does exist a tautological nodal family over that moduli space; see [11]. This shows that any family of nodal curves can be extended after an alteration of the base to a nodal family over a complete base, and that is what we need in the proof!

One could ask whether a family of stable curves (or of semi-abelian varieties) given over an open set U of a base variety S can be extended to a stable family over S; this question is discussed in [16]. Under certain conditions this is possible. In general the answer is negative. However, after an alteration on S this extension is possible, as explained above, or as explained in [9].

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