Jet Schemes and Singularities

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1. Introduction

The study of singularities of pairs is fundamental for higher dimensional birational geometry. The usual approach to invariants of such singularities is via divisorial valuations, as in [**Kol**]. In this paper we give a self-contained presentation of an alternative approach, via contact loci in spaces of arcs. Our main application is a version of Inversion of Adjunction for a normal **Q**–Gorenstein variety embedded in a nonsingular variety.

The invariants we study are the minimal log discrepancies. Their systematic study is due to Shokurov and Ambro, who made in particular several conjectures, whose solution would imply the remaining step in the Minimal Model Program, the Termination of Flips (see [**Amb**] and [**Sho**]). We work in the following setting: we have a pair (X, Y), where X is a normal, **Q**–Gorenstein variety and Y is equal to a formal linear combination $\sum_{i=1}^{s} q_i Y_i$, where all q_i are non-negative real numbers, and all Y_i are proper closed subschemes of X. To every closed subset W of X one associates an invariant, the minimal log discrepancy $\mathrm{mld}(W; X, Y)$, obtained by taking the minimum of the so-called log discrepancies of the pair (X, Y) with respect to all divisors E over X whose image lies in W. We do not give here the precise definition, but refer instead to §7.

The space of arcs $J_\infty(X)$ of X parametrizes morphisms $\operatorname{Spec} k[\![t]\!] \to X$, where k is the ground field. It consists of the k–valued points of a scheme that is in general not of finite type over k. This space is studied by looking at its image in the jet schemes of X via the truncation maps. The mth jet scheme $J_m(X)$ is a scheme of finite type that parametrizes morphisms $\operatorname{Spec} k[t]/(t^{m+1}) \to X$. It was shown in [**EMY**] that the minimal log discrepancies can be computed in terms of the codimensions of certain contact loci in $J_\infty(X)$, defined by the order of vanishing along various subschemes of X. As an application it was shown in [**EMY**] and [**EM**] that a precise form of Inversion of Adjunction holds for locally complete intersection varieties. In practice one always works at the finite level, in a suitable jet scheme, and therefore in order to apply the above-mentioned criterion one has to find (a small number of) equations for the jets that can be lifted to the space of arcs. This was the technical core of the argument in [**EM**]. In the present paper we simplify this approach by giving first an interpretation of minimal log discrepancies

in terms of the dimensions of certain contact loci in the jet schemes, as opposed to such loci in the space of arcs (see Theorem 7.9 for the precise statement). We apply this point of view to give a proof of the following version of Inversion of Adjunction. This has been proved independently also by Kawakita in [**Kaw2**].

THEOREM 1.1. *Let A be a nonsingular variety and $Y = \sum_{i=1}^{s} q_i Y_i$, where the q_i are non-negative real numbers and the Y_i are proper closed subschemes of A. If X is a closed normal subvariety of A of codimension c such that X is not contained in the support of any Y_i, and if rK_X is Cartier, then there is an ideal J_r on X whose support is the non-locally complete intersection locus of X such that*

$$(1.1) \qquad \mathrm{mld}(W; A, Y + cX) = \mathrm{mld}\left(W; X, Y|_X + \frac{1}{r} V(J_r)\right)$$

for every proper closed subset W of X.

When X is locally complete intersection, this recovers the main result from [**EM**]. We want to emphasize that from the point of view of jet schemes the ideal J_r in the above theorem appears quite naturally. In fact, the reduction to complete intersection varieties is a constant feature in the the study of jet schemes (see, for example, the results in §4). On the other hand, the appearance of $\frac{1}{r} V(J_r)$ on the right-hand side of (1.1) is the reason why the jet-theoretic approach has failed so far to prove the general case of Inversion of Adjunction.

The main ingredients in the arc-interpretation of the invariants of singularities are the results of Denef and Loeser from [**DL**]. In particular, we use their version of the Birational Transformation Theorem, extending the so-called Change of Variable Theorem for motivic integration, due to Kontsevich [**Kon**]. We have strived to make this paper self-contained, and therefore we have reproved the results we needed from [**DL**]. One of our goals was to avoid the formalism of semi-algebraic sets and work entirely in the context of algebraic-geometry, with the hope that this will be useful to some of the readers. In addition to the results needed for our purpose, we have included a few other fundamental results when we felt that our treatment simplifies the presentation available in the literature. For example, we have included proofs of Kolchin's Irreducibility Theorem and of Greenberg's Theorem on the constructibility of the images of the truncation maps.

A great part of the results on spaces of jets are characteristic–free. In particular, the Birational Transformation Theorem holds also in positive characteristic in a form that is slightly weaker than its usual form, but which suffices for our applications (see Theorem 6.2 below for the precise statement). On the other hand, all our applications depend on the existence of resolutions of singularities. Therefore we did not shy away from using resolutions whenever this simplified the arguments. We emphasize, however, that results such as Theorem 1.1 above depend only on having resolutions of singularities.

While there are no motivic integrals in these notes, the setup we discuss has strong connections with motivic integration (in fact, the first proofs of the results connecting invariants of singularities with spaces of arcs used this framework, see [**Mus**] and [**EMY**]). For a beautiful introduction to the circle of ideas around motivic integration, we refer the reader to Loeser's Seattle lecture notes [**Los**], in this volume.

The paper is organized as follows. The sections §2–§6 are devoted to the general theory of jet schemes and spaces of arcs. In §2 we construct the jet schemes and prove their basic properties. In the next section we treat the spaces of arcs and give a proof of Kolchin's Theorem saying that in characteristic zero the space of arcs of an irreducible variety is again irreducible. Section 4 contains two key technical results concerning the fibers of the truncation morphisms between jet schemes. These are applied in §5 to study cylinders in the space of arcs of an arbitrary variety. In particular, we prove Greenberg's Theorem and discuss the codimension of cylinders. In §6 we present the Birational Transformation Theorem of Denef-Loeser, with a simplified proof following [**Loj**]. This is the crucial ingredient for relating the codimensions of cylinders in the spaces of arcs of X' and of X, when X' is a resolution of singularities of X.

The reader already familiar with the basics about the codimension of cylinders in spaces of arcs can jump directly to §7. Here we give the interpretation of minimal log discrepancies from [**EMY**], but without any recourse to motivic integration. In addition, we prove our new description of these invariants in terms of contact loci in the jet schemes. We apply this description in §8 to prove the version of Inversion of Adjunction in Theorem 1.1. The last section is an appendix in which we collect some general facts that we use in the main body of the paper. In particular, in §9.2 we describe the connection between the Jacobian subscheme of a variety and the subscheme $V(J_r)$ that appears in Theorem 1.1.

Acknowledgements. The debt we owe to the paper [**DL**] of Denef and Loeser can not be overestimated. In addition, we have received a lot of help from Bernd Ulrich. He explained to us the material in §9.2, which got us started in our present treatment. We are grateful to Kyle Hofmann for pointing out several typos in a preliminary version. These notes were written while the second author visited the Institute for Advanced Study. He would like to thank his hosts for the stimulating environment.

2. Jet schemes: construction and basic properties

We work over an algebraically closed field k of arbitrary characteristic. A variety is an integral scheme, separated and of finite type over k. The set of nonnegative integers is denoted by $\mathbf{N}$.

Let X be a scheme of finite type over k, and $m \in \mathbf{N}$. We call a scheme $J_m(X)$ over k the mth *jet scheme* of X if for every k–algebra A we have a functorial bijection

$$(2.1) \qquad \mathrm{Hom}(\mathrm{Spec}(A), J_m(X)) \simeq \mathrm{Hom}(\mathrm{Spec}\ A[t]/(t^{m+1}), X).$$

In particular, the k–valued points of $J_m(X)$ are in bijection with the $k[t]/(t^{m+1})$–valued points of X. The bijections (2.1) describe the functor of points of $J_m(X)$. It follows that if $J_m(X)$ exists, then it is unique up to a canonical isomorphism.

Note that if the jet schemes $J_m(X)$ and $J_p(X)$ exist and if $m > p$, then we have a canonical projection $\pi_{m,p} \colon J_m(X) \to J_p(X)$. This can be defined at the level of the functor of points via (2.1): the induced map

$$\mathrm{Hom}(\mathrm{Spec}\ A[t]/(t^{m+1}), X) \to \mathrm{Hom}(\mathrm{Spec}\ A[t]/(t^{p+1}), X)$$

is induced by the truncation morphism $A[t]/(t^{m+1}) \to A[t]/(t^{p+1})$. It is clear that these morphisms are compatible whenever they are defined: $\pi_{m,p} \circ \pi_{q,m} = \pi_{q,p}$ if

$p < m < q$. If the scheme X is not clear from the context, then we write $\pi_{m,p}^X$ instead of $\pi_{m,p}$.

EXAMPLE 2.1. We clearly have $J_0(X) = X$. For every m, we denote the canonical projection $\pi_{m,0} \colon J_m(X) \to X$ by π_m.

PROPOSITION 2.2. For every scheme X of finite type over k, and for every nonnegative integer m, there is an mth jet scheme $J_m(X)$ of X, and this is again a scheme of finite type over k.

Before proving the proposition we give the following lemma.

LEMMA 2.3. If $U \subseteq X$ is an open subset and if $J_m(X)$ exists, then $J_m(U)$ exists and $J_m(U) = \pi_m^{-1}(U)$.

PROOF. Indeed, let A be a k–algebra and let $\iota_A \colon \operatorname{Spec}(A) \to \operatorname{Spec} A[t]/(t^{m+1})$ be induced by truncation. Note that a morphism $f \colon \operatorname{Spec} A[t]/(t^{m+1}) \to X$ factors through U if and only if the composition $f \circ \iota_A$ factors through U (factoring through U is a set-theoretic statement). Therefore the assertion of the lemma follows from definitions. □

PROOF OF PROPOSITION 2.2. Suppose first that X is affine, and consider a closed embedding $X \hookrightarrow \mathbf{A}^n$ such that X is defined by the ideal $I = (f_1, \ldots, f_q)$. For every k–algebra A, giving a morphism $\operatorname{Spec} A[t]/(t^{m+1}) \to X$ is equivalent with giving a morphism $\phi \colon k[x_1, \ldots, x_n]/I \to A[t]/(t^{m+1})$. Such a morphism is determined by $u_i = \phi(x_i) = \sum_{j=0}^m a_{i,j} t^j$ such that $f_\ell(u_1, \ldots, u_n) = 0$ for every ℓ. We can write

$$f_\ell(u_1, \ldots, u_n) = \sum_{p=0}^m g_{\ell,p}((a_{i,j})_{i,j})t^p,$$

for suitable polynomials $g_{\ell,p}$ depending only on f_ℓ. It follows that $J_m(X)$ can be defined in $\mathbf{A}^{(m+1)n}$ by the polynomials $g_{\ell,p}$ for $1 \le \ell \le q$ and $0 \le p \le m$.

Suppose now that X is an arbitrary scheme of finite type over k. Consider an affine cover $X = U_1 \cup \ldots \cup U_r$. As we have seen, we have an mth jet scheme $\pi_m^i \colon J_m(U_i) \to U_i$ for every i. Moreover, by Lemma 2.3, for every i and j, the inverse images $(\pi_m^i)^{-1}(U_i \cap U_j)$ and $(\pi_m^j)^{-1}(U_i \cap U_j)$ give the mth jet scheme of $U_i \cap U_j$. Therefore they are canonically isomorphic. This shows that we may construct a scheme $J_m(X)$ by glueing the schemes $J_m(U_i)$ along the canonical isomorphisms of $(\pi_m^i)^{-1}(U_i \cap U_j)$ with $(\pi_m^j)^{-1}(U_i \cap U_j)$. Moreover, the projections π_m^i also glue to give a morphism $\pi_m \colon J_m(X) \to X$. It is now straightforward to check that $J_m(X)$ has the required property. □

REMARK 2.4. It follows from the description in the above proof that for every X, the projection $\pi_m \colon J_m(X) \to X$ is affine.

EXAMPLE 2.5. The first jet scheme $J_1(X)$ is isomorphic to the total tangent space $TX := \mathcal{S}pec(\operatorname{Sym}(\Omega_{X/k}))$. Indeed, arguing as in the proof of Proposition 2.2, we see that it is enough to show the assertion when $X = \operatorname{Spec}(R)$ is affine, in which case $TX = \operatorname{Spec}(\operatorname{Sym}(\Omega_{R/k}))$. In this case, if A is a k–algebra, then giving a morphism of schemes $f \colon \operatorname{Spec}(A) \to \operatorname{Spec}(\operatorname{Sym}(\Omega_{R/k}))$ is equivalent with giving a morphism of k–algebras $\phi \colon R \to A$ and a k-derivation $D \colon R \to A$ (where A becomes an R-module via ϕ). This is the same as giving a ring homomorphism $f \colon R \to A[t]/(t^2)$, where $f(u) = \phi(u) + tD(u)$.

If $f\colon X \to Y$ is a morphism of schemes, then we get a corresponding morphism $f_m\colon J_m(X) \to J_m(Y)$. At the level of A–valued points, this takes an $A[t]/(t^{m+1})$–valued point γ of X to $f \circ \gamma$. Taking X to $J_m(X)$ gives a functor from the category of schemes of finite type over k to itself. Note also that the morphisms f_m are compatible in the obvious sense with the projections $J_m(X) \to J_{m-1}(X)$ and $J_m(Y) \to J_{m-1}(Y)$.

REMARK 2.6. The jet schemes of the affine space are easy to describe: we have an isomorphism $J_m(\mathbf{A}^n) \simeq \mathbf{A}^{(m+1)n}$ such that the projection $J_m(\mathbf{A}^n) \to J_{m-1}(\mathbf{A}^n)$ corresponds to the projection onto the first mn coordinates. Indeed, an A–valued point of $J_m(\mathbf{A}^n)$ corresponds to a ring homomorphism $\phi\colon k[x_1, \ldots, x_n] \to A[t]/(t^{m+1})$, which is uniquely determined by giving each $\phi(X_i) \in A[t]/(t^{m+1}) \simeq A^{m+1}$.

REMARK 2.7. In light of the previous remark, we see that the proof of Proposition 2.2 showed that if $i\colon X \hookrightarrow \mathbf{A}^n$ is a closed immersion, then the induced morphism $i_m\colon J_m(X) \to J_m(\mathbf{A}^n)$ is also a closed immersion. Using the description of the equations of $J_m(X)$ in $J_m(\mathbf{A}^n)$ we see that more generally, if $f\colon X \hookrightarrow Y$ is a closed immersion, then f_m is a closed immersion, too.

REMARK 2.8. The following are some direct consequences of the definition.
  i) For every schemes X and Y and for every m, there is a canonical isomorphism $J_m(X \times Y) \simeq J_m(X) \times J_m(Y)$.
  ii) If G is a group scheme over k, then $J_m(G)$ is also a group scheme over k. Moreover, if G acts on X, then $J_m(G)$ acts on $J_m(X)$.
  iii) If $f\colon Y \to X$ is a morphism of schemes and $Z \hookrightarrow X$ is a closed subscheme, then we have a canonical isomorphism $J_m(f^{-1}(Z)) \simeq f_m^{-1}(J_m(Z))$.

The following lemma generalizes Lemma 2.3 to the case of an étale morphism.

LEMMA 2.9. *If $f\colon X \to Y$ is an étale morphism, then for every m the commutative diagram*

$$
\begin{array}{ccc}
J_m(X) & \xrightarrow{\ f_m\ } & J_m(Y) \\
\downarrow{\scriptstyle \pi_m^X} & & \downarrow{\scriptstyle \pi_m^Y} \\
X & \xrightarrow{\ f\ } & Y
\end{array}
$$

is Cartesian.

PROOF. From the description of the A–valued points of $J_m(X)$ and $J_m(Y)$ we see that it is enough to show that for every k–algebra A and every commutative diagram

$$
\begin{array}{ccc}
\mathrm{Spec}(A) & \longrightarrow & X \\
\downarrow & & \downarrow \\
\mathrm{Spec}\, A[t]/(t^{m+1}) & \longrightarrow & Y
\end{array}
$$

there is a unique morphism $\mathrm{Spec}\, A[t]/(t^{m+1}) \to X$ making the two triangles commutative. This is a consequence of the fact that f is formally étale. $\quad\square$

REMARK 2.10. A similar argument shows that if $f\colon Y \to X$ is a smooth surjective morphism, then f_m is surjective for every m. Moreover, f_m is again

smooth: this follows from Lemma 2.9 and the fact that f can be locally factored as $U \xrightarrow{g} V \times \mathbb{A}^n \xrightarrow{p} V$, where g is étale and p is the projection onto the first component.

We say that a morphism of schemes $g \colon V' \to V$ is *locally trivial* with fiber F if there is a cover by Zariski open subsets $V = U_1 \cup \ldots \cup U_r$ such that $g^{-1}(U_i) \simeq U_i \times F$, with the restriction of g corresponding to the projection onto the first component.

COROLLARY 2.11. If X is a nonsingular variety of dimension n, then all projections $\pi_{m,m-1} \colon J_m(X) \to J_{m-1}(X)$ are locally trivial with fiber $\mathbf{A}^n$. In particular, $J_m(X)$ is a nonsingular variety of dimension $(m+1)n$.

PROOF. Around every point in X we can find an open subset U and an étale morphism $U \to \mathbf{A}^n$. Using Lemma 2.9 we reduce our assertion to the case of the affine space, when it follows from Remark 2.6. $\qquad\square$

REMARK 2.12. If X and Y are schemes and $x \in X$ and $y \in Y$ are points such that the completions $\widehat{\mathcal{O}}_{X,x}$ and $\widehat{\mathcal{O}}_{Y,y}$ are isomorphic, then the fiber of $J_m(X)$ over x is isomorphic to the fiber of $J_m(Y)$ over y. Indeed, the A–valued points of the fiber of $J_m(X)$ over x are in natural bijection with

$$\{\phi \colon \mathcal{O}_{X,x} \to A[t]/(t^{m+1}) \mid \phi(\mathfrak{m}_x) \subseteq (t)\} = \{\hat{\phi} \colon \widehat{\mathcal{O}}_{X,x} \to A[t]/(t^{m+1}) \mid \hat{\phi}(\widehat{\mathfrak{m}}_x) \subseteq (t)\}$$

$$\simeq \{\hat{\psi} \colon \widehat{\mathcal{O}}_{Y,y} \to A[t]/(t^{m+1}) \mid \hat{\psi}(\widehat{\mathfrak{m}}_y) \subseteq (t)\} = \{\psi \colon \mathcal{O}_{Y,y} \to A[t]/(t^{m+1}) \mid \psi(\mathfrak{m}_y) \subseteq (t)\}.$$

EXAMPLE 2.13. Suppose that X is a reduced curve having a node at p, i.e. we have $\widehat{\mathcal{O}}_{X,p} \simeq k[\![x,y]\!]/(xy)$. By the previous remark, in order to compute the fiber of $J_m(X)$ over p we may assume that $X = \operatorname{Spec} k[x,y]/(xy)$ and that p is the origin. We see that this fiber consists of the union of m irreducible components, each of them (with the reduced structure) being isomorphic to $\mathbf{A}^{m+1}$. Indeed, the ith such component corresponds to morphisms $\phi \colon k[x,y] \to k[t]/(t^{m+1})$ such that $\operatorname{ord}(\phi(x)) \geq i$ and $\operatorname{ord}(\phi(y)) \geq m+1-i$.

If C is an irreducible component of X passing through p and C_{reg} is its nonsingular locus, then Corollary 2.11 implies that $\overline{J_m(C_{\text{reg}})}$ is an irreducible component of $J_m(X)$ of dimension $(m+1)$. Therefore all the above components of the fiber of $J_m(X)$ over p are irreducible components of $J_m(X)$. In particular, $J_m(X)$ is not irreducible for every $m \geq 1$.

EXAMPLE 2.14. Let X be an arbitrary scheme and p a point in X. If all projections $(\pi_m^X)^{-1}(p) \to (\pi_{m-1}^X)^{-1}(p)$ are surjective, then p is a nonsingular point. To see this, it is enough to show that if a tangent vector in $T_p X$ can be lifted to any $J_m(X)$, then it lies in the tangent cone of X at p. We may assume that X is a closed subscheme of $\mathbf{A}^n$ and that p is the origin. The tangent cone of X at p is the intersection of the tangent cone at p to each hypersurface H containing X. Since $J_m(X) \subseteq J_m(H)$ for every m and every such H, it is enough to prove our assertion when X is a hypersurface. Let f be an equation defining X, and write $f = f_r + f_{r+1} + \ldots$, where f_i has degree i and $f_r \neq 0$. By considering the equations defining $J_r(X)$ in $J_r(\mathbf{A}^n)$, we see that the commutative diagram

$$
\begin{array}{ccc}
(\pi_r^X)^{-1}(p) & \longrightarrow & (\pi_r^{\mathbf{A}^n})^{-1}(p) = \mathbf{A}^n \times \mathbf{A}^{(r-1)n} \\
\downarrow & & \downarrow {\scriptstyle \mathrm{pr}_1} \\
T_p X & \longrightarrow & T_p \mathbf{A}^n = \mathbf{A}^n
\end{array}
$$

identifies the fiber of $J_r(X)$ over p with $T \times \mathbf{A}^{(r-1)n} \hookrightarrow \mathbf{A}^n \times \mathbf{A}^{(r-1)n}$, where T is defined by f_r in $\mathbf{A}^n$. Since T is the tangent cone to X at p, this completes the proof of our assertion.

3. Spaces of arcs

We now consider the projective limit of the jet schemes. Suppose that X is a scheme of finite type over k. Since the projective system

$$\cdots \to J_m(X) \to J_{m-1}(X) \to \cdots \to J_0(X) = X$$

consists of affine morphisms, the projective limit exists in the category of schemes over k. It is denoted by $J_\infty(X)$ and it is called the space of arcs of X. In general, it is not of finite type over k.

The space of arcs comes equipped with projection morphisms $\psi_m \colon J_\infty(X) \to J_m(X)$ that are affine. In particular, we have $\psi_0 \colon J_\infty(X) \to X$. Over an affine open subset $U \subseteq X$, the space of arcs is described by

$$\mathcal{O}(\psi_0^{-1}(U)) = \varinjlim \mathcal{O}(\pi_m^{-1}(U)).$$

It follows from the projective limit definition and the functorial description of the jet schemes that if X is affine, then for every k–algebra A we have
(3.1)
$$\mathrm{Hom}(\mathrm{Spec}(A), J_\infty(X)) \simeq \underleftarrow{\mathrm{Hom}}(\mathrm{Spec}\, A[t]/(t^{m+1}), X) \simeq \mathrm{Hom}(\mathrm{Spec}\, A[\![t]\!], X).$$

If X is not necessarily affine, note that every morphism $\mathrm{Spec}\, k[t]/(t^{m+1}) \to X$ or $\mathrm{Spec}\, k[\![t]\!] \to X$ factors through any affine open neighborhood of the image of the closed point. It follows that for every X, the k–valued points of $J_\infty(X)$ correspond to *arcs in X*

$$\mathrm{Hom}(\mathrm{Spec}(k), J_\infty(X)) \simeq \mathrm{Hom}(\mathrm{Spec}\, k[\![t]\!], X).$$

If $f \colon X \to Y$ is a morphism of schemes, by taking the projective limit of the morphisms f_m we get a morphism $f_\infty \colon J_\infty(X) \to J_\infty(Y)$. We get in this way a functor from k-schemes of finite type over k to arbitrary k-schemes (in fact, to quasicompact and quasiseparated k-schemes).

The properties we have discussed in the previous section for jet schemes induce corresponding properties for the spaces of arcs. For example, if $f \colon X \to Y$ is an étale morphism, then we have a Cartesian diagram

$$
\begin{array}{ccc}
J_\infty(X) & \xrightarrow{\ f_\infty\ } & J_\infty(Y) \\
\downarrow{\scriptstyle \psi_0^X} & & \downarrow{\scriptstyle \psi_0^Y} \\
X & \xrightarrow{\ f\ } & Y.
\end{array}
$$

If $i \colon X \hookrightarrow Y$ is a closed immersion, then i_∞ is also a closed immersion. Moreover, if $Y = \mathbf{A}^n$, then $J_\infty(Y) \simeq \mathbf{A}^\mathbf{N} = \mathrm{Spec}\, k[x_1, x_2, \ldots]$, such that ψ_m corresponds to the projection onto the first $(m+1)n$ components. As in the proof of Proposition 2.2, starting with equations for a closed subscheme X of $\mathbf{A}^n$ we can write down equations for $J_\infty(X)$ in $J_\infty(\mathbf{A}^n)$.

Note that the one-dimensional torus k^* has a natural action on jet schemes induced by reparametrization the jets. In fact, for every scheme X we have a morphism

$$\Phi_m \colon \mathbf{A}^1 \times J_m(X) \to J_m(X)$$

described at the level of functors of points as follows. For every k–algebra A, an A–valued point of $\mathbf{A}^1 \times J_m(X)$ corresponds to a pair (a, ϕ), where $a \in A$ and $\phi \colon \operatorname{Spec} A[t]/(t^{m+1}) \to X$. This pair is mapped by Φ_m to the A–valued point of $J_m(X)$ given by the composition

$$\operatorname{Spec} A[t]/(t^{m+1}) \to \operatorname{Spec} A[t]/(t^{m+1}) \xrightarrow{\phi} X,$$

where the first arrow corresponds to the ring homomorphism induced by $t \to at$.

It is clear that Φ_m induces an action of k^* on $J_m(X)$. The fixed points of this action are given by $\Phi_m(\{0\} \times J_m(X))$. These are the *constant jets* over the points in X: over a point $x \in X$ the constant m–jet is the composition

$$\gamma_m^x \colon \operatorname{Spec} k[t]/(t^{m+1}) \to \operatorname{Spec} k \to X,$$

where the second arrow gives x. We have a *zero-section* $s_m \colon X \to J_m(X)$ of the projection π_m that takes x to γ_m^x. If A is a k–algebra, then s_m takes an A–valued point of X given by $u \colon \operatorname{Spec} A \to X$ to the composition

$$\operatorname{Spec} A[t]/(t^{m+1}) \to \operatorname{Spec} A \xrightarrow{u} X,$$

the first arrow being induced by the inclusion $A \hookrightarrow A[t]/(t^{m+1})$.

Note that if $\gamma \in J_m(X)$ is a jet lying over $x \in X$, then γ_m^x lies in the closure of $\Phi_m(k^* \times \{\gamma\})$. Since every irreducible component Z of $J_m(X)$ is preserved by the k^*–action, this implies that if γ is an m–jet in Z that lies over $x \in X$, then also γ_m^x is in Z. This will be very useful for the applications in §8.

Both the morphisms Φ_m and the zero-sections s_m are functorial. Moreover, they satisfy obvious compatibilities with the projections $J_m(X) \to J_{m-1}(X)$. Therefore we get a morphism

$$\Phi_\infty \colon \mathbf{A}^1 \times J_\infty(X) \to J_\infty(X)$$

inducing an action of k^* on $J_\infty(X)$, and a zero-section $s_\infty \colon X \to J_\infty(X)$.

If $\operatorname{char}(k) = 0$, then one can write explicit equations for $J_\infty(X)$ and $J_m(X)$ by "formally differentiating", as follows. If $S = k[x_1, \ldots, x_n]$, let us write $S_\infty = k[x_i^{(m)} \mid 1 \le i \le n, m \in \mathbf{N}]$, so that $\operatorname{Spec}(S_\infty) = J_\infty(\mathbf{A}^n)$ (in practice, we simply write $x_i = x_i^{(0)}$, $x_i' = x_i^{(1)}$, and so on). The identification is made as follows: for a k–algebra A, a morphism $\phi \colon k[x_1, \ldots, x_n] \to A[\![t]\!]$ determined by

$$(3.2) \qquad\qquad \phi(x_i) = \sum_{m \in \mathbf{N}} \frac{a_i^{(m)}}{m!} t^m$$

corresponds to the A–valued point $(a_i^{(m)})$ of $\operatorname{Spec}(S_\infty)$.

Note that on S_∞ we have a k–derivation D characterized by $D(x_i^{(m)}) = x_i^{(m+1)}$. If $f \in R$, then we put $f^{(0)} := f$, and we define recursively $f^{(m)} := D(f^{(m-1)})$ for $m \ge 1$. Suppose now that $R = S/I$, where I is generated by $f_1, \ldots, f_r$. We claim that if

$$(3.3) \qquad\qquad R_\infty := S_\infty/(f_i^{(m)} \mid 1 \le i \le r, m \in \mathbf{N}),$$
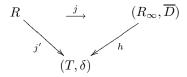
then $J_\infty(\operatorname{Spec} R) \simeq \operatorname{Spec}(R_\infty)$.

Indeed, given A and ϕ as above, for every $f \in k[x_1, \ldots, x_n]$ we have

$$\phi(f) = \sum_{m \in \mathbf{N}} \frac{f^{(m)}(a, a', \ldots, a^{(m)})}{m!} t^m$$

(note that both sides are additive and multiplicative in f, hence it is enough to check this for $f = x_i$, when it is trivial). It follows that ϕ induces a morphism $R \to A[\![t]\!]$ if and only if $f_i^{(m)}(a, a', \ldots, a^{(m)}) = 0$ for every m and every $i \le r$. This completes the proof of the above claim.

REMARK 3.1. Note that D induces a k–derivation $\overline{D}$ on R_∞. Moreover, $(R_\infty, \overline{D})$ is universal in the following sense: we have a k–algebra homomorphism $j \colon R \to R_\infty$ such that if (T, δ) is another k–algebra with a k–derivation δ, and if $j' \colon R \to T$ is a k–algebra homomorphism, then there is a unique k–algebra homomorphism $h \colon R_\infty \to T$ making the diagram

$$
\begin{array}{ccc}
R & \xrightarrow{\ j\ } & (R_\infty, \overline{D}) \\
& {\scriptstyle j'}\searrow \quad \swarrow {\scriptstyle h} & \\
& (T, \delta) &
\end{array}
$$

commutative, and such that h commutes with the derivations, i.e. $\delta(h(u)) = h(\overline{D}(u))$ for every $u \in R_\infty$. This is the starting point for the applications of spaces of arcs in differential algebra, see [**Bui**].

Of course, if we consider finite level truncations, then we obtain equations for the jet schemes. More precisely, if we put $S_m := k[x_i^{(j)} \mid i \le n, 0 \le j \le m]$ and

$$R_m := S_m/(f_i, f_i', \ldots, f_i^{(m)} \mid 1 \le i \le r),$$

then $\mathrm{Spec}(R_m) \simeq J_m(\mathrm{Spec}\, R)$. Moreover, the obvious morphisms $R_{m-1} \to R_m$ induce the projections $J_m(\mathrm{Spec}\, R) \to J_{m-1}(\mathrm{Spec}\, R)$.

From now on, whenever dealing with the schemes $J_m(X)$ and $J_\infty(X)$ we will restrict to their k–valued points. Of course, for $J_m(X)$ this causes no ambiguity since this is a scheme of finite type over k. Note that the Zariski topology on $J_\infty(X)$ is the projective limit topology of $J_\infty(X) \simeq \varprojlim J_m(X)$. Moreover, since we consider only k–valued points, we have $J_\infty(X) = J_\infty(X_{\mathrm{red}})$ (note that the analogous assertion is false for the spaces $J_m(X)$). Indeed, since $k[\![t]\!]$ is a domain, we have $\mathrm{Hom}(\mathrm{Spec}\, k[\![t]\!], X) = \mathrm{Hom}(\mathrm{Spec}\, k[\![t]\!], X_{\mathrm{red}})$. Similarly, if $X = X_1 \cup \ldots \cup X_r$, where all X_i are closed in X, then $J_\infty(X) = J_\infty(X_1) \cup \ldots \cup J_\infty(X_r)$.

To a closed subscheme Z of a scheme X we associate subsets of the spaces of arcs and jets of X by specifying the vanishing order along Z. If $\gamma \colon \mathrm{Spec}\, k[\![t]\!] \to X$ is an arc on X, then the inverse image of Z by γ is defined by an ideal in $k[\![t]\!]$. If this ideal is generated by t^r, then we put $\mathrm{ord}_\gamma(Z) = r$ (if the ideal is zero, then we put $\mathrm{ord}_\gamma(Z) = \infty$). The *contact locus* of order e with Z in $J_\infty(X)$ is the set

$$\mathrm{Cont}^e(Z) := \{\gamma \in J_\infty(X) \mid \mathrm{ord}_\gamma(Z) = e\}.$$

We similarly define

$$\mathrm{Cont}^{\ge e}(Z) := \{\gamma \in J_\infty(X) \mid \mathrm{ord}_\gamma(Z) \ge e\}.$$

We can define in the obvious way also subsets $\mathrm{Cont}^e(Z)_m$ (if $e \le m$) and $\mathrm{Cont}^{\ge e}(Z)_m$ (if $e \le m + 1$) of $J_m(X)$ and we have

$$\mathrm{Cont}^e(Z) = \psi_m^{-1}(\mathrm{Cont}^e(Z)_m), \ \ \mathrm{Cont}^{\ge e}(Z) = \psi_m^{-1}(\mathrm{Cont}^{\ge e}(Z)_m).$$

Note that $\mathrm{Cont}^{\geq(m+1)}(Z)_m = J_m(Z)$. If $\mathcal{I}$ is the ideal sheaf in $\mathcal{O}_X$ defining Z, then we sometimes write $\mathrm{ord}_\gamma(\mathcal{I})$, $\mathrm{Cont}^e(\mathcal{I})$ and $\mathrm{Cont}^{\geq e}(\mathcal{I})$.

The next proposition gives the first hint of the relevance of spaces of arcs to birational geometry. A key idea is that certain subsets in the space of arcs are "small" and they can be ignored. A subset of $J_\infty(X)$ is called *thin* if it is contained in $J_\infty(Y)$, where Y is a closed subset of X that does not contain an irreducible component of X. It is clear that a finite union of thin subsets is again thin. If $f\colon X' \to X$ is a dominant morphism with X and X' irreducible, and $A \subseteq J_\infty(X)$ is thin, then $f_\infty^{-1}(A)$ is thin. If f is in addition generically finite, and $B \subseteq J_\infty(X')$ is thin, then $f_\infty(B)$ is thin.

We show that a proper birational morphism induces a bijective map on the complements of suitable thin sets.

PROPOSITION 3.2. *Let $f\colon X' \to X$ be a proper morphism. If Z is a closed subset of X such that f is an isomorphism over $X \smallsetminus Z$, then the induced map*

$$J_\infty(X') \smallsetminus J_\infty(f^{-1}(Z)) \to J_\infty(X) \smallsetminus J_\infty(Z)$$

is bijective. In particular, if f is a proper birational morphism of reduced schemes, then f_∞ gives a bijection on the complements of suitable thin subsets.

PROOF. Let $U = X \smallsetminus Z$. Since f is proper, the Valuative Criterion for Properness implies that an arc $\gamma\colon \mathrm{Spec}\, k[\![t]\!] \to X$ lies in the image of f_∞ if and only if the induced morphism $\overline{\gamma}\colon \mathrm{Spec}\, k(\!(t)\!) \to X$ can be lifted to X' (moreover, if the lifting of $\overline{\gamma}$ is unique, then the lifting of γ is also unique). On the other hand, γ does not lie in $J_\infty(Z)$ if and only if $\overline{\gamma}$ factors through $U \hookrightarrow X$. In this case, the lifting of $\overline{\gamma}$ exists and is unique since f is an isomorphism over U. $\qquad\square$

We use the above proposition to prove the following result of Kolchin.

THEOREM 3.3. *If X is irreducible and $\mathrm{char}(k) = 0$, then $J_\infty(X)$ is irreducible.*

PROOF. Since $J_\infty(X) = J_\infty(X_{\mathrm{red}})$, we may assume that X is also reduced. If X is nonsingular, then the assertion in the theorem is easy: we have seen that every jet scheme $J_m(X)$ is a nonsingular variety. Since the projections $J_\infty(X) \to J_m(X)$ are surjective, and $J_\infty(X) = \varprojlim J_m(X)$ with the projective limit topology, it follows that $J_\infty(X)$, too, is irreducible.

In the general case we do induction on $n = \dim(X)$, the case $n = 0$ being trivial. By Hironaka's Theorem we have a resolution of singularities $f\colon X' \to X$, that is, a proper birational morphism, with X' nonsingular. Suppose that Z is a proper closed subset of X such that f is an isomorphism over $U = X \smallsetminus Z$. It follows from Proposition 3.2 that

$$J_\infty(X) = J_\infty(Z) \cup \mathrm{Im}(f_\infty).$$

Moreover, the nonsingular case implies that $J_\infty(X')$, hence also $\mathrm{Im}(f_\infty)$, is irreducible. Therefore, in order to complete the proof it is enough to show that $J_\infty(Z)$ is contained in the closure of $\mathrm{Im}(f_\infty)$.

Consider the irreducible decomposition $Z = Z_1 \cup \ldots \cup Z_r$, inducing $J_\infty(Z) = J_\infty(Z_1) \cup \ldots \cup J_\infty(Z_r)$. Since f is surjective, for every i there is an irreducible component Z_i' of $f^{-1}(Z_i)$ such that the induced map $Z_i' \to Z_i$ is surjective. We are in characteristic zero, hence by the Generic Smoothness Theorem we can find

open subsets U_i' and U_i in Z_i' and Z_i, respectively, such that the induced morphisms $g_i \colon U_i' \to U_i$ are smooth and surjective. In particular, we have

$$J_\infty(U_i) = \mathrm{Im}((g_i)_\infty) \subseteq \mathrm{Im}(f_\infty).$$

On the other hand, every $J_\infty(Z_i)$ is irreducible by induction. Since $J_\infty(U_i)$ is a nonempty open subset of $J_\infty(Z_i)$, it follows that

$$J_\infty(Z_i) \subseteq \overline{\mathrm{Im}(f_\infty)}$$

for every i. This completes the proof of the theorem. $\qquad\square$

REMARK 3.4. In fact, Kolchin's Theorem holds in a much more general setup, see [**Kln**] and also [**Gil**] for a scheme-theoretic aproach. In fact, we proved a slightly weaker statement even in our restricted setting. Kolchin's result says that the scheme $J_\infty(X)$ is irreducible, while we only proved that its k–valued points form an irreducible set. In fact, one can deduce the stronger statement from ours by showing that the k–valued points are dense in $J_\infty(X)$. In turn, this can be proved in a similar way with Theorem 3.3 above. For a different proof of (the stronger version of) Kolchin's Theorem, without using resolution of singularities, see [**IK**] and [**NS**]. Note also that Remark 1 in [**NS**] gives a counterexample in positive characteristic.

4. Truncation maps between spaces of jets

In what follows we will encounter morphisms that are not locally trivial, but that satisfy this property after passing to a stratification. Suppose that $g \colon V' \to V$ is a morphism of schemes, $W' \subseteq V'$ and $W \subseteq V$ are constructible subsets such that $g(W') \subseteq W$, and F is a reduced scheme. We will say that g gives a *piecewise trivial fibration* $W' \to W$ with fiber F if there is a decomposition $W = T_1 \sqcup \ldots \sqcup T_r$, with all T_i locally closed subsets of W (with the reduced scheme structure) such that each $W' \cap g^{-1}(T_i)$ is locally closed in V and, with the reduced scheme structure, it is isomorphic to $T_i \times F$ (with the restriction of g corresponding to the projection onto the first component). It is clear that if $g \colon V' \to V$ is locally trivial with fiber F, then it gives a piecewise trivial fibration with fiber F_{red} from $g^{-1}(W)$ to W for every constructible subset W of V.

If in the definition of piecewise trivial fibrations we assume only that $W' \cap g^{-1}(T_i) \to T_i$ factors as

$$W' \cap g^{-1}(T_i) \xrightarrow{u} T_i' \times F \xrightarrow{v} T_i' \xrightarrow{w} T_i,$$

where u is an isomorphism, v is the projection, and w is bijective, then we say that $W' \to W$ is a *weakly piecewise trivial* fibration with fiber F. If $\mathrm{char}(k) = 0$, then every bijective morphism is piecewise trivial with fiber $\mathrm{Spec}(k)$, and therefore the two notions coincide.

We have seen in Corollary 2.11 that if X is a nonsingular variety of dimension n, then the truncation maps $J_m(X) \to J_{m-1}(X)$ are locally trivial with fiber $\mathbf{A}^n$. In order to generalize this to more general schemes, we need to introduce the *Jacobian subscheme*. If X is a scheme of pure dimension n, then its Jacobian subscheme is defined by Jac_X, the Fitting ideal $\mathrm{Fitt}^n(\Omega_X)$. For the basics on Fitting ideals we refer to [**Eis**]. A basic property of Fitting ideals that we will keep using is that they commute with pull-back: if $f \colon X' \to X$ is a morphism and if $\mathcal{M}$ is a coherent sheaf on X, then $\mathrm{Fitt}^i(f^*\mathcal{M}) = (\mathrm{Fitt}^i(\mathcal{M})) \cdot \mathcal{O}_{X'}$ for every i.

The ideal Jac_X can be explicitly computed as follows. Suppose that U is an open subset of X that admits a closed immersion $U \hookrightarrow \mathbf{A}^N$. We have a surjection

$$\Omega_{\mathbf{A}^N}|_X = \oplus_{j=1}^N \mathcal{O}_X dx_j \to \Omega_X$$

with the kernel generated by the $df = \sum_{j=1}^N \frac{\partial f}{\partial x_j} dx_j$, where f varies over a system of generators $f_1, \ldots, f_d$ for the ideal of U in $\mathbf{A}^N$. If $r = N - n$, then Jac_X is generated over U by the image in $\mathcal{O}_U$ of the r–minors of the Jacobian matrix $(\partial f_i / \partial x_j)_{i,j}$.

It is well-known that the support of the Jacobian subscheme is the singular locus X_{sing} of X. Most of the time we will assume that X is reduced, hence its singular locus does not contain any irreducible component of X. Note also that $\mathrm{Fitt}^{n-1}(\Omega_X) = 0$ if either X is locally a complete intersection (when the above Jacobian matrix has r rows) or if X is reduced (when the $(r + 1)$–minors of the Jacobian matrix vanish at the generic points of the irreducible components of X, hence are zero in $\mathcal{O}_X$).

We start by describing the fibers of the truncation morphisms when we restrict to jets that can be lifted to the space of arcs.

PROPOSITION 4.1. ([**DL**]) Let X be a reduced scheme of pure dimension n and e a nonnegative integer. Fix $m \geq e$ and let $\pi_{m+e,m} \colon J_{m+e}(X) \to J_m(X)$ be the canonical projection.

    i) We have $\psi_m(\mathrm{Cont}^e(\mathrm{Jac}_X)) = \pi_{m+e,m}(\mathrm{Cont}^e(\mathrm{Jac}_X)_{m+e})$, i.e. an m–jet on $J_m(X)$ that vanishes with order e along Jac_X can be lifted to $J_\infty(X)$ if and only if it can be lifted to $J_{m+e}(X)$. In particular, $\psi_m(\mathrm{Cont}^e(\mathrm{Jac}_X))$ is a constructible set.

    ii) The projection $J_{m+1}(X) \to J_m(X)$ induces a piecewise trivial fibration

$$\alpha \colon \psi_{m+1}(\mathrm{Cont}^e(\mathrm{Jac}_X)) \to \psi_m(\mathrm{Cont}^e(\mathrm{Jac}_X))$$

    with fiber $\mathbf{A}^n$.

Before giving the proof of Proposition 4.1 we make some general considerations that will be used again later. A key point for the proof of Proposition 4.1 is the reduction to the complete intersection case. We present now the basic setup, leaving the proof of a technical result for the Appendix.

Let X be a reduced scheme of pure dimension n. All our statements are local over X, hence we may assume that X is affine. Fix a closed embedding $X \hookrightarrow \mathbf{A}^N$ and let $f_1, \ldots, f_d$ be generators of the ideal I_X of X. Consider $F_1, \ldots, F_d$ with $F_i = \sum_{j=1}^d a_{i,j} f_j$ for general $a_{i,j} \in k$. Note that we still have $I_X = (F_1, \ldots, F_d)$, but in addition we have the following properties. Let us denote by M the subscheme defined by the ideal $I_M = (F_1, \ldots, F_r)$, where $r = N - n$.

    1) All irreducible components of M have dimension n, hence M is a complete intersection.

    2) X is a closed subscheme of M and $X = M$ at the generic point of every irreducible component of X.

    3) There is an r–minor of the Jacobian matrix of $F_1, \ldots, F_r$ that does not vanish at the generic point of any irreducible component of X.

Of course, every r elements of $\{F_1, \ldots, F_d\}$ satisfy analogous properties.

Suppose now that e is a nonnegative integer, $m \geq e$ and we want to study $\mathrm{Cont}^e(\mathrm{Jac}_X)_m$. If M is as above, then we have an open subset U_M of $\mathrm{Cont}^e(\mathrm{Jac}_X)_m$ that is contained in $\mathrm{Cont}^e(\mathrm{Jac}_M)_m$ (the latter contact locus is a subset of $J_m(M)$).

Moreover, when varying the subsets of $\{1, \ldots, d\}$ with r elements, the corresponding open subsets cover $\mathrm{Cont}^e(\mathrm{Jac}_X)_m$.

LEMMA 4.2. *If $\gamma \in \mathrm{Cont}^e(\mathrm{Jac}_M) \subseteq J_\infty(M)$ is such that its projection to $J_m(M)$ lies in $J_m(X)$, then γ lies in $J_\infty(X)$.*

PROOF. Let $X' \subseteq \mathbf{A}^N$ be defined by $(I_M : I_X)$, hence set-theoretically X' is the union of the irreducible components in M that are not contained in X. We have $J_\infty(M) = J_\infty(X) \cup J_\infty(X')$, and therefore it is enough to show that γ does not lie in $J_\infty(X')$.

It follows from Corollary 9.2 in the Appendix that if we denote by J_F the ideal generated by the r–minors of the Jacobian matrix of $(F_1, \ldots, F_r)$ (hence $\mathrm{Jac}_M = (J_F + I_M)/I_M$), then
$$J_F \subseteq I_{X'} + I_X.$$
By assumption $\mathrm{ord}_\gamma(J_F) = e < m + 1 \leq \mathrm{ord}_\gamma(I_X)$, hence $\mathrm{ord}_\gamma(I_{X'}) \leq e$. In particular, γ is not in $J_\infty(X')$. $\qquad\square$

PROOF OF PROPOSITION 4.1. We may assume that X is affine, and let $X \hookrightarrow \mathbf{A}^N$ be a closed immersion of codimension r. Let $F_1, \ldots, F_d$ be general elements in the ideal of I_X as in the above discussion. Consider the subscheme M of $\mathbf{A}^N$ defined by $F_1, \ldots, F_r$ and let U_M be the open subset of $\mathrm{Cont}^e(\mathrm{Jac}_X)_m$ that is contained in $\mathrm{Cont}^e(\mathrm{Jac}_M)_m$. When we vary the subsets with r elements of $\{1, \ldots, d\}$, the corresponding open subsets cover $\mathrm{Cont}^e(\mathrm{Jac}_X)_m$. Therefore it is enough to prove the two assertions in the proposition over U_M.

We claim that it is enough to prove i) and ii) for M. Indeed, if $\gamma \in U_M$ can be lifted to $J_{m+e}(X)$, then in particular it can be lifted to $J_{m+e}(M)$. If we know i) for M, it follows that γ can be lifted to an arc $\delta \in J_\infty(M)$. Lemma 4.2 implies that δ lies in $J_\infty(X)$, hence we have i) for X. Moreover, suppose that ii) holds for M, hence the projection
$$\beta \colon \psi_{m+1}^M(\mathrm{Cont}^e(\mathrm{Jac}_M)) \to \psi_m^M(\mathrm{Cont}^e(\mathrm{Jac}_M))$$
is piecewise trivial with fiber $\mathbf{A}^n$. Again, Lemma 4.2 implies that the restriction of β over $U_M \cap \psi_m^M(\mathrm{Cont}^e(\mathrm{Jac}_M))$ coincides with the restriction of α over $U_M \cap \psi_m(\mathrm{Cont}^e(\mathrm{Jac}_X))$. Therefore X also satisfies ii).

We now prove the proposition for a subscheme M defined by a regular sequence $F_1, \ldots, F_r$ (M might not be reduced, but we do not need this assumption anymore). Consider an element $u = (u_1, \ldots, u_N) \in J_m(M)$, where all u_i lie in $k[t]/(t^{m+1})$ (for the matrix computations that will follow we consider u as a column vector). We denote by $\widetilde{u}_i \in k[\![t]\!]$ the lifting of u_i that has degree $\leq m$. Our assumption is that $\mathrm{ord}(F_i(\widetilde{u})) \geq m + 1$ for every i. An element in the fiber $(\psi_m^M)^{-1}(u)$ is an N–uple $w = \widetilde{u} + t^{m+1}v$ where $v = (v_1, \ldots, v_N) \in (k[\![t]\!])^N$, such that $F_i(w) = 0$ for every i. Using the Taylor expansion, we get

$$(4.1) \qquad F_i(w) = F_i(\widetilde{u}) + t^{m+1} \cdot \sum_{j=1}^N \frac{\partial F_i}{\partial x_j}(\widetilde{u})v_j + t^{2(m+1)} A_i(\widetilde{u}, v),$$

where each A_i has all terms of degree ≥ 2 in the v_j. We write F and A for the column vectors $(F_1, \ldots, F_r)$ and $(A_1, \ldots, A_r)$, respectively.

Let $J(\widetilde{u})$ denote the Jacobian matrix $(\partial F_i(\widetilde{u})/\partial x_j)_{i \leq r, j \leq N}$. Since u lies in $\mathrm{Cont}^e(\mathrm{Jac}_M)_m$, all r–minors of this matrix have order $\geq e$. Moreover, after taking

a suitable open cover of $\mathrm{Cont}^e(\mathrm{Jac}_M)_m$ and reordering the variables, we may assume that the determinant of the submatrix $R(\widetilde{u})$ on the first r columns of $J(\widetilde{u})$ has order precisely e. If $R^*(\widetilde{u})$ denotes the classical adjoint of the matrix $R(\widetilde{u})$, then

$$R^*(\widetilde{u}) \cdot J(\widetilde{u}) = (t^e \cdot I_r, t^e \cdot J'(\widetilde{u}))$$

for some $r \times (N-r)$ matrix $J'(\widetilde{u})$. Indeed, for every $i \le r$ and $r+1 \le j \le N$, the (i,j) entry of $R^*(\widetilde{u}) \cdot J(\widetilde{u})$ is equal, up to a sign, with the r–minor of $J(\widetilde{u})$ on the columns $1, \ldots, i-1, i+1, \ldots, r, j$. Therefore its order is $\ge e$.

Since the determinant of $R^*(\widetilde{u})$ is nonzero, it follows that $F(w) = 0$ if and only if $R^*(\widetilde{u}) \cdot F(w) = 0$. By equation (4.1) we have

(4.2) $R^*(\widetilde{u}) \cdot F(w) = R^*(\widetilde{u}) \cdot F(\widetilde{u}) + t^{m+e+1} \cdot (I_r, J'(\widetilde{u})) \cdot v + t^{2m+2} \cdot R^*(\widetilde{u}) \cdot A(\widetilde{u}, v).$

Note that since $m \ge e$ we have $2m + 2 > m + e + 1$.

We claim that there is v such that $F(\widetilde{u} + t^{m+1}v) = 0$ if and only if

(4.3) $\mathrm{ord}(R^*(\widetilde{u}) \cdot F(\widetilde{u})) \ge m + e + 1.$

Indeed, the fact that this condition is necessary follows immediately from (4.2). To see that it is also sufficient, suppose that (4.3) holds, and let us show that we can find v such that $F(\widetilde{u} + t^{m+1}v) = 0$. We write $v_i = \sum_j v_i^{(j)} t^j$ and determine inductively the $v_i^{(j)}$. If we consider the term of order $m + e + 1$ on the right-hand side of (4.2), then we see that we can choose $v_{r+1}^{(0)}, \ldots, v_N^{(0)}$ arbitrarily, and then the other $v_i^{(0)}$ are uniquely determined. In the term of order t^{m+e+2}, the contribution of the part coming from $R^*(\widetilde{u}) \cdot A(\widetilde{u}, v)$ involves only the $v_i^{(0)}$. It follows that again we may choose $v_{r+1}^{(1)}, \ldots, v_N^{(1)}$ arbitrarily, and then $v_1^{(1)}, \ldots, v_r^{(1)}$ are determined uniquely such that the coefficient of t^{m+e+2} in $R^*(\widetilde{u}) \cdot F(\widetilde{u} + t^{m+1}v)$ is zero. Continuing this way we see that we can find v such that $F(\widetilde{u} + t^{m+1}v) = 0$. This concludes the proof of our claim. Since the fiber over u in $\psi_{m+1}(J_\infty(M))$ corresponds to those $(v_1^{(0)}, \ldots, v_N^{(0)})$ such that there is v with $F(\widetilde{u} + t^{m+1}v) = 0$, it follows from our description that this is a linear subspace of codimension r of $\mathbf{A}^N$.

Note that if there is v such that $\mathrm{ord}\, F(\widetilde{u} + t^{m+1}v) \ge m + e + 1$, then as above we get that $\mathrm{ord}(R^*(\widetilde{u}) \cdot F(\widetilde{u})) \ge m + e + 1$. We deduce that if u can be lifted to $J_{m+e}(M)$, then u can be lifted to $J_\infty(M)$, which proves i). Moreover, the above computation shows that over the set W defined by (4.3) in our locally closed subset of $J_m(M)$, the inclusion

$$\psi_{m+1}(\mathrm{Cont}^e(\mathrm{Jac}_M)) \subseteq J_m(M) \times \mathbf{A}^N$$

is, at least set-theoretically, an affine bundle with fiber $\mathbf{A}^{N-r}$. This proves ii) and completes the proof of the proposition. $\square$

REMARK 4.3. It follows from the above proof that the assertions of the proposition hold also for a locally complete intersection scheme (the scheme does not have to be reduced).

We now discuss the fibers of the truncation maps between jet spaces without restricting to the jets that can be lifted to the space of arcs.

PROPOSITION 4.4. ([**Loj**]) Let X be a scheme of finite type over k. For every nonnegative integers m and p, with $p \le m \le 2p + 1$, consider the projection

$$\pi_{m,p} \colon J_m(X) \to J_p(X).$$

i) If $\gamma \in J_p(X)$ is such that $\pi_{m,p}^{-1}(\gamma)$ is non-empty, then scheme-theoretically we have

(4.4) $$\pi_{m,p}^{-1}(\gamma) \simeq \mathrm{Hom}_{k[t]/(t^{p+1})}(\gamma^*\Omega_X, (t^{p+1})/(t^{m+1})).$$

ii) Suppose that X has pure dimension n and that for $e = \mathrm{ord}_\gamma(\mathrm{Jac}_X)$ we have $2p \geq m \geq e + p$. If X is either locally complete intersection or reduced, and if $\pi_{m,p}^{-1}(\gamma) \neq \emptyset$, then

$$\pi_{m,p}^{-1}(\gamma) \simeq \mathbf{A}^{e+(m-p)n}.$$

PROOF. Note that γ corresponds to a ring homomorphism $\mathcal{O}_{X,x} \to k[t]/(t^{m+1})$, for some $x \in X$. Our assumption on m and p implies that $(t^{p+1})/(t^{m+1})$ is a $k[t]/(t^{p+1})$–module. Therefore the right-hand side of (4.4) is well-defined. It is a finite-dimensional k–vector space, hence it is an affine space.

In order to describe it, we use the structure of finitely generated modules over $k[\![t]\!]$ to write a free presentation

$$(k[t]/(t^{p+1})^{\oplus N} \xrightarrow{A} (k[t]/(t^{p+1}))^{\oplus N} \to \gamma^*\Omega_X \to 0,$$

where A is the diagonal matrix $\mathrm{diag}(t^{a_1}, \ldots, t^{a_N})$, with $0 \leq a_1 \leq \ldots \leq a_N \leq p + 1$. In this case the right-hand side of (4.4) is isomorphic to $\mathbf{A}^\ell$, where $\ell = \sum_i \min\{a_i, m - p\}$. Note also that its R–valued points are in natural bijection with

$$\mathrm{Der}_k(\mathcal{O}_{X,x}, t^{p+1}R[t]/t^{m+1}R[t]).$$

We first show that it is enough to prove i). Suppose that we are in the setting of ii). We use the above description of the right-hand side of (4.4). It follows from the definition of e that $\sum_{i=1}^{N-n} a_i = e$. In particular, $a_i \leq e \leq m - p$ for $i \leq N - n$. In order to deduce ii) from i) it is enough to show that $a_i = p + 1$ for $i > N - n$. If δ is an element in $\pi_{m,p}^{-1}(\gamma)$, then by taking a free presentation of $\delta^*\Omega_X$, we see that A is the reduction mod (t^{p+1}) of a matrix $\mathrm{diag}(t^{b_1}, \ldots, t^{b_N})$ with $0 \leq b_1 \leq \ldots \leq b_N \leq m + 1$. We have $a_i = b_i$ if $b_i \leq p$ and $a_i = p + 1$ otherwise. Either of our two conditions on X implies that $\mathrm{Fitt}^{n-1}(\Omega_X) = 0$, hence

$$\mathrm{ord}_\delta(\mathrm{Fitt}^{n-1}(\Omega_X)) \geq m + 1,$$

and therefore $b_1 + \ldots + b_{N-n+1} \geq m + 1$. We deduce that for every $i \geq N - n + 1$ we have $b_i \geq m + 1 - e \geq p + 1$, hence $a_i = p + 1$.

Therefore it is enough to prove i). We may clearly assume that $X = \mathrm{Spec}(S)$ is affine. We start with the following observation. If β is an R–valued point in $J_p(X)$, then either the fiber $\pi_{m,p}^{-1}(\beta)$ is empty, or it is a principal homogeneous space over $\mathrm{Der}_k(S, t^{p+1}R[t]/t^{m+1}R[t])$, where $t^{p+1}R[t]/t^{m+1}R[t]$ becomes an S–module via $\beta\colon S \to R[t]/(t^{p+1})$. Indeed, if $D \in \mathrm{Der}_k(S, t^{p+1}R[t]/t^{m+1}R[t])$ and if $\alpha\colon S \to R[t]/(t^{m+1})$ corresponds to an R–valued point in $J_m(X)$ lying over β, then $\alpha + D$ gives another R–valued point over β. Moreover, every other element in $\pi_{m,p}^{-1}(\beta)$ arises in this way for a unique derivation D.

We see that if δ is a fixed k–valued point in $\pi_{m,p}^{-1}(\gamma)$, then we get a morphism

$$\mathrm{Hom}_{k[t]/(t^{p+1})}(\Omega_S \otimes_S k[t]/(t^{p+1}), (t^{p+1})/(t^{m+1})) \to \pi_{m,p}^{-1}(\gamma).$$

This is an isomorphism since it induces a bijection at the level of R–valued points for every R. $\square$

REMARK 4.5. Let X be a reduced scheme of pure dimension n. Suppose that m, p and e are nonnegative integers such that $2p \geq m \geq e + p$ and $\gamma \in J_p(X)$ is such that $\mathrm{ord}_\gamma(\mathrm{Jac}_X) = e$. Assume also that X is a closed subscheme of a locally complete intersection scheme M of the same dimension such that $\mathrm{ord}_\gamma(\mathrm{Jac}_M) = e$ (if X is embedded in some $\mathbf{A}^N$, then one can take M to be generated by $(N - n)$ general elements in the ideal of X). Consider the commutative diagram

$$
\begin{array}{ccc}
J_m(X) & \longrightarrow & J_m(M) \\
\downarrow{\scriptstyle \pi_{m,p}^X} & & \downarrow{\scriptstyle \pi_{m,p}^M} \\
J_p(X) & \longrightarrow & J_p(M)
\end{array}
$$

where the horizontal maps are inclusions. It follows from Proposition 4.4 that the scheme-theoretic fibers of $\pi_{m,p}^X$ and $\pi_{m,p}^M$ over γ are equal.

Indeed, note first that if $(\pi_{m,p}^M)^{-1}(\gamma) \neq \emptyset$, then γ can be lifted to $J_\infty(M)$ by Proposition 4.1 (see also Remark 4.3). On the other hand such a lifting would lie in $J_\infty(X)$ by Lemma 4.2, hence $(\pi_{m,p}^X)^{-1}(\gamma) \neq \emptyset$. In this case, it follows from Proposition 4.4 that both fibers are affine spaces of the same dimension, one contained in the other, hence they are equal.

REMARK 4.6. Suppose that X is a nonsingular variety of dimension n, and suppose that $m \leq 2p + 1$. On $J_p(X)$ we have a geometric vector bundle E whose fiber over γ is $\mathrm{Hom}_{k[t]/(t^{p+1})}(\gamma^* \Omega_X, (t^{p+1})/(t^{m+1}))$. If we consider this as a group scheme over $J_p(X)$, then the argument in the proof of Proposition 4.4 shows that we have an action of E on $J_m(X)$ over $J_p(X)$. Moreover, whenever we have a section of the projection of $\pi_{m,p}$ we get an isomorphism of $J_m(X)$ with E. We can always find such a section if we restrict to an affine open subset of X on which Ω_X is trivial.

We will need later the following global version of the assertion in Proposition 4.4 ii).

PROPOSITION 4.7. Let X be a scheme of pure dimension n that is either reduced or a locally complete intersection. If m, p and e are nonnegative integers such that $2p \geq m \geq p + e$, then the canonical projection $\pi_{m,p} \colon J_m(X) \to J_p(X)$ induces a piecewise trivial fibration

$$
\mathrm{Cont}^e(\mathrm{Jac}_X)_m \to \mathrm{Cont}^e(\mathrm{Jac}_X)_p \cap \mathrm{Im}(\pi_{m,p})
$$

with fiber $\mathbf{A}^{(m-p)n+e}$.

PROOF. We need to "globalize" the argument in the proof of Proposition 4.4. Note that we may assume that X is locally a complete intersection. Indeed, we may assume first that X is affine. If X is reduced, arguing as in the proof of Proposition 4.1 we may cover $\mathrm{Cont}^e(\mathrm{Jac}_X)_p$ by open subsets U_i such that there are n–dimensional locally complete intersection schemes M_i containing X, with

$$
U_i \subseteq \mathrm{Cont}^e(\mathrm{Jac}_{M_i})_p \subseteq J_p(M_i).
$$

It follows from Remark 4.5 that knowing the assertion in the proposition for each M_i, we get it also for X.

Therefore we may assume that X is a closed subscheme of $\mathbf{A}^N$ of codimension r, defined by $f_1, \ldots, f_r$. Write $f = (f_1, \ldots, f_r)$, which we consider as a vertical

vector. Suppose that

$$u = (u_1, \ldots, u_N) \in \mathrm{Cont}^e(\mathrm{Jac}_X)_p,$$

where $u_i \in k[t]/(t^{p+1})$ for every i. We denote by $\widetilde{u} \in (k[t]/(t^{m+1}))^N$ the lifting of u having each entry of degree $\leq p$. The fiber of $\pi_{m,p}$ over u consists of those $\widetilde{u} + t^{p+1}v$ such that $f(\widetilde{u} + t^{p+1}v) = 0$ in $(k[t]/(t^{m+1}))^N$. Here $v = (v_1, \ldots, v_N)$ where $v_i = \sum_{j=0}^{m-p-1} v_i^{(j)} t^j$.

Denote by $J(\widetilde{u})$ the Jacobian matrix $(\partial f_i(\widetilde{u})/\partial x_j)_{i \leq r, j \leq N}$. Using the Taylor expansion we see that

(4.5) $$f(\widetilde{u} + t^{p+1}v) = f(\widetilde{u}) + t^{p+1} \cdot J(\widetilde{u})v$$

(there are no further terms since $2(p+1) \geq m+1$).

Note that by assumption we can write $f(\widetilde{u}) = t^{p+1}g(u)$ where $g(u) = (\sum_{j=0}^{m-p-1} g_{i,j}(u)t^j)_i$. If we denote by $\overline{J}(u)$ the reduction of $J(\widetilde{u})$ mod t^{m-p}, we see that the condition on v becomes

(4.6) $$-g(u) = \overline{J}(u) \cdot v,$$

where the equality is in $(k[t]/(t^{m-p}))^r$.

It follows from the structure theory of matrices over principal ideal domains, applied to a lifting of $\overline{J}(u)$ to a matrix over $k[\![t]\!]$, that we can find invertible matrices A and B over $k[t]/(t^{m-p})$ such that $A \cdot \overline{J}(u) \cdot B = (\mathrm{diag}(t^{a_1}, \ldots, t^{a_r}), \mathbf{0})$, with $0 \leq a_i \leq m-p$. Moreover, after partitioning $\mathrm{Cont}^e(\mathrm{Jac}_X)_p$ into suitable locally closed subsets, we may assume that the a_i are independent of u and that $A = A(u)$ and $B = B(u)$, where the entries of $A(u)$ and $B(u)$ are regular functions of u.

Since the ideal generated by the r–minors of $\overline{J}(u)$ is (t^e), we see that $a_1 + \ldots + a_r = e$. If we write $A(u) \cdot g(u) = (h_1(u), \ldots, h_r(u))$, we see that u lies in the image of $\pi_{m,p}$ if and only if $\mathrm{ord}(h_i(u)) \geq a_i$ for every $i \leq r$. Moreover, if we put $v' = B(u)^{-1}v$, then we see that our condition gives the values of $t^{a_i}v_i'$ for $i \leq r$. Therefore the set of possible v is isomorphic to an affine space of dimension $(N-r)(m-p) + \sum_{i=1}^{r} a_i = n(m-p) + e$. Since the equations defining the fiber over u depend algebraically on u, we get the assertion of the proposition. $\square$

5. Cylinders in spaces of arcs

We start by giving some applications of Proposition 4.1. For every scheme X, a *cylinder* in $J_\infty(X)$ is a subset of the form $C = \psi_m^{-1}(S)$, for some m and some constructible subset $S \subseteq J_m(X)$. From now on, unless explicitly mentioned otherwise, we assume that X is reduced and of pure dimension n.

LEMMA 5.1. *If $C \subseteq J_\infty(X)$ is a cylinder, then C is not thin if and only if it is not contained in $J_\infty(X_{\mathrm{sing}})$.*

PROOF. We need to show that for every closed subset Y of X with $\dim(Y) < \dim(X)$, and every cylinder $C \nsubseteq J_\infty(X_{\mathrm{sing}})$, we have $C \nsubseteq J_\infty(Y)$. If this is not the case, then arguing by Noetherian induction we may choose a minimal Y for which there is a cylinder $C \nsubseteq J_\infty(X_{\mathrm{sing}})$ with $C \subseteq J_\infty(Y)$. After replacing C by a suitable $C \cap \mathrm{Cont}^e(\mathrm{Jac}_X)$, we may assume that $C \subseteq \mathrm{Cont}^e(\mathrm{Jac}_X)$. It follows from Proposition 4.1 that if $m \gg 0$, then the maps $\psi_{m+1}(C) \to \psi_m(C)$ are piecewise trivial, with fiber $\mathbf{A}^n$.

Note that Y has to be irreducible. Indeed, if $Y = Y_1 \cup Y_2$, with Y_1 and Y_2 both closed and different from Y, then either $C \cap J_\infty(Y_1)$ or $C \cap J_\infty(Y_2)$ is not contained in $J_\infty(X_{\mathrm{sing}})$. This contradicts the minimality of Y.

Using again the fact that Y is minimal, we see that $C \not\subseteq J_\infty(Y_{\mathrm{sing}})$ (we consider Y with the reduced structure). After replacing C with some $C \cap \mathrm{Cont}^{e'}(\mathrm{Jac}_Y)$, we may assume that $C \subseteq \mathrm{Cont}^{e'}(\mathrm{Jac}_Y)$. Since C is a cylinder also in $J_\infty(Y)$, it follows from Proposition 4.1 that if $m \gg 0$, then the projection $\psi_{m+1}(C) \to \psi_m(C)$ is piecewise trivial with fiber $\mathbf{A}^{\dim(Y)}$. This is a contradiction, and completes the proof of the lemma. $\qquad\square$

COROLLARY 5.2. Let $f \colon X' \to X$ be a proper birational morphism of reduced, pure-dimensional schemes. If $\gamma \in \psi_m^X(J_\infty(X) \smallsetminus J_\infty(X_{\mathrm{sing}}))$, then γ lies in the image of f_m.

PROOF. If $C = (\psi_m^X)^{-1}(\gamma)$, then C is a cylinder that is not contained in $J_\infty(X_{\mathrm{sing}})$. Let $Z \subset X$ be a closed subset with $\dim(Z) < \dim(X)$ such that f is an isomorphism over $X \smallsetminus Z$. It follows from Proposition 3.2 that $J_\infty(X) \smallsetminus J_\infty(Z) \subseteq \mathrm{Im}(f_\infty)$. Since $C \not\subseteq J_\infty(Z)$ by the lemma, we deduce that there is $\delta \in J_\infty(X')$ such that $f_m(\psi_m^{X'}(\delta)) = \psi_m^X(f_\infty(\delta)) = \gamma$. $\qquad\square$

COROLLARY 5.3. Let f be as in the previous corollary, with X' nonsingular. If k is uncountable, then $J_\infty(X) \smallsetminus J_\infty(X_{\mathrm{sing}}) \subseteq \mathrm{Im}(f_\infty)$.

PROOF. Let $\gamma \in J_\infty(X) \smallsetminus J_\infty(X_{\mathrm{sing}})$. It follows from Corollary 5.2 that for every m we have $\gamma_m := \psi_m^X(\gamma) \in \mathrm{Im}(f_m)$. Therefore we get a decreasing sequence

$$\cdots \supseteq (\psi_m^{X'})^{-1}(f_m^{-1}(\gamma_m)) \supseteq (\psi_{m+1}^{X'})^{-1}(f_{m+1}^{-1}(\gamma_{m+1})) \supseteq \cdots$$

of nonempty cylinders. Lemma 5.4 below implies that there is δ in the intersection of all these cylinders. Therefore $\psi_m^X(f_\infty(\delta)) = \gamma_m$ for all m, hence $\gamma = f_\infty(\delta)$. $\qquad\square$

LEMMA 5.4. ([**Bat**]) *If X is nonsingular and k is uncountable, then every decreasing sequence of cylinders*

$$C_1 \supseteq \cdots \supseteq C_m \supseteq \cdots$$

has nonempty intersection.

PROOF. Since the projections ψ_m are surjective, it follows from Chevalley's Constructibility Theorem that the image of every cylinder in $J_m(X)$ is constructible. Consider the decreasing sequence

$$\psi_0(C_1) \supseteq \psi_0(C_2) \supseteq \cdots$$

of nonempty constructible subsets. Since k is uncountable, the intersection of this sequence is nonempty. Let γ_0 be an element in this intersection.

Since γ_0 lies in the image of every C_m, we see that all the constructible subsets in the decreasing sequence

$$\psi_1(C_1) \cap \pi_{1,0}^{-1}(\gamma_0) \supseteq \psi_1(C_2) \cap \pi_{1,0}^{-1}(\gamma_0) \supseteq \cdots$$

are nonempty. Therefore there is γ_1 contained in their intersection. Continuing in this way we get $\gamma_m \in J_m(X)$ for every m such that $\pi_{m,m-1}(\gamma_m) = \gamma_{m-1}$ for every m and $\gamma_m \in \psi_m(C_p)$ for every p. Therefore $(\gamma_m)_m$ determines an arc $\gamma \in J_\infty(X)$ whose image in $J_m(X)$ is equal to γ_m. Since each C_p is a cylinder and $\psi_m(\gamma) \in \psi_m(C_p)$ for every m, we see that $\gamma \in C_p$. Hence $\gamma \in \cap_{p \geq 1} C_p$. $\qquad\square$

REMARK 5.5. Note that in the above lemma, the hypothesis that X is nonsingular was used only to ensure that the image in $J_m(X)$ of a cylinder is a constructible set. We will prove this below for an arbitrary scheme X (see Corollary 5.8), and therefore the lemma will hold in this generality.

REMARK 5.6. If $\mathrm{char}(k) = 0$, then the assumption that X' is nonsingular is not necessary in Corollary 5.3. Indeed, we can take a resolution of singularities $g \colon X'' \to X'$ and we clearly have $\mathrm{Im}(f \circ g)_\infty \subseteq \mathrm{Im}(f_\infty)$.

We have seen in Proposition 4.1 that for a reduced pure-dimensional scheme X the set $\psi_m(\mathrm{Cont}^e(\mathrm{Jac}_X))$ is constructible. In fact, the image of every cylinder is constructible, as follows from the following result of Greenberg.

PROPOSITION 5.7. ([**Gre**]) For an arbitrary scheme X and every m, the image of $J_\infty(X) \to J_m(X)$ is constructible.

PROOF. We give the proof assuming that $\mathrm{char}(k) = 0$. For a proof in the general case, see [**Gre**]. We do induction on $\dim(X)$, the case $\dim(X) = 0$ being trivial. If $X_1, \ldots, X_r$ are the irreducible components of X, with the reduced structure, then $J_\infty(X) = J_\infty(X_1) \cup \ldots \cup J_\infty(X_r)$. Hence the image of $J_\infty(X)$ is equal to the union of the images of the $J_\infty(X_i)$ in $J_m(X_i) \subseteq J_m(X)$. Therefore we may assume that X is reduced and irreducible.

Let $f \colon X' \to X$ be a resolution of singularities. Since X' is nonsingular, the projection $J_\infty(X') \to J_m(X')$ is surjective, hence $\mathrm{Im}(f_m) \subseteq \mathrm{Im}(\psi_m^X)$. Moreover, Corollary 5.2 gives $\psi_m^X(J_\infty(X) \smallsetminus J_\infty(X_{\mathrm{sing}})) \subseteq \mathrm{Im}(f_m)$. Therefore

$$\psi_m^X(J_\infty(X)) = \mathrm{Im}(f_m) \cup \psi_m^X(J_\infty(X_{\mathrm{sing}})).$$

The first term on the right-hand side is constructible by Chevalley's Constructibility Theorem, while the second term is constructible by induction. This implies that $\psi_m^X(J_\infty(X))$ is constructible. □

COROLLARY 5.8. For an arbitrary scheme X, the image of a cylinder C by the projection $J_\infty(X) \to J_m(X)$ is constructible.

PROOF. Let $C = \psi_p^{-1}(A)$, where $A \subseteq J_p(X)$ is constructible. If $m \geq p$, then $\psi_m(C) = \psi_m(J_\infty(X)) \cap \pi_{m,p}^{-1}(A)$, hence it is constructible by the proposition. The constructibility for $m < p$ now follows from Chevalley's Theorem. □

Proposition 5.7 is deduced in [**Gre**] from the fact that for every m there is $p \geq m$ such that the image of the projection $\psi_m \colon J_\infty(X) \to J_m(X)$ is equal to the image of $\pi_{p,m} \colon J_p(X) \to J_m(X)$ (in fact, Greenberg also shows that one can take $p = L(m)$ for a suitable linear function L). We now show that if we assume k uncountable, then this follows from the above proposition.

COROLLARY 5.9. If k is uncountable, then for an arbitrary scheme X and every m there is $p \geq m$ such that the image of ψ_m is equal to the image of $\pi_{p,m}$.

PROOF. Since $\mathrm{Im}(\psi_m)$ is constructible by Proposition 5.7 and each $\mathrm{Im}(\pi_{p,m})$ is constructible by Chevalley's Theorem, the assertion follows if we show

$$(5.1) \qquad\qquad \mathrm{Im}(\psi_m) = \bigcap_{p \geq m} \mathrm{Im}(\pi_{p,m})$$

(we use the fact that k is uncountable). The inclusion "$\subseteq$" is obvious. For the reverse inclusion we argue as in the proof of Lemma 5.4 to show that if

$\gamma_m \in \cap_{p \geq m} \text{Im}(\pi_{p,m})$, then we can find $\gamma_q \in J_q(X)$ for every $q \geq m+1$ such that $\pi_{q,q-1}(\gamma_q) = \gamma_{q-1}$. The sequence $(\gamma_q)_q$ defines an element $\gamma \in J_\infty(X)$ lying over γ_m. □

We give one more result about the fibers of the truncation maps between the images of the spaces of arcs (one should compare this with Proposition 4.1).

PROPOSITION 5.10. ([**DL**]) If X is a scheme of dimension n, then for every $m \geq p$, all fibers of the truncation map

$$\phi_{m,p} \colon \psi_m(J_\infty(X)) \to \psi_p(J_\infty(X))$$

have dimension $\leq (m-p)n$.

PROOF. Note that the sets in the statement are constructible by Proposition 5.7. Clearly, it is enough to prove the proposition when $m = p+1$. We may assume that X is a closed subscheme of $\mathbf{A}^N$ defined by $F_1, \ldots, F_r$. Consider $\gamma_p \in J_p(X)$ given by $u = (u_1, \ldots, u_N)$ where $u_i \in k[t]$ with $\deg(u_i) \leq p$.

Let $T = \text{Spec}\, k[t]$. Consider the subscheme $\mathcal{Z}$ of $T \times \mathbf{A}^N$ defined by $I_\mathcal{Z} = (F_1(u + t^{p+1}x), \ldots, F_r(u + t^{p+1}x))$. We have a subscheme $\mathcal{Z}' \subseteq \mathcal{Z}$ defined by

$$I_{\mathcal{Z}'} = (f \mid hf \in I_\mathcal{Z} \text{ for some nonzero } h \in k[t]).$$

Note that by construction $\mathcal{Z}'$ is flat over T, and $\mathcal{Z} = \mathcal{Z}'$ over the generic point of T.

The generic fiber of $\mathcal{Z}$ over T is isomorphic to $X \times_k k(t)$. Since $\mathcal{Z}'$ is flat over T, it follows that the fiber of $\mathcal{Z}'$ over the origin is either empty or has dimension n.

On the other hand, an element in the fiber of $\phi_{p+1,p}$ over γ_p is the $(p+1)$–jet of an arc in X given by $u + t^{p+1}w$ for some $w \in (k[[t]])^N$. Since $F_i(u + t^{p+1}w) = 0$ for every i, it follows from the definition of $I_{\mathcal{Z}'}$ that if $f \in I_{\mathcal{Z}'}$, then $f(t, w) = 0$. Hence the fiber of $\phi_{p+1,p}$ over γ_p can be embedded in the fiber of $\mathcal{Z}'$ over the origin, and its dimension is $\leq n$. □

We now discuss the notion of codimension for cylinders in spaces of arcs. In the remaining part of this section we assume that k is uncountable, and also that $\text{char}(k) = 0$ (this last condition is due only to the fact that we use resolutions of singularities).

Let X be a scheme of pure dimension n that is either reduced or locally complete intersection, and let $C = \psi_p^{-1}(A)$ be a cylinder, where A is a constructible subset of $J_p(X)$. If $C \subseteq \text{Cont}^e(\text{Jac}_X)$ and $m \geq \max\{p, e\}$, then we put $\text{codim}(C) := (m+1)n - \dim(\psi_m(C))$. We refer to §9.1 for a quick review of some basic facts about the dimension of constructible subsets. Note that by Proposition 4.1 (see also Remark 4.3), this is well-defined. Moreover, it is a nonnegative integer: by Theorem 3.3, the closure of $\psi_m(J_\infty(X))$ is equal to the closure in $J_m(X)$ of the mth jet scheme of the nonsingular locus of X_{red}. Therefore it is a set of pure dimension $(m+1)n$ (the fact that $\dim \psi_m(J_\infty(X)) = (m+1)n$ follows also from Proposition 5.10).

For an arbitrary cylinder C we put $C^{(e)} := C \cap \text{Cont}^e(\text{Jac}_X)$ and

$$\text{codim}(C) := \min\{\text{codim}(C^{(e)}) \mid e \in \mathbf{N}\}$$

(by convention, if $C \subseteq J_\infty(X_{\text{sing}})$, we have $\text{codim}(C) = \infty$). It is clear that if C_1 and C_2 are cylinders, then $\text{codim}(C_1 \cup C_2) = \min\{\text{codim}(C_1), \text{codim}(C_2)\}$. In particular, if $C_1 \subseteq C_2$, then $\text{codim}(C_1) \geq \text{codim}(C_2)$.

PROPOSITION 5.11. Suppose that X is reduced and let C be a cylinder in $J_\infty(X)$. If we have disjoint cylinders $C_i \subseteq C$ for $i \in \mathbf{N}$ such that the complement $C \smallsetminus \bigsqcup_{i \in \mathbf{N}} C_i$ is thin, then $\lim_{i \to \infty} \operatorname{codim}(C_i) = \infty$ and $\operatorname{codim}(C) = \min_i \operatorname{codim}(C_i)$.

Note that the proposition implies that for every cylinder C we have

$$\lim_{e \to \infty} \operatorname{codim}(C^{(e)}) = \infty.$$

We will prove Proposition 5.11 at the same time with the following proposition.

PROPOSITION 5.12. If X is reduced and Y is a closed subscheme of X with $\dim(Y) < \dim(X)$, then

$$\lim_{m \to \infty} \operatorname{codim}(\operatorname{Cont}^{\geq m}(Y)) = \infty.$$

We first show that these results hold when X is nonsingular. Let us start by making some comments about this special case. Suppose for the moment that X is nonsingular of pure dimension n. Since the projections $J_{m+1}(X) \to J_m(X)$ are locally trivial with fiber $\mathbf{A}^n$, cylinders are much easier to understand in this case. We say that a cylinder $C = \psi_m^{-1}(S)$ is *closed*, *locally closed* or *irreducible* if S is (the definition does not depend on m by the local triviality of the projection). Moreover, if S is closed and $S = S_1 \cup \ldots \cup S_r$ is the irreducible decomposition of S, then we get a unique decomposition into maximal irreducible closed cylinders $C = \psi_m^{-1}(S_1) \cup \ldots \cup \psi_m^{-1}(S_r)$. The cylinders $\psi_m^{-1}(S_i)$ are the *irreducible components* of C.

Note that if $C = \psi_m^{-1}(S)$, then by definition $\operatorname{codim}(C) = \operatorname{codim}(S, J_m(X))$. If $C \subseteq C'$ are closed cylinders with $\operatorname{codim}(C) = \operatorname{codim}(C')$, then every irreducible component of C whose codimension is equal to $\operatorname{codim}(C)$ is also an irreducible component of C'.

PROOF OF PROPOSITIONS 5.11 AND 5.12. We start by noting that if Proposition 5.12 holds on X, then Proposition 5.11 holds on X, too. Indeed, suppose that $\bigsqcup_{i \in \mathbf{N}} C_i \subseteq C$, where all C_i and C are cylinders, and that $C \smallsetminus \bigsqcup_i C_i$ is contained in $J_\infty(Y)$, where $\dim(Y) < \dim(X)$. For every m we have

$$C \subseteq \operatorname{Cont}^{\geq m}(Y) \cup \bigcup_{i \in \mathbf{N}} C_i.$$

It follows from Lemma 5.4 that there is an integer $i(m)$ such that

$$(5.2) \qquad\qquad C \subseteq \operatorname{Cont}^{\geq m}(Y) \cup \bigcup_{i \leq i(m)} C_i.$$

In particular, for every $i > i(m)$ we have $C_i \subseteq \operatorname{Cont}^{\geq m}(Y)$, hence $\operatorname{codim}(C_i) \geq \operatorname{codim} \operatorname{Cont}^{\geq m}(Y)$. If Proposition 5.12 holds on X, it follows that

$$\lim_{i \to \infty} \operatorname{codim}(C_i) = \infty.$$

The second assertion in Proposition 5.11 follows, too. Indeed, note first that if all $C_i \subseteq J_\infty(X_{\operatorname{sing}})$, then $C \subseteq J_\infty(Y \cup X_{\operatorname{sing}})$. Therefore $C \subseteq J_\infty(X_{\operatorname{sing}})$ by Lemma 5.1, and the assertion is clear in this case. If $C_i \not\subseteq J_\infty(X_{\operatorname{sing}})$ for some i, then $\operatorname{codim}(C) < \infty$. The assertion in Proposition 5.12 implies that there is m such that $\operatorname{codim} \operatorname{Cont}^{\geq m}(Y) > \operatorname{codim}(C)$. We deduce from (5.2) that

$$\operatorname{codim}(C) \geq \min\{\operatorname{codim}(C_0), \ldots, \operatorname{codim}(C_{i(m)}), \operatorname{codim} \operatorname{Cont}^{\geq m}(Y)\}.$$

Therefore $\operatorname{codim}(C) \geq \min_i \operatorname{codim}(C_i)$ and the reverse inequality is trivial.

We now prove Proposition 5.12 when X is nonsingular. We have a decreasing sequence of closed cylinders $\{\operatorname{Cont}^{\geq m}(Y)\}_{m \in \mathbf{N}}$. Since

$$\operatorname{codim} \operatorname{Cont}^{\geq m}(Y) \leq \operatorname{codim} \operatorname{Cont}^{\geq m+1}(Y)$$

for every m, it follows that if the limit in the proposition is not infinity, then there is m_0 such that $\operatorname{codim} \operatorname{Cont}^{\geq m}(Y) = \operatorname{codim} \operatorname{Cont}^{\geq m_0}(Y)$ for every $m \geq m_0$. Hence for all such m, the irreducible components of $\operatorname{Cont}^{\geq m+1}(Y)$ of minimal codimension are also components of $\operatorname{Cont}^{\geq m}(Y)$. It is easy to see that this implies that there is an irreducible component C of all $\operatorname{Cont}^{\geq m}(Y)$ for $m \geq m_0$. Therefore $C \subseteq J_\infty(Y)$, which contradicts Lemma 5.1. By our discussion at the beginning of the proof we see that both propositions hold on nonsingular varieties.

In order to complete the proof it is enough to show that Proposition 5.12 holds for an arbitrary reduced pure-dimensional scheme X. Let $f \colon X' \to X$ be a resolution of singularities of X (in other words X' is the disjoint union of resolutions of the irreducible components of X). Since $f_\infty^{-1}(\operatorname{Cont}^{\geq m}(Y)) = \operatorname{Cont}^{\geq m}(f^{-1}(Y))$ and since we know that Proposition 5.12 holds on X', we see that it is enough to prove that for every cylinder $C \subseteq J_\infty(X)$, we have $\operatorname{codim}(f_\infty^{-1}(C)) \leq \operatorname{codim}(C)$.

We clearly have $\bigsqcup_{e \in \mathbf{N}} f_\infty^{-1}(C^{(e)}) \subseteq f_\infty^{-1}(C)$ and the complement of this union is contained in $J_\infty(f^{-1}(X_{\mathrm{sing}}))$. Since Proposition 5.11 holds on X', we see that $\operatorname{codim}(f_\infty^{-1}(C)) = \min_e \operatorname{codim} f_\infty^{-1}(C^{(e)})$. Therefore we may assume that $C = C^{(e)}$ for some e. In this case, if $m \gg 0$, then

$$\operatorname{codim}(C) = (m+1)\dim(X) - \dim \psi_m^X(C) \geq (m+1)\dim(X) - \dim f_m^{-1}(\psi_m^X(C))$$

$$= \operatorname{codim}(f_m \circ \psi_m^{X'})^{-1}(\psi_m^X(C)) = \operatorname{codim} f_\infty^{-1}(C).$$

We have used the fact that $\psi_m^X(C) \subseteq \operatorname{Im}(f_m)$ by Corollary 5.2. This completes the proof of the two propositions. $\qquad \square$

EXAMPLE 5.13. Let $Z \subseteq \mathbf{A}^2$ be the curve defined by $x^2 - y^3 = 0$. The Jacobian ideal of Z is $\operatorname{Jac}_Z = (x, y^2)$. Let $\pi \colon J_\infty(Z) \to Z$ be the projection map. If $z \in Z$ is different from the origin, then z is a smooth point of Z and $\operatorname{codim}(\pi^{-1}(z)) = 1$. On the other hand, if z is the origin, then we can decompose $C = \pi^{-1}(z)$ as

$$C = J_\infty(z) \bigsqcup \left(\bigsqcup_{e > 0} \{(u(t), v(t)) \mid u(t)^2 = v(t)^3, \operatorname{ord} u(t) = 3e, \operatorname{ord} v(t) = 2e\} \right).$$

Note that the set corresponding to e is precisely $C^{(3e)}$. If we take $m = 3e$, we see that $\psi_m(C^{(3e)})$ is equal to

$$\{(at^{3e}, b_0 t^{2e} + \ldots + b_e t^{3e} \mid a^2 = b_0^3, a \neq 0, b_0 \neq 0\}.$$

Therefore $\operatorname{codim}(C^{(3e)}) = (3e+1) - (e+1) = 2e$ for every $e \geq 1$, and $\operatorname{codim}(C) = 2$. Note that in this case the codimension of the special fiber of π is larger than that of the general fiber (compare with the behavior of dimensions of fibers of morphisms of algebraic varieties).

Proposition 5.11 is a key ingredient in setting up motivic integration (see [**Bat**] and [**DL**]). We describe one elementary application of this proposition to the definition of another invariant of a cylinder, the "number of components of minimal codimension".

Let X be a reduced pure-dimensional scheme and C a cylinder in $J_\infty(X)$. If $C \subseteq \mathrm{Cont}^e(\mathrm{Jac}_X)$, then we take $m \gg 0$ and define $|C|$ to be the number of irreducible components of $\psi_m(C)$ whose codimension is $\mathrm{codim}(C)$. Note that by Proposition 4.1 this number is independent of m. For an arbitrary C, we put $|C| := \sum_{e \in \mathbf{N}} |C^{(e)}|$, where the sum is over those e such that $\mathrm{codim}(C^{(e)}) = \mathrm{codim}(C)$ (Proposition 5.11 implies that this is a finite sum). With this definition, we see that under the hypothesis of Proposition 5.11 we have $|C| = \sum_i |C_i|$, the sum being over the finite set of those i with $\mathrm{codim}(C_i) = \mathrm{codim}(C)$.

If X is a nonsingular variety and C is a closed cylinder in $J_\infty(X)$, then $|C|$ is equal to the number of irreducible components of C of minimal codimension.

6. The Birational Transformation Theorem

We now present the fundamental result of the theory. Suppose that $f \colon X' \to X$ is a proper birational morphism, with X' nonsingular and X reduced and of pure dimension n. The Birational Transformation Theorem shows that in this case f_∞ induces at finite levels weakly piecewise trivial fibrations.

The dimension of the fibers of these fibrations depends on the order of vanishing along the *Jacobian ideal* Jac_f of f. Consider the morphism induced by pulling-back n–forms

$$f^* \Omega_X^n \to \Omega_{X'}^n.$$

Since X' is nonsingular, $\Omega_{X'}^n$ is locally free of rank one, hence the image of the above morphism can be written as $\mathrm{Jac}_f \otimes \Omega_{X'}^n$ for a unique ideal Jac_f of $\mathcal{O}_{X'}$. In other words, we have $\mathrm{Jac}_f = \mathrm{Fitt}^0(\Omega_{X'/X})$.

If X is nonsingular, too, then Jac_f is locally principal, and it defines a subscheme supported on the exceptional locus of f. In this case, Proposition 3.2 implies that f_∞ is injective on $J_\infty(X') \smallsetminus J_\infty(V(\mathrm{Jac}_f))$. In general, we have the following.

LEMMA 6.1. *If $f \colon X' \to X$ is a proper birational morphism, with X' nonsingular and X reduced and pure-dimensional, and if $\gamma, \gamma' \in J_\infty(X')$ are such that*

$$\gamma \notin J_\infty(V(\mathrm{Jac}_f)) \cup f_\infty^{-1}(J_\infty(X_{\mathrm{sing}}))$$

and $f_\infty(\gamma) = f_\infty(\gamma')$, then $\gamma = \gamma'$.

PROOF. We argue as in the proof of Proposition 3.2. Since f is separated, it is enough to show that if $j \colon \mathrm{Spec}\, k(\!(t)\!) \to \mathrm{Spec}\, k[\![t]\!]$ corresponds to $k[\![t]\!] \subset k(\!(t)\!)$, then $\gamma \circ j = \gamma' \circ j$.

Note that $U := f^{-1}(X_{\mathrm{reg}}) \smallsetminus V(\mathrm{Jac}_f)$ is an open subset of X' that is the inverse image of an open subset of X. Moreover, f is invertible on U. By assumption, $\gamma \circ j$ factors through U and $f \circ \gamma \circ j = f \circ \gamma' \circ j$. Therefore $\gamma' \circ j$ also factors through U and $\gamma \circ j = \gamma' \circ j$. $\square$

THEOREM 6.2. *Let $f \colon X' \to X$ be a proper birational morphism, with X' nonsingular and X reduced and of pure dimension n. For nonnegative integers e and e', we put*

$$C_{e,e'} := \mathrm{Cont}^e(\mathrm{Jac}_f) \cap f_\infty^{-1}(\mathrm{Cont}^{e'}(\mathrm{Jac}_X)).$$

Fix $m \geq \max\{2e, e + e'\}$.

    i) *$\psi_m^{X'}(C_{e,e'})$ is a union of fibers of f_m.*
    ii) *f_m induces a weakly piecewise trivial fibration with fiber $\mathbf{A}^e$*

$$\psi_m^{X'}(C_{e,e'}) \to f_m(\psi_m^{X'}(C_{e,e'})).$$

In the case when also X is nonsingular, this theorem is due to Kontsevich [**Kon**]. The case of singular X is due to Denef and Loeser [**DL**], while the proof we give below follows [**Loj**]. Note that in these references one makes the assumption that the base field has characteristic zero, and therefore one gets piecewise trivial fibrations in ii) above. For a version in the context of formal schemes, allowing also positive characteristic, but with additional assumptions on the morphism, see [**Seb**]. The above theorem is at the heart of the Change of Variable Formula in motivic integration (see [**Bat**], [**DL**], and also [**Los**]).

We start with some preliminary remarks. Let f be as in the theorem, and suppose that $\alpha \in J_\infty(X')$, with $\operatorname{ord}_\alpha(\operatorname{Jac}_f) = e$ and $\operatorname{ord}_{f_\infty(\alpha)}(\operatorname{Jac}_X) = e'$. Pulling-back via α the right exact sequence of sheaves of differentials associated to f, we get an exact sequence of $k[\![t]\!]$–modules

$$\alpha^*(f^*\Omega_X) \xrightarrow{h} \alpha^*\Omega_{X'} \to \alpha^*\Omega_{X'/X} \to 0.$$

By assumption $\operatorname{Fitt}^0(\alpha^*\Omega_{X'/X}) = (t^e)$, hence

$$\alpha^*(\Omega_{X'/X}) \simeq k[t]/(t^{a_1}) \oplus \ldots \oplus k[t]/(t^{a_n})$$

for some $0 \le a_1 \le \ldots \le a_n$ with $\sum_i a_i = e$.

It follows that if $T = \operatorname{Im}(h)$, then T is free of rank n, and in suitable bases of T and $\alpha^*\Omega_{X'}$, the induced map $g \colon T \to \alpha^*\Omega_{X'}$ is given by the diagonal matrix with entries $t^{a_1}, \ldots, t^{a_n}$. We get a decomposition $\alpha^*(f^*\Omega_X) \simeq T \oplus \operatorname{Ker}(h)$, and therefore

$$\operatorname{Fitt}^0(\operatorname{Ker}(h)) = \operatorname{Fitt}^n(\alpha^*(f^*\Omega_X)) = (t^{e'}).$$

Hence $\operatorname{Ker}(h) \simeq k[t]/(t^{b_1}) \oplus \ldots \oplus k[t]/(t^{b_r})$ for some $0 \le b_1 \le \ldots \le b_r$ with $\sum_i b_i = e'$.

Suppose now that $p \ge \max\{e, e'\}$ and that α_p is the image of α in $J_p(X)$. If we tensor everything with $k[t]/(t^{p+1})$, we get the following factorization of the pull-back map $h_p \colon \alpha_p^* f^*\Omega_X \to \alpha_p^*\Omega_{X'}$

$$(6.1) \qquad \alpha_p^* f^*\Omega_X \xrightarrow{g'_p} T_p = T \otimes_{k[\![t]\!]} k[t]/(t^{p+1}) \xrightarrow{g_p} \alpha_p^*\Omega_{X'},$$

with g'_p surjective and $\operatorname{Ker}(g'_p) = \operatorname{Ker}(h) \otimes_{k[\![t]\!]} k[t]/(t^{p+1}) \simeq \oplus_i k[t]/(t^{b_i})$.

The following lemma will be needed in the proof of Theorem 6.2.

LEMMA 6.3. *Let $f \colon X' \to X$ be as in the theorem. Suppose that γ_m, $\gamma'_m \in J_m(X')$ are such that $\operatorname{ord}_{\gamma_m}(\operatorname{Jac}_f) = e$ and $\operatorname{ord}_{f_m(\gamma_m)}(\operatorname{Jac}_X) = e'$, with $m \ge \max\{2e, e + e'\}$. If $f_m(\gamma_m) = f_m(\gamma'_m)$, then γ_m and γ'_m have the same image in $J_{m-e}(X')$.*

PROOF. For an arc δ we will denote by δ_m its image in the space of m–jets. It is enough to show the following claim: if $q \ge \max\{2e, e + e'\}$, and if we have

$$\alpha \in J_\infty(X'), \ \beta \in J_\infty(X),$$

with $\operatorname{ord}_\alpha(\operatorname{Jac}_f) = e$, $\operatorname{ord}_\beta(\operatorname{Jac}_X) = e'$ and $f_q(\alpha_q) = \beta_q$, then there is $\delta \in J_\infty(X')$ having the same image as α in $J_{q-e}(X')$ and such that $f_{q+1}(\delta_{q+1}) = \beta_{q+1}$.

Indeed, in the situation in the lemma, let us choose arbitrary liftings γ and γ' of γ_m and γ'_m, respectively, to $J_\infty(X')$. We use the above claim to construct recursively $\alpha^{(q)} \in J_\infty(X')$ for $q \ge m$ such that $\alpha^{(m)} = \gamma$ and $\alpha^{(q+1)}$, $\alpha^{(q)}$ have the same image in $J_{q-e}(X')$ and

$$f_q(\alpha_q^{(q)}) = \psi_q^X(f_\infty(\gamma'))$$

for every $q \geq m$ (note that since $m \geq \max\{2e, e + e'\}$ each $\alpha^{(q)}$ vanishes along Jac_f and $f^{-1}(\mathrm{Jac}_X)$ with the same order as γ). The sequence given by the image of each $\alpha^{(q)}$ in $J_q(X')$ defines a unique $\alpha \in J_\infty(X')$ such that α and $\alpha^{(q)}$ have the same image in $J_{q-e}(X')$ for every $q \geq m$. We deduce that $f_\infty(\alpha) = f_\infty(\gamma')$. Since α has the same image as γ in $J_{m-e}(X')$, and since $m - e \geq \max\{e, e'\}$, it follows that

$$\alpha \notin J_\infty(V(\mathrm{Jac}_f)) \cup f_\infty^{-1}(J_\infty(X_{\mathrm{sing}})),$$

hence $\alpha = \gamma'$ by Lemma 6.1. In particular, γ and γ' have the same image in $J_{m-e}(X')$.

We now prove the claim made at the beginning of the proof. It follows from Proposition 4.4 i) that using $\alpha_{q+1} \in (\pi_{q+1,q-e}^{X'})^{-1}(\alpha_{q-e})$ we get an isomorphism

$$(\pi_{q+1,q-e}^{X'})^{-1}(\alpha_{q-e}) \simeq \mathrm{Hom}_{k[t]/(t^{q-e+1})}(\alpha_{q-e}^* \Omega_{X'}, (t^{q-e+1})/(t^{q+2})).$$

Similarly, using $f_{q+1}(\alpha_{q+1})$ we see that

$$(\pi_{q+1,q-e}^X)^{-1}(\beta_{q-e})) \simeq \mathrm{Hom}_{k[t]/(t^{q-e+1})}(\beta_{q-e}^* \Omega_X, (t^{q-e+1})/(t^{q+2})).$$

Via this isomorphism β_{q+1} corresponds to $w\colon \beta_{q-e}^* \Omega_X \to (t^{q-e+1})/(t^{q+2})$. Note that since $\beta_q = f_q(\alpha_q)$, the image of w lies in $(t^{q+1})/(t^{q+2})$. We now use the factorization (6.1) with $p = q - e$. If we construct a morphism $u\colon \alpha_{q-e}^* \Omega_{X'} \to (t^{q-e+1})/(t^{q+2})$ such that $u \circ h_{q-e} = w$, then u corresponds to an element $\delta_{q+1} \in J_{q+1}(X')$ such that any lifting δ of δ_{q+1} to $J_\infty(X')$ satisfies our requirement.

We first show that w is zero on $\mathrm{Ker}(g'_{q-e})$. Note that by using $f_{2q+1}(\alpha) \in (\pi_{2q+1,q}^X)^{-1}(f_q(\alpha_q))$ we see that β_{2q+1} corresponds to a morphism

$$w'\colon \beta_q^* \Omega_X \to (t^{q+1})/(t^{2q+2}),$$

such that w is obtained by tensoring w' with $k[t]/(t^{q-e+1})$ and composing with the natural map $(t^{q+1})/(t^{2q-e+2}) \to (t^{q-e+1})/(t^{q+2})$. Therefore in order to show that w is zero on $\mathrm{Ker}(g'_{q-e})$ it is enough to show that w' maps $\mathrm{Ker}(g'_{q-e})$ to $(t^{q+2})/(t^{2q+2})$. Since $\mathrm{Ker}(g'_{q-e})$ is a direct sum of $k[t]/(t^{q+1})$–modules of the form $k[t]/(t^b)$ with $b \leq e'$, it follows that $w'(\mathrm{Ker}(h'))$ is contained in $(t^{2q+2-e'})/(t^{2q+2})$. We have $2q + 2 - e' \geq q + 2$, hence w is zero on $\mathrm{Ker}(g'_{q-e})$.

Therefore w induces a morphism $\overline{w}\colon T_{q-e} \to (t^{q+1})/(t^{q+2})$. We know that in suitable bases of T_{q-e} and $\beta_{q-e}^* \Omega_X$ the map g_{q-e} is given by the diagonal matrix with entries $t^{a_1}, \ldots, t^{a_n}$, with all $a_i \leq e$. It follows that we can find $u\colon \alpha_{q-e}^* \Omega_{X'} \to (t^{q+1-e})/(t^{q+2})$ such that $u \circ h_{q-e} = w$, which completes the proof of the lemma. $\square$

PROOF OF THEOREM 6.2. The assertion in i) follows from Lemma 6.3, and we now prove ii). We first show that every fiber of the restriction of f_m to $\psi_m^{X'}(C_{e,e'})$ is isomorphic to $\mathbf{A}^e$, and we then explain how to globalize the argument. Note first that since X' is nonsingular, every jet in $J_m(X')$ can be lifted to $J_\infty(X')$, hence an element in $J_m(X')$ lies in $\psi_m^{X'}(C_{e,e'})$ if and only if its projection to $J_{m-e}(X')$ lies in $\psi_{m-e}^{X'}(C_{e,e'})$.

Let $\gamma_m' \in \psi_m^{X'}(C_{e,e'})$ and γ_{m-e}' its image in $J_{m-e}(X')$. We denote by γ_m and γ_{m-e} the images of γ_m' and γ_{m-e}' by f_m and f_{m-e}, respectively. It follows from Lemma 6.3 that $f_m^{-1}(\gamma_m)$ is contained in the fiber of $\pi_{m,m-e}^{X'}$ over γ_{m-e}'. Using the identifications of the fibers of $\pi_{m,m-e}^{X'}$ and $\pi_{m,m-e}^X$ over γ_{m-e}' and, respectively,

γ_{m-e} given by Proposition 4.4, we get an isomorphism of $f_m^{-1}(\gamma_m)$ with the kernel of

$$(6.2) \quad \mathrm{Hom}((\gamma'_{m-e})^*\Omega_{X'}, (t^{m-e+1})/(t^{m+1})) \to \mathrm{Hom}(\gamma_{m-e}^*\Omega_X, (t^{m-e+1})/(t^{m+1})),$$

where the Hom groups are over $k[t]/(t^{m-e+1})$. This gives an isomorphism

$$f_m^{-1}(\gamma_m) \simeq \mathrm{Hom}((\gamma'_{m-e})^*\Omega_{X'/X}, (t^{m-e+1})/(t^{m+1})).$$

Since $(\gamma'_{m-e})^*\Omega_{X'/X} \simeq k[t]/(t^{a_1}) \oplus \ldots \oplus k[t]/(t^{a_n})$, with $0 \le a_i \le \ldots \le a_n \le e$, with $\sum_i a_i = e$, we deduce

$$f_m^{-1}(\gamma_m) \simeq \oplus_{i=1}^n (t^{m+1-a_i})/(t^{m+1}) \simeq \mathbf{A}^e.$$

We now show that the above argument globalizes to give the full assertion in ii). Note first that after restricting to an affine open subset of X', we may assume that we have a section of $\pi_{m,m-e}^{X'}$. By Remark 4.6, it follows that $J_m(X')$ becomes isomorphic to a geometric vector bundle E over $J_{m-e}(X')$ whose fiber over some γ'_{m-e} is isomorphic to $\mathrm{Hom}((\gamma'_{m-e})^*\Omega_{X'}, (t^{m-e+1})/(t^{m+1}))$. Moreover, after restricting to a suitable locally closed cover of $\psi_{m-e}^{X'}(C_{e,e'})$, we may assume that, in the above notation, the integers $a_1, \ldots, a_n$ do not depend on γ'_{m-e}. It follows that we get a geometric subbundle F of E over this subset of $J_{m-e}(X')$ whose fiber over γ'_{m-e} is $\mathrm{Hom}((\gamma'_{m-e})^*\Omega_{X'/X}, (t^{m-e+1})/(t^{m+1}))$. It follows from the above discussion that we get a one-to-one map from the quotient bundle E/F to $J_m(X)$. This completes the proof of the theorem. $\square$

COROLLARY 6.4. Suppose that k is uncountable. With the notation in Theorem 6.2, if $A \subseteq C_{e,e'}$ is a cylinder in $J_\infty(X')$, then $f_\infty(A)$ is a cylinder in $J_\infty(X)$.

PROOF. Suppose that $A = \psi_p^{-1}(S)$, and let $m \ge \max\{2e, e + e', e + p\}$. It is enough to show that

$$f_\infty(A) = (\psi_m^X)^{-1}(f_m(\psi_m^{X'}(A))).$$

The inclusion "$\subseteq$" is trivial, hence it is enough to show the reverse inclusion. Consider $\delta \in (\psi_m^X)^{-1}(f_m(\psi_m^{X'}(A)))$. In particular $\delta \notin J_\infty(X_{\mathrm{sing}})$, and by Corollary 5.3 there is $\gamma \in J_\infty(X')$ such that $\delta = f_\infty(\gamma)$. Since $f_m(\psi_m^{X'}(\gamma)) \in f_m(\psi_m^{X'}(A))$, it follows from Lemma 6.3 that the image of γ in $J_p(X')$ lies in S, hence $\gamma \in A$. $\square$

COROLLARY 6.5. Suppose that k is uncountable and of characteristic zero. With the notation in the theorem, if $B \subseteq J_\infty(X)$ is a cylinder, then

$$\mathrm{codim}(B) = \min\{\mathrm{codim}(f_\infty^{-1}(B) \cap C_{e,e'}) + e | e, e' \in \mathbf{N}\}.$$

Moreover, we have

$$|B| = \sum_{e,e'} |f_\infty^{-1}(B) \cap C_{e'e'}|,$$

where the sum is over those $e, e' \in \mathbf{N}$ such that $\mathrm{codim}(f_\infty^{-1}(B) \cap C_{e,e'}) + e = \mathrm{codim}(B)$.

PROOF. It follows from the previous corollary that each $B \cap f_\infty(C_{e,e'})$ is a cylinder and Lemma 6.1 implies that these cylinders are disjoint. Moreover, the complement in B of their union is thin, so $\mathrm{codim}(B) = \min_{e,e'} \mathrm{codim}(B \cap f_\infty(C_{e,e'}))$ by Proposition 5.11 and $|B| = \sum_{e,e'} |B \cap f_\infty(C_{e,e'})|$, the sum being over those e and e' such that $\mathrm{codim}(f_\infty(C_{e,e'}) \cap B) = \mathrm{codim}(B)$. The fact that

$$\mathrm{codim}(f_\infty(C_{e,e'}) \cap B) = \mathrm{codim}(C_{e,e'} \cap f_\infty^{-1}(B)) + e, \quad |f_\infty(C_{e,e'}) \cap B| = |C_{e,e'} \cap f_\infty^{-1}(B)|$$

is a direct consequence of Theorem 6.2.   □

REMARK 6.6. Note that we needed to assume $\text{char}(k) = 0$ simply because we used existence of resolution of singularities in proving the basic properties of codimension of cylinders.

7. Minimal log discrepancies via arcs

From now on we assume that the characteristic of the ground field is zero, as we will make systematic use of existence of resolution of singularities. We start by recalling some basic definitions in the theory of singularities of pairs.

We work with pairs (X, Y), where X is a normal $\mathbf{Q}$–Gorenstein n–dimensional variety and $Y = \sum_{i=1}^{s} q_i Y_i$ is a formal combination with real numbers q_i and proper closed subschemes Y_i of X. An important special case is when Y is an $\mathbf{R}$-*Cartier divisor*, i.e. when all Y_i are defined by locally principal ideals. We say that a pair (X, Y) is effective if all q_i are nonnegative. Since X is normal, we have a Weil divisor K_X on X, uniquely defined up to linear equivalence, such that $\mathcal{O}(K_X) \simeq i_* \Omega_{X_{\text{reg}}}^n$, where $i \colon X_{\text{reg}} \hookrightarrow X$ is the inclusion of the smooth locus. Moreover, since X is $\mathbf{Q}$–Gorenstein, we may and will fix a positive integer r such that rK_X is a Cartier divisor.

Invariants of the singularities of (X, Y) are defined using *divisors over X*: these are prime divisors $E \subset X'$, where $f \colon X' \to X$ is a birational morphism and X' is normal. Every such divisor E gives a discrete valuation ord_E of the function field $K(X') = K(X)$, corresponding to the DVR $\mathcal{O}_{X', E}$. We identify two divisors over X if they give the same valuation of $K(X)$. In particular, we may always assume that X' and E are both smooth. The *center* of E is the closure of $f(E)$ in X and it is denoted by $c_X(E)$.

Let E be a divisor over X. If Z is a closed subscheme of X, then we define $\text{ord}_E(Z)$ as follows: we may assume that E is a divisor on X' and that the scheme-theoretic inverse image $f^{-1}(Z)$ is a divisor. Then $\text{ord}_E(Z)$ is the coefficient of E in $f^{-1}(Z)$. If (X, Y) is a pair as above, then we put $\text{ord}_E(Y) := \sum_i q_i \text{ord}_E(Y_i)$. We also define $\text{ord}_E(K_{-/X})$ as the coefficient of E in $K_{X'/X}$. Recall that $K_{X'/X}$ is the unique $\mathbf{Q}$–divisor supported on the exceptional locus of f such that $rK_{X'/X}$ is linearly equivalent with $rK_{X'} - f^*(rK_X)$. Note that both $\text{ord}_E(Y)$ and $\text{ord}_E(K_{-/X})$ do not depend on the particular X' we have chosen.

Suppose now that (X, Y) is a pair and that E is a divisor over X. The *log discrepancy* of (X, Y) with respect to E is

$$a(E; X, Y) := \text{ord}_E(K_{-/X}) - \text{ord}_E(Y) + 1.$$

If W is a closed subset of X, and $\dim(X) \geq 2$, then the *minimal log discrepancy* of (X, Y) along W is defined by

$$\text{mld}(W; X, Y) := \inf\{a(E; X, Y) \mid E \text{ divisor over} X, \, c_X(E) \subseteq W\}.$$

When $\dim(X) = 1$ we use the same definition of minimal log discrepancy, unless the infimum is negative, in which case we make the convention that $\text{mld}(W; X, Y) = -\infty$ (see below for motivation). There are also other versions of minimal log discrepancies (see [**Amb**]), but the study of all these variants can be reduced to the study of the above one. In what follows we give a quick introduction to minimal log discrepancies, and refer for proofs and details to *loc. cit.*

REMARK 7.1. If $\widetilde{Y} := \sum_i q_i \widetilde{Y}_i$, where each $\widetilde{Y}_i$ is defined by the integral closure of the ideal defining Y_i, then $\mathrm{ord}_E(Y) = \mathrm{ord}_E(\widetilde{Y})$ for every divisor E over X. For basic facts about integral closure of ideals, see for example [**Laz**], §9.6.A. We deduce that we have $\mathrm{mld}(W; X, Y) = \mathrm{mld}(W; X, \widetilde{Y})$.

It is an easy computation to show that if E and F are divisors with simple normal crossings on X' above X, and if F_1 is the exceptional divisor of the blowing-up of X' along $E \cap F$ (we assume that this is nonempty and connected), then

$$a(F_1; X, Y) = a(E; X, Y) + a(F; X, Y).$$

We may repeat this procedure, blowing-up along the intersection of F_1 with the proper transform of E. In this way we get divisors F_m over X for every $m \geq 1$ with

$$a(F_m; X, Y) = m \cdot a(E; X, Y) + a(F; X, Y).$$

In particular, this computation shows that if $\dim(X) \geq 2$ and $\mathrm{mld}(W; X, Y) < 0$, then $\mathrm{mld}(W; X, Y) = -\infty$ (which explains our convention in the one-dimensional case).

A pair (X, Y) is *log canonical* (*Kawamata log terminal*, or klt for short) if and only if $\mathrm{mld}(X; X, Y) \geq 0$ (respectively, $\mathrm{mld}(X; X, Y) > 0$). Note that for a closed subset W, if $\mathrm{mld}(W; X, Y) \geq 0$ then for every divisor E over X such that $c_X(E) \cap W \neq \emptyset$ we have $a(E; X, Y) \geq 0$. Indeed, if this is not the case, then we can find a divisor F on some X' with $c_X(F) \subseteq W$ and such that E and F have simple normal crossings and nonempty intersection. As above, we produce a sequence of divisors F_m with $c_X(F_m) \subseteq W$ and $\lim_{m \to \infty} a(F_m; X, Y) = -\infty$.

This assertion can be used to show that $\mathrm{mld}(W; X, Y) \geq 0$ if and only if there is an open subset U of X containing W such that $(U, Y|_U)$ is log canonical. In fact, we have the following more precise proposition that allows computing minimal log discrepancies via log resolutions.

PROPOSITION 7.2. Let (X, Y) be a pair as above and $W \subseteq X$ a closed subset. Suppose that $f \colon X' \to X$ is a proper birational morphism with X' nonsingular, and such that the union of $\cup_i f^{-1}(Y_i)$, of the exceptional locus of f and of $f^{-1}(W)$ (in case $W \neq X$) is a divisor with simple normal crossings. Write

$$f^{-1}(Y) := \sum_i q_i f^{-1}(Y_i) = \sum_{j=1}^d \alpha_j E_j, \ \ K_{X'/X} = \sum_{j=1}^d \kappa_j E_j.$$

For a nonnegative real number τ, we have $\mathrm{mld}(W; X, Y) \geq \tau$ if and only if the following conditions hold:

(1) For every j such that $f(E_j) \cap W \neq \emptyset$ we have $\kappa_j + 1 - \alpha_j \geq 0$.
(2) For every j such that $f(E_j) \subseteq W$ we have $\kappa_j + 1 - \alpha_j \geq \tau$.

We now turn to the description of minimal log discrepancies in terms of codimensions of contact loci from [**EMY**]. We assume that k is uncountable. If (X, Y) is a pair with $Y = \sum_{i=1}^s q_i Y_i$ and if $w = (w_i) \in \mathbf{N}^s$, then we put $\mathrm{Cont}^{\geq w}(Y) := \cap_i \mathrm{Cont}^{\geq w_i}(Y_i)$, which is clearly a cylinder. We similarly define $\mathrm{Cont}^w(Y)$, $\mathrm{Cont}^w(Y)_m$ and $\mathrm{Cont}^{\geq w}(Y)_m$.

Recall that rK_X is a Cartier divisor. We have a canonical map

$$\eta_r \colon (\Omega_X^n)^{\otimes r} \to \mathcal{O}(rK_X) = i_*((\Omega_{X_{\mathrm{reg}}}^n)^{\otimes r}).$$

We can write $\mathrm{Im}(\eta_r) = I_{Z_r} \otimes \mathcal{O}(rK_X)$, and the subscheme Z_r defined by I_{Z_r} is the rth *Nash subscheme* of X. It is clear that $I_{Z_{rs}} = I_{Z_r}^s$ for every $s \geq 1$.

Suppose that W is a proper closed subset of X, and let $f\colon X' \to X$ be a resolution of singularities as in Proposition 7.2 such that, in addition, $f^{-1}(V(\mathrm{Jac}_X))$ and $f^{-1}(Z_r)$ are divisors, having simple normal crossings with the exceptional locus of f, with $f^{-1}(Y)$ and with $f^{-1}(W)$.

LEMMA 7.3. ([**EMY**]) *Let (X, Y) be a pair and $f\colon X' \to X$ a resolution as above. Write*

$$f^{-1}(Y_i) = \sum_{j=1}^{d} \alpha_{i,j} E_j, \ \ K_{X'/X} = \sum_{j=1}^{d} \kappa_j E_j, \ \ f^{-1}(Z_r) = \sum_{j=1}^{d} z_j E_j.$$

For every $w = (w_i) \in \mathbf{N}^s$ and $\ell \in \mathbf{N}$ we have

$$\mathrm{codim}(\mathrm{Cont}^w(Y) \cap \mathrm{Cont}^{\ell}(Z_r) \cap \mathrm{Cont}^{\geq 1}(W)) = \frac{\ell}{r} + \min_{\nu} \sum_j (\kappa_j + 1)\nu_j,$$

where the minimum is over those $\nu = (\nu_j) \in \mathbf{N}^d$ with $\sum_j \alpha_{i,j}\nu_j = w_i$ for all i and $\sum_j z_j \nu_j = \ell$, and such that $\cap_{\nu_j \geq 1} E_j \neq \emptyset$ and $\nu_j \geq 1$ for at least one j with $f(E_j) \subseteq W$.

PROOF. For every $\nu = (\nu_j) \in \mathbf{N}^d$ we put $\mathrm{Cont}^{\nu}(E) = \cap_j \mathrm{Cont}^{\nu_j}(E_j)$. Since $\sum_j E_j$ has simple normal crossings, we see that $\mathrm{Cont}^{\nu}(E)$ is nonempty if and only if $\cap_{\nu_j \geq 1} E_j \neq \emptyset$, and in this case all irreducible components of $\mathrm{Cont}^{\nu}(E)$ have codimension $\sum_j \nu_j$. Indeed, by Lemma 2.9 it is enough to check this when $X = \mathbf{A}^n$ and the E_j are coordinate hyperplanes, in which case the assertion is clear.

Suppose that $\gamma \in \mathrm{Cont}^{\nu}(E)$, hence $\mathrm{ord}_{f_\infty(\gamma)}(Y_i) = \sum_j \alpha_{i,j}\nu_j$ and $\mathrm{ord}_{f_\infty(\gamma)}(Z_r) = \sum_j z_j \nu_j$. It is clear that $f_\infty(\gamma) \in \mathrm{Cont}^{\geq 1}(W)$ if and only if there is j such that $\nu_j \geq 1$ and $E_j \subseteq f^{-1}(W)$.

By the definition of Z_r we have $\mathrm{Jac}_f^r = f^{-1}(I_{Z_r}) \cdot \mathcal{O}(-rK_{X'/X})$, hence

$$\mathrm{ord}_\gamma(\mathrm{Jac}_f) = \sum_j \left(\frac{z_j}{r} + \kappa_j \right) \nu_j.$$

Moreover, by our assumption, the order of vanishing of arcs in $\mathrm{Cont}^{\nu}(E)$ along $f^{-1}V(\mathrm{Jac}_X)$ is finite and constant. It follows from Corollary 6.4 and Theorem 6.2 that $f_\infty(\mathrm{Cont}^{\nu}(E))$ is a cylinder with

$$\mathrm{codim}\ f_\infty(\mathrm{Cont}^{\nu}(E)) = \sum_j \frac{z_j}{r}\nu_j + \sum_j (\kappa_j + 1)\nu_j.$$

By Lemma 6.1 the cylinders $f_\infty(\mathrm{Cont}^{\nu}(E))$ for various ν are mutually disjoint. If we take the union over those ν such that $\sum_j \alpha_{i,j}\nu_j = w_i$ for all i and $\sum_j z_j \nu_j = \ell$, with $\nu_j \geq 1$ for some $E_j \subseteq f^{-1}(W)$, this union is contained in $\mathrm{Cont}^w(Y) \cap \mathrm{Cont}^{\ell}(Z_r) \cap \mathrm{Cont}^{\geq 1}(W)$. Moreover, its complement is contained in $\cup_j J_\infty(f(E_j))$, hence it is thin. The formula in the lemma now follows from Proposition 5.11. $\square$

THEOREM 7.4. ([**EMY**]) *If (X, Y) is a pair as above, and $W \subset X$ is a proper closed subset, then*

$$\mathrm{mld}(W; X, Y) = \inf_{w, \ell} \left\{ \mathrm{codim}\left(\mathrm{Cont}^w(Y) \cap \mathrm{Cont}^{\ell}(Z_r) \cap \mathrm{Cont}^{\geq 1}(W) \right) - \frac{\ell}{r} - \sum_{i=1}^{s} q_i w_i \right\},$$

where the minimum is over the $w = (w_i) \in \mathbf{N}^s$ and $\ell \in \mathbf{N}$. Moreover, if this minimal log discrepancy is finite, then the infimum on the right-hand side is a minimum.

If X is nonsingular, then $Z_r = \emptyset$ and the description of minimal log discrepancies in the theorem takes a particularly simple form.

PROOF OF THEOREM 7.4. Let f be a resolution as in Lemma 7.3. We keep the notation in that lemma and its proof. We also put $f^{-1}(Y) = \sum_j \alpha_j E_j$. Note that $\alpha_j = \sum_i \alpha_{i,j} q_i$. After restricting to an open neighborhood of W we may assume that all $f(E_j)$ intersect W.

We first show that $\mathrm{mld}(W; X, Y)$ is bounded above by the infimum in the theorem. Of course, we may assume that $\mathrm{mld}(W; X, Y)$ is finite. Therefore $\kappa_j + 1 - \alpha_j \geq \mathrm{mld}(W; X, Y)$ if $f(E_j) \subseteq W$ and $\kappa_j + 1 - \alpha_j \geq 0$ for every j.

Let $\nu = (\nu_j) \in \mathbf{N}^s$ be such that $\cap_{\nu_j \geq 1} E_j \neq \emptyset$, and $\nu_j \geq 1$ for some j with $f(E_j) \subseteq W$. In this case we have

$$\sum_{j=1}^{s} (k_j + 1)\nu_j \geq \sum_j \alpha_j \nu_j + \mathrm{mld}(W; X, Y) \cdot \sum_{f(E_j) \subseteq W} \nu_j \geq \sum_j \alpha_j \nu_j + \mathrm{mld}(W; X, Y).$$

If $\sum_j \alpha_{i,j} \nu_j = w_i$ for every i, and $\sum_j z_j \nu_j = \ell$, then $\sum_j \alpha_j \nu_j = \sum_i q_i w_i$, and the formula in Lemma 7.3 gives

$$\mathrm{mld}(W; X, Y) \leq \mathrm{codim}\left(\mathrm{Cont}^w(Y) \cap \mathrm{Cont}^\ell(Z_r) \cap \mathrm{Cont}^{\geq 1}(W) \right) - \sum_i q_i w_i - \frac{\ell}{r}.$$

Suppose now that we fix E_j such that $f(E_j) \subseteq W$. If $w_i = \alpha_{i,j}$ for every i and if $\ell = z_j$, then it follows from Lemma 7.3 that

$$\mathrm{codim}(\mathrm{Cont}^w(Y) \cap \mathrm{Cont}^\ell(Z_r) \cap \mathrm{Cont}^{\geq 1}(W)) \leq k_j + 1 + \frac{\ell}{r}$$

$$= \sum_i q_i w_i + \frac{\ell}{r} + a(E_j; X, Y).$$

Such an inequality holds for every divisor over X whose center is contained in W, and we deduce that if $\dim(X) \geq 2$, then the infimum in the theorem is $\leq \mathrm{mld}(W; X, Y)$ (note that the infimum does not depend on the particular resolution, and every divisor with center in W appears on some resolution). Moreover, we see that if $a(E_j; X, Y) = \mathrm{mld}(W; X, Y)$, then the infimum is obtained for the above intersection of contact loci.

In order to complete the proof of the theorem, it is enough to show that if X is a curve, and if $a(W; X, Y) < 0$, then the infimum in the theorem is $-\infty$. Note that in this case W is a (smooth) point on X, and we may assume that $Y_i = n_i W$ for some $n_i \in \mathbf{Z}$. Therefore our condition says that $\sum_i q_i n_i > 1$. Since $\mathrm{codim}(\mathrm{Cont}^m(W)) = m$, we see by taking $w_i = m n_i$ for all i that

$$\mathrm{codim}\left(\mathrm{Cont}^w(Y) \right) - \sum_i q_i w_i = m \left(1 - \sum_i q_i n_i \right) \to -\infty,$$

when we let m go to infinity.   □

REMARK 7.5. If (X, Y) is an effective pair and $W \subset X$ is a proper closed subset, then $\mathrm{mld}(W; X, Y)$ is equal to

$$\inf_{w, \ell} \left\{ \mathrm{codim} \left(\mathrm{Cont}^{\geq w}(Y) \cap \mathrm{Cont}^{\ell}(Z_r) \cap \mathrm{Cont}^{\geq 1}(W) \right) - \frac{\ell}{r} - \sum_{i=1}^{s} q_i w_i \right\},$$

where the infimum is over all $w \in \mathbf{N}^s$ and $\ell \in \mathbf{N}$. Indeed, note that we have

$$\bigsqcup_{w'} \left(\mathrm{Cont}^{w'}(Y) \cap \mathrm{Cont}^{\ell}(Z_r) \cap \mathrm{Cont}^{\geq 1}(W) \right) \subseteq \mathrm{Cont}^{\geq w}(Y) \cap \mathrm{Cont}^{\ell}(Z_r) \cap \mathrm{Cont}^{\geq 1}(W),$$

where the disjoint union is over $w' \in \mathbf{N}^s$ such that $w'_i \geq w_i$ for every i. Since the complement of this union is contained in $\cup_i J_\infty(Y_i)$, hence it is thin, our assertion follows from Theorem 7.4 via Proposition 5.11 (we also use the fact that since (X, Y) is effective, if $w'_i \geq w_i$ for all i, then $\sum_i q_i w'_i \geq \sum_i q_i w_i$).

REMARK 7.6. We have assumed in Theorem 7.4 that W is a proper closed subset. In general, it is easy to reduce computing minimal log discrepancies to this case, using the fact that if X is nonsingular and if Y is empty, then $\mathrm{mld}(X; X, Y) = 1$. Indeed, this implies that if (X, Y) is an arbitrary pair and if we take $W = X_{\mathrm{sing}} \cup \bigcup_i Y_i$, then

$$\mathrm{mld}(X; X, Y) = \min\{\mathrm{mld}(W; X, Y), 1\}.$$

For example, one can use this (or alternatively, one could just follow the proof of Theorem 7.4) to show that the pair (X, Y) is log canonical if and only if for every $w \in \mathbf{N}^s$ and every $\ell \in \mathbf{N}$, we have

$$\mathrm{codim} \left(\mathrm{Cont}^{w}(Y) \cap \mathrm{Cont}^{\ell}(Z_r) \right) \geq \frac{\ell}{r} + \sum_i q_i w_i.$$

REMARK 7.7. ([**EMY**]) The usual set-up in Mori Theory is to work with a normal variety X and a $\mathbf{Q}$–divisor D such that $K_X + D$ is $\mathbf{Q}$–Cartier (see for example [**Kol**]). The results in this section have analogues in that context. Suppose for simplicity that D is effective, giving an embedding $\mathcal{O}_X \hookrightarrow \mathcal{O}_X(rD)$, and that $r(K_X + D)$ is Cartier. The image of the composition

$$(\Omega_X^n)^{\otimes r} \to (\Omega_X^n)^{\otimes r} \otimes \mathcal{O}_X(rD) \to \mathcal{O}_X(r(K_X + D))$$

can be written an $I_T \otimes \mathcal{O}_X(r(K_X + D))$, for a closed subscheme T of X. Arguing as above, one can then show that if W is a proper closed subset of X, then

$$\mathrm{mld}(W; X, D) = \inf_{e \in \mathbf{N}} \left\{ \mathrm{codim} \left(\mathrm{Cont}^{e}(T) \cap \mathrm{Cont}^{\geq 1}(W) \right) - \frac{e}{r} \right\}.$$

EXAMPLE 7.8. Suppose that X is nonsingular and Y, Y' are effective combinations of closed subschemes of X. If P is a point on X, then

$$(7.1) \qquad \mathrm{mld}(P; X, Y + Y') \leq \mathrm{mld}(P; X, Y) + \mathrm{mld}(P; X, Y') - \dim(X).$$

Indeed, let us write $Y = \sum_i q_i Y_i$ and $Y' = \sum_i q'_i Y_i$, where the q_i and the q'_i are nonnegative real numbers. If one of the minimal log discrepancies on the right-hand side of (7.1) is $-\infty$, then $\mathrm{mld}(P; X, Y + Y') = -\infty$, as well. Otherwise, we can find w and $w' \in \mathbf{N}^s$ and irreducible components C of $\mathrm{Cont}^{\geq w}(Y)$ and C' of $\mathrm{Cont}^{\geq w'}(Y')$ such that $\mathrm{codim}(C) = \sum_i q_i w_i + \mathrm{mld}(P; X, Y)$ and $\mathrm{codim}(C') = \sum_i q_i w'_i + \mathrm{mld}(P; X, Y')$. Note that $C \cap C'$ is nonempty, since it contains the

constant arc over P. If $m \gg 0$, then $\psi_m(C \cap C') = \psi_m(C) \cap \psi_m(C')$, and using the fact that the fiber $\pi_m^{-1}(P)$ of $J_m(X)$ over P is nonsingular, we deduce

$$\mathrm{codim}(\psi_m(C) \cap \psi_m(C'), \pi_m^{-1}(P)) \leq \mathrm{codim}(\psi_m(C), \pi_m^{-1}(P)) + \mathrm{codim}(\psi_m(C'), \pi_m^{-1}(P)).$$

Since $C \cap C' \subseteq \mathrm{Cont}^{\geq w+w'}(Y + Y')$, we deduce our assertion from Remark 7.5.

Our next goal is to give a different interpretation of minimal log discrepancies that is better suited for applications. The main difference is that we replace cylinders in the space of arcs by suitable locally closed subsets in the spaces of jets.

Recall that Z_r is the rth Nash subscheme of X. The *non-lci subscheme of X of level r* is defined by the ideal $J_r = (\overline{\mathrm{Jac}_X^r} : I_{Z_r})$, where we denote by $\overline{\mathfrak{a}}$ the integral closure of an ideal $\mathfrak{a}$. It is shown in Corollary 9.4 in the Appendix that $J_r \cdot I_{Z_r}$ and Jac_X^r have the same integral closure. Note also that by Remark 9.6, the subscheme defined by J_r is supported on the set of points $x \in X$ such that $\mathcal{O}_{X,x}$ is not locally complete intersection. It follows from the basic properties of integral closure that given any ideal $\mathfrak{a}$, we have $\mathrm{ord}_\gamma(\mathfrak{a}) = \mathrm{ord}_\gamma(\overline{\mathfrak{a}})$ for every arc $\gamma \in J_\infty(X)$. In particular, $\mathrm{ord}_\gamma(J_r) + \mathrm{ord}_\gamma(I_{Z_r}) = r \cdot \mathrm{ord}_\gamma(\mathrm{Jac}_X)$.

THEOREM 7.9. *Let (X, Y) be an effective pair and r and J_r as above. If W is a proper closed subset of X, then*

$$\mathrm{mld}(W; X, Y) = \inf\{(m+1)\dim(X) + \frac{e'}{r} - \sum_i q_i w_i$$

$$- \dim(\mathrm{Cont}^{\geq w}(Y)_m \cap \mathrm{Cont}^e(\mathrm{Jac}_X)_m \cap \mathrm{Cont}^{e'}(J_r)_m \cap \mathrm{Cont}^{\geq 1}(W))_m\},$$

where the infimum is over those $w \in \mathbf{N}^s$, and $e, e', m \in \mathbf{N}$ such that $m \geq \max\{2e, e + e', e + w_i\}$. Moreover, if this minimal log discrepancy is finite, then the above infimum is a minimum.

It will follow from the proof that the expression in the above infimum does not depend on m, as long as $m \geq \max\{2e, e + e', e + w_i\}$. Note also that e comes up only in the condition on m. The condition in the theorem simplifies when X is locally complete intersection, since $J_r = \mathcal{O}_X$ by Remark 9.6 in the Appendix.

PROOF OF THEOREM 7.9. It follows from Theorem 7.4 (see also Remark 7.5) that $\mathrm{mld}(W; X, Y) = \inf_{w,\ell}\{\mathrm{codim}(C_{w,\ell}) - \frac{\ell}{r} - \sum_i q_i w_i\}$, where $w \in \mathbf{N}^s$, $\ell \in \mathbf{N}$, and

$$C_{w,\ell} = \mathrm{Cont}^{\geq w}(Y) \cap \mathrm{Cont}^\ell(Z_r) \cap \mathrm{Cont}^{\geq 1}(W).$$

On the other hand, Proposition 5.11 gives

$$\mathrm{codim}(C_{w,\ell}) = \min_{e \in \mathbf{N}} \mathrm{codim}(C_{w,\ell} \cap \mathrm{Cont}^e(\mathrm{Jac}_X)),$$

and for every e we can write

$$C_{w,\ell} \cap \mathrm{Cont}^e(\mathrm{Jac}_X) = \mathrm{Cont}^{\geq w}(Y) \cap \mathrm{Cont}^e(\mathrm{Jac}_X) \cap \mathrm{Cont}^{e'}(J_r) \cap \mathrm{Cont}^{\geq 1}(W),$$

where $e' = re - \ell$.

Suppose now that w, e and ℓ are fixed, $e' = re - \ell$, and let $m \geq \max\{2e, e + e', e + w_i\}$. Consider

$$S := \mathrm{Cont}^{\geq w}(Y)_m \cap \mathrm{Cont}^e(\mathrm{Jac}_X)_m \cap \mathrm{Cont}^{e'}(J_r)_m \cap \mathrm{Cont}^{\geq 1}(W)_m.$$

If we apply Proposition 4.7 for the morphism $\pi_{m,m-e}\colon J_m(X) \to J_{m-e}(X)$, we see that $\dim(S) = \dim(\pi_{m,m-e}(S)) + e(\dim(X) + 1)$. Moreover, $\pi_{m,m-e}(S) \subseteq \mathrm{Im}(\psi_{m-e}^X)$ by Proposition 4.1. It follows that

$$\mathrm{codim}(C_{w,\ell} \cap \mathrm{Cont}^e(\mathrm{Jac}_X)) = (m - e + 1)\dim(X) - \dim(\pi_{m,m-e}(S))$$

$$= (m + 1)\dim(X) + e - \dim(S).$$

This gives the formula in the theorem. $\square$

REMARK 7.10. If the pair (X, Y) is not necessarily effective, then we can get an analogue of Theorem 7.9, but involving contact loci of specified order along each Y_i, as in Theorem 7.4.

In this section we have related the codimensions of various contact loci with the numerical data of a log resolution. One can use, in fact, Theorem 6.2 to interpret also the "number of irreducible components of minimal dimension" in the corresponding contact loci. We illustrate this in the following examples. The proofs are close in spirit to the proof of the other results in this section, so we leave them for the reader.

EXAMPLE 7.11. Consider an effective pair (X, Y) as above and $W \subset X$ a proper closed subset. Suppose that $\tau := \mathrm{mld}(W; X, Y) \geq 0$. We say that a divisor E over X computes $\mathrm{mld}(W; X, Y)$ if $c_X(E) \subseteq W$ and $a(E; X, Y) = \tau$. There is only one divisor over X computing $\mathrm{mld}(W; X, Y)$ if and only if for every $w \in \mathbf{N}^s$ and $m, e, e' \in \mathbf{N}$ with $m \geq \max\{2e, e + e', e + w_i\}$, there is at most one irreducible component of

$$\mathrm{Cont}^{\geq w}(Y)_m \cap \mathrm{Cont}^e(\mathrm{Jac}_X)_m \cap \mathrm{Cont}^{e'}(J_r)_m \cap \mathrm{Cont}^{\geq 1}(W)_m$$

of dimension $(m + 1)\dim(X) + \frac{e'}{r} - \tau - \sum_i q_i w_i$. A similar equivalence holds when $W = X$ and $\mathrm{mld}(X; X, Y) = 0$.

EXAMPLE 7.12. ([**Mus**]) Let X be a nonsingular variety, and $Y \subset X$ a closed subvariety of codimension c, which is reduced and irreducible. Since

$$\dim J_m(Y) \geq \dim J_m(Y_{\mathrm{reg}}) = (m + 1)\dim(Y)$$

for every m, it follows from Theorem 7.4 that $\mathrm{mld}(X; X, cY) \leq 0$, with equality if and only if $\dim J_m(Y) = (m + 1)\dim(Y)$ for every m. In fact, note that if X' is the blowing-up of X along Y, and if E is the component of the exceptional divisor that dominates Y, then $a(E; X, cY) = 0$.

Suppose now that (X, cY) is log canonical. The assertion in the previous example implies that E is the unique divisor over X with $a(E; X, cY) = 0$ if and only if for every m, the unique irreducible component of $J_m(Y)$ of dimension $(m + 1)\dim(Y)$ is $\overline{J_m(Y_{\mathrm{reg}})}$.

Assume now that Y is locally complete intersection. Since $J_m(Y)$ can be locally defined in $J_m(X)$ by $c(m+1)$ equations, it follows that every irreducible component of $J_m(Y)$ has dimension at least $(m + 1)\dim(Y)$. Hence (X, cY) is log canonical if and only if $J_m(Y)$ has pure dimension for every m. In addition, we deduce from the above discussion that $J_m(Y)$ is irreducible for every m if and only if (X, cY) is log canonical and E is the only divisor over X such that $a(E; X, cY) = 0$. It is shown in [**Mus**] that this is equivalent with Y having rational singularities.

EXAMPLE 7.13. Let (X, Y) be an effective log canonical pair that is *strictly log canonical*, that is $\mathrm{mld}(X; X, Y) = 0$. A *center of non-klt singularities* is a closed subset of X of the form $c_X(F)$, where F is a divisor over X such that $a(F; X, Y) = 0$. One can show that an irreducible closed subset $T \subset X$ is such a center if and only if there are $w \in \mathbf{N}^s$, and $e, e' \in \mathbf{N}$ not all zero, such that for $m \geq \max\{2e, e + e', e + w_i\}$, some irreducible component of

$$\mathrm{Cont}^{\geq w}(Y)_m \cap \mathrm{Cont}^e(\mathrm{Jac}_X)_m \cap \mathrm{Cont}^{e'}(J_r)_m$$

has dimension $(m + 1)\dim(X) + \frac{e'}{r} - \sum_i q_i w_i$ and dominates T.

8. Inversion of Adjunction

We apply the description of minimal log discrepancies from the previous section to prove the following version of Inversion of Adjunction. This result has been proved also by Kawakita in [**Kaw1**].

THEOREM 8.1. *Let A be a nonsingular variety and $X \subset A$ a closed normal subvariety of codimension c. Suppose that $W \subset X$ is a proper closed subset and $Y = \sum_{i=1}^s q_i Y_i$ where all $q_i \in \mathbf{R}_+$ and the $Y_i \subset A$ are closed subschemes not containing X in their support. If r is a positive integer such that rK_X is Cartier and if J_r is the ideal defining the non-lci subscheme of level r of X, then*

$$\mathrm{mld}\left(W; X, \frac{1}{r}V(J_r) + Y|_X\right) = \mathrm{mld}(W; A, cX + Y),$$

where $Y|_X := \sum_i q_i(Y_i \cap X)$.

When X is locally complete intersection, then $J_r = \mathcal{O}_X$, and we recover the result from [**EM**] saying that $\mathrm{mld}(W; X, Y|_X) = \mathrm{mld}(W; A, cX + Y)$. It is shown in *loc. cit.* that this is equivalent with the following version of Inversion of Adjunction for locally complete intersection varieties.

COROLLARY 8.2. Let X be a normal locally complete intersection variety and $H \subset X$ a normal Cartier divisor. If $W \subset H$ is a proper closed subset, and if $Y = \sum_{i=1}^s q_i Y_i$, where all $q_i \in \mathbf{R}_+$ and Y_i are closed subsets of X not containing H in their support, then

$$\mathrm{mld}(W; H, Y|_H) = \mathrm{mld}(W; X, Y + H).$$

For motivation and applications of the general case of the Inversion of Adjunctions Conjecture, we refer to [**K+**]. For results in the klt and the log canonical cases, see [**Kol**] and [**Kaw1**].

We start with two lemmas. Recall that for every scheme X we have a morphism $\Phi_\infty \colon \mathbf{A}^1 \times J_\infty(X) \to J_\infty(X)$ such that if γ is an arc lying over $x \in X$, then $\Phi_\infty(0, \gamma)$ is the constant arc over x.

LEMMA 8.3. *Let X be a reduced, pure-dimensional scheme and $C \subseteq J_\infty(X)$ a nonempty cylinder. If $\Phi_\infty(\mathbf{A}^1 \times C) \subseteq C$, then $C \not\subseteq J_\infty(X_{\mathrm{sing}})$.*

PROOF. Write $C = (\psi_m^X)^{-1}(S)$, for some $S \subseteq J_m(X)$. Let $\gamma \in C$ be an arc lying over $x \in X$. By hypothesis, the constant m–jet γ_m^x over x lies in S. We take a resolution of singularities $f \colon X' \to X$. It is enough to show that $f_\infty^{-1}(C)$ is not contained in $f_\infty^{-1}(J_\infty(X_{\mathrm{sing}}))$.

Let x' be a point in $f^{-1}(x)$. The constant jet $\gamma_m^{x'}$ lies in $f_m^{-1}(S)$, hence $C' := (\psi_m^{X'})^{-1}(\gamma_m^{x'})$ is contained in $f_\infty^{-1}(C)$. On the other hand, X' is nonsingular, hence C' is not contained in $f_\infty^{-1}(J_\infty(X_{\mathrm{sing}})) = J_\infty(f^{-1}(X_{\mathrm{sing}}))$ by Lemma 5.1. $\qquad\square$

We will apply this lemma as follows. We will consider a reduced and irreducible variety X embedded in a nonsingular variety A. In $J_\infty(A)$ we will take a finite intersection of closed cylinders of the form $\mathrm{Cont}^{\geq m}(Z)$. Such an intersection is preserved by Φ_∞, and therefore so is each irreducible component $\widetilde{C}$. The lemma then implies that $C := \widetilde{C} \cap J_\infty(X)$ is not contained in $J_\infty(X_{\mathrm{sing}})$.

LEMMA 8.4. *Let A be a nonsingular variety and $M = H_1 \cap \ldots \cap H_c$ a codimension c complete intersection in A. If C is an irreducible locally closed cylinder in $J_\infty(A)$ such that*

$$C \subseteq \bigcap_{i=1}^c \mathrm{Cont}^{\geq d_i}(H_i),$$

and if there is $\gamma \in C \cap J_\infty(M)$ with $\mathrm{ord}_\gamma(\mathrm{Jac}_M) = e$, then

$$\mathrm{codim}(C \cap J_\infty(M), J_\infty(M)) \leq \mathrm{codim}(C, J_\infty(A)) + e - \sum_{i=1}^c d_i.$$

PROOF. We may assume that e is the smallest order of vanishing along Jac_M of an arc in $C \cap J_\infty(M)$. Let $m \geq \max\{2e, e + d_i\}$ be such that $C = (\psi_{m-e}^A)^{-1}(S)$ for some irreducible locally closed subset S in $J_{m-e}(A)$. Let S' be the inverse image of S in $J_m(A)$ and S'' an irreducible component of $S' \cap J_m(M)$ containing some jet having order e along Jac_M. Every jet in S' has order $\geq d_i$ along H_i, hence $S' \cap J_m(M)$ is cut out in S' by $\sum_i (m - d_i + 1)$ equations, and therefore

$$\dim(S'') \geq \dim(S') - (m+1)c + \sum_{i=1}^c d_i = \dim(S) + e \cdot \dim(A) - (m+1)c + \sum_{i=1}^c d_i.$$

Let S_0'' be the open subset of S'' consisting of jets having order $\leq e$ along Jac_M. It follows from Proposition 4.1 (see also Remark 4.3) that the image in $J_{m-e}(M)$ of any element in S_0'' can be lifted to $J_\infty(M) \cap C$, hence by assumption its order of vanishing along Jac_M is e. Moreover, Proposition 4.7 implies that the image of S_0'' in $J_{m-e}(M)$ has dimension $\dim(S_0'') - e(\dim(M) + 1)$. We conclude that

$$\mathrm{codim}(C \cap J_\infty(M), J_\infty(M)) \leq (m - e + 1)\dim(M) - \dim(S_0'') + e(\dim(M) + 1) \leq$$

$$(m - e + 1)\dim(A) + e - \dim(S) - \sum_{i=1}^c d_i = \mathrm{codim}(C, J_\infty(A)) + e - \sum_{i=1}^c d_i.$$

$\qquad\square$

PROOF OF THEOREM 8.1. The assertion is local, hence we may assume that A is affine. We first show that $\mathrm{mld}(W; X, \frac{1}{r}V(J_r) + Y|_X) \geq \mathrm{mld}(W; A, cX + Y)$. Suppose that this is not the case, and let us use Theorem 7.9 for $(X, \frac{1}{r}V(J_r) + Y|_X)$. We get $w \in \mathbf{N}^s$ and e, e', $m \in \mathbf{N}$ such that $m \geq \max\{2e, e + e', e + w_i\}$ and $S \subseteq J_m(X)$ with

$$S \subseteq \mathrm{Cont}^{\geq w}(Y)_m \cap \mathrm{Cont}^e(\mathrm{Jac}_X)_m \cap \mathrm{Cont}^{e'}(J_r) \cap \mathrm{Cont}^{\geq 1}(W)$$

such that $\dim(S) > (m+1)\dim(X) - \mathrm{mld}(W; A, cX+Y) - \sum_i q_i w_i$. We may consider S as a subset of $J_m(A)$ contained in $\mathrm{Cont}^{\geq(m+1)}(X)$, and applying Theorem 7.9 for the pair $(A, cX+Y)$ we see that

$$\dim(S) \leq (m+1)\dim(A) - c(m+1) - \sum_i q_i w_i - \mathrm{mld}(W; A, cX+Y).$$

This gives a contradiction.

We now prove the reverse inequality

$$\tau := \mathrm{mld}(W; X, \frac{1}{r}V(J_r) + Y|_X) \leq \mathrm{mld}(W; A, cX+Y).$$

If this does not hold, then we apply Theorem 7.4 (see also Remark 7.5) to find $w \in \mathbf{N}^s$ and $d \in \mathbf{N}$ such that for some irreducible component C of $\mathrm{Cont}^{\geq w}(Y) \cap \mathrm{Cont}^{\geq d}(X)$ we have $\mathrm{codim}(C) < cd + \sum_i q_i w_i + \tau$. It follows from Lemma 8.3 that $C \cap J_\infty(X) \not\subseteq J_\infty(X_{\mathrm{sing}})$. Let e be the smallest order of vanishing along Jac_X of an arc in $C \cap J_\infty(X)$. Fix such an arc γ_0.

Consider the closed subscheme $M \subset A$ whose ideal I_M is generated by c general linear combinations of the generators of the ideal I_X of X. Therefore M is a complete intersection and $\mathrm{ord}_{\gamma_0}(\mathrm{Jac}_M) = e$. By Corollary 9.2 in the Appendix, we have $\mathrm{Jac}_M \cdot \mathcal{O}_X \subseteq ((I_M : I_X) + I_X)/I_X$. It follows that γ_0 lies in the cylinder

$$C_0 := C \cap \mathrm{Cont}^{\leq e}(\mathrm{Jac}_M) \cap \mathrm{Cont}^{\leq e}(I_M : I_X).$$

C_0 is a nonempty open subcylinder of C, hence $\mathrm{codim}(C) = \mathrm{codim}(C_0)$. On the other hand, Lemma 8.4 gives

$$\mathrm{codim}(C_0 \cap J_\infty(M), J_\infty(M)) \leq \mathrm{codim}(C_0) + e - cd.$$

If $\gamma \in J_\infty(M)$, then $\mathrm{ord}_\gamma(\mathrm{Jac}_X) \leq \mathrm{ord}_\gamma(\mathrm{Jac}_M)$. If γ lies also in C_0, then γ can't lie in the space of arcs of any other irreducible component of M but X (we use the fact that γ has finite order along $(I_M : I_X)$, and the support of the scheme defined by $(I_M : I_X)$ is the union of the irreducible components of M different from X). Therefore $C_0 \cap J_\infty(M) = C_0 \cap J_\infty(X)$, and for every arc γ in this intersection we have $\mathrm{ord}_\gamma(\mathrm{Jac}_X) = \mathrm{ord}_\gamma(\mathrm{Jac}_M) = e$. We deduce that

$$\mathrm{codim}(C_0 \cap J_\infty(X), J_\infty(X)) = \mathrm{codim}(C_0 \cap J_\infty(M), J_\infty(M)) < \sum_i q_i w_i + \tau + e.$$

Since $C_0 \cap J_\infty(X) = \cup_{e'=0}^{re} \left(C_0 \cap J_\infty(X) \cap \mathrm{Cont}^{e'}(J_r) \right)$, it follows that there is e' such that $\mathrm{codim}(C_0 \cap J_\infty(X) \cap \mathrm{Cont}^{e'}(J_r)) < \sum_i q_i w_i + \tau + e$. On the other hand, this cylinder is contained in $\mathrm{Cont}^{re-e'}(Z_r)$. We deduce from Theorem 7.4 (see also Remark 7.5) that $\mathrm{mld}(W; X, \frac{1}{r}V(J_r) + Y|_X) < \tau$, a contradiction. This completes the proof of the theorem. $\qquad\square$

REMARK 8.5. It follows from the above proof that even if the coefficients of Y are negative, we still have the inequality

$$\mathrm{mld}\left(W; X, \frac{1}{r}V(J_r) + Y|_X\right) \geq \mathrm{mld}(W; A, cX+Y)$$

(it is enough to use the description of minimal log discrepancies mentioned in Remark 7.10).

9. Appendix

9.1. Dimension of constructible subsets. We recall here a few basic facts about the dimension of constructible subsets. Let X be a scheme of finite type over k, and $W \subseteq X$ a constructible subset, with the induced Zariski topology from X. If A is a closed subset of W, we have $\overline{A} \cap W = A$. Since X is a Noetherian topological space of bounded dimension, it follows that so is W.

Note that we have $\dim(W) = \dim(\overline{W})$. Indeed, the inequality $\dim(W) \le \dim(\overline{W})$ follows as above, while the reverse inequality is a consequence of the fact that W contains a subset U that is open and dense in $\overline{W}$. We see that if $W = T_1 \cup \ldots \cup T_r$, where all T_i are locally closed (or more generally, constructible) in X, then $\dim(W) = \max_i \{\dim(T_i)\}$.

Since W is Noetherian, we have a unique decomposition $W = W_1 \cup \ldots \cup W_s$ in irreducible components. If $\dim(W) = n$ and if we have a decomposition $W = T_1 \sqcup \ldots \sqcup T_r$ into disjoint constructible subsets of X, then every irreducible component A of some T_i, with $\dim(A) = n$ gives an irreducible component of W of dimension n, namely $\overline{A} \cap W$. Moreover, every n–dimensional irreducible component of W comes from a unique T_i and a unique such irreducible component of T_i.

If $f \colon X' \to X$ is a morphism of schemes that induces a bijection between the constructible subsets $V' \subseteq X'$ and $V \subseteq X$, then $\dim(V) = \dim(V')$ and $T \to V \cap \overline{f(T)}$ gives a bijection between the irreducible components of maximal dimension of V' and those of V. It follows that if we have a morphism of schemes $g \colon X' \to X$ and constructible subsets $V' \subseteq X'$ and $V \subseteq X$ such that we get a weakly piecewise trivial fibration $V' \to V$ with fiber F, then $\dim(V) = \dim(V') - \dim(F)$. Moreover, if F is irreducible, then we have a bijection between the irreducible components of maximal dimension of V' and those of V.

9.2. Differentials and the canonical sheaf. We start by reviewing the definition and some basic properties of the canonical sheaf. The standard reference for this is [**Har**]. To every pure-dimensional scheme over k one associates a coherent sheaf ω_X with the following properties:

  i) If X is nonsingular of dimension n, then there is a canonical isomorphism $\omega_X \simeq \Omega_X^n$.
  ii) The definition is local: if U is an open subset of X, then there is a canonical isomorphism $\omega_U \simeq \omega_X|_U$.
  iii) If $X \hookrightarrow M$ is a closed subscheme of codimension c, where M is a pure-dimensional Cohen-Macaulay scheme, then there is a canonical isomorphism
  $$\omega_X \simeq \mathcal{E}xt_{\mathcal{O}_M}^c(\mathcal{O}_X, \omega_M).$$
  iv) If $f \colon X \to M$ is a finite surjective morphism of equidimensional schemes, then
  $$f_* \omega_X \simeq \mathcal{H}om_{\mathcal{O}_M}(f_* \mathcal{O}_X, \omega_M).$$
  v) If X is normal of dimension $n \ge 2$, then $\mathrm{depth}(\omega_X) \ge 2$. Therefore there is a canonical isomorphism
  $$\omega_X \simeq i_* \Omega_{X_{\mathrm{reg}}}^n,$$
  where $i \colon X_{\mathrm{reg}} \hookrightarrow X$ is the inclusion of the nonsingular locus of X.
  vi) If X is Gorenstein, then ω_X is locally free of rank one.

Note that ω_X is uniquely determined by properties i), ii) and iii) above. Indeed, by ii) it is enough to describe ω_{U_i} for the elements of an affine open cover U_i of X. On the other hand, if we embed U_i as a closed subscheme of codimension c of an affine space $\mathbf{A}^N$, then we have

$$\omega_{U_i} \simeq \mathcal{E}xt^c_{\mathcal{O}_{\mathbf{A}^N}}(\mathcal{O}_{U_i}, \Omega^N_{\mathbf{A}^N}).$$

Note also that if Z is an irreducible component of X that is generically reduced, then by i) and ii) we see that the stalk of ω_X at the generic point of Z is $\Omega^n_{K/k}$, where $n = \dim(X)$ and K is the residue field at the generic point of Z.

Suppose now that we are in the following setting. X is a reduced scheme of pure dimension n, and we have a closed embedding $X \hookrightarrow A$, where A is nonsingular of dimension N and has global algebraic coordinates $x_1, \ldots, x_N \in \Gamma(\mathcal{O}_A)$ (that is, $dx_1, \ldots, dx_N$ trivialize Ω_A). We assume that the ideal I_X of X in A is generated by $f_1, \ldots, f_d \in \Gamma(\mathcal{O}_A)$. For example, if X is affine we may consider a closed embedding in an affine space.

Let $c = N - n$. As in §4, for $1 \le i \le d$ we take $F_i := \sum_{j=1}^d a_{i,j} f_j$, where the $a_{i,j}$ are general elements in k. If M is the closed subscheme defined by $I_M = (F_1, \ldots, F_c)$, then we have the following properties.

1) All irreducible components of M have dimension n, hence M is a complete intersection.
2) X is a closed subscheme of M and $X = M$ at the generic point of every irreducible component of X.
3) Some minor Δ of the Jacobian matrix of $F_1, \ldots, F_c$ with respect to the coordinates $x_1, \ldots, x_N$ (let's say $\Delta = \det(\partial F_i/\partial x_j)_{i,j \le c}$) does not vanish at the generic point of any irreducible component of X.

Moreover, every c of the F_i will satisfy similar properties. Let us fix now $F_1, \ldots, F_c$ as above, generating the ideal I_M. We also consider the residue scheme X' of X in M defined by the ideal $(I_M : I_X)$. Note that X' is supported on the union of the irreducible components of M that are not contained in X. The intersection of X and X' is cut out in X by the ideal $((I_M : I_X) + I_X)/I_X$.

Let K denote the fraction field of X, i.e. K is the product of the residue fields of the generic points of the irreducible components of X. We have a localization map $\Omega^n_X \to \Omega^n_{K/k}$ given by taking a section of Ω^n_X to its images in the corresponding stalks. By our assumption Δ is an invertible element in K, and $\Omega^n_{K/k}$ is freely generated over K by $dx_{c+1} \wedge \ldots \wedge dx_N$.

PROPOSITION 9.1. With the above notation, there are canonical morphisms

$$\Omega^n_X \xrightarrow{\eta} \omega_X \xrightarrow{u} \omega_M|_X \xrightarrow{w} \Omega^n_{K/k}$$

with the following properties:

a) If X is normal, then η is given by the canonical isomorphism $\omega_X \simeq i_* \Omega^n_{X_{\mathrm{reg}}}$, where $i\colon X_{\mathrm{reg}} \hookrightarrow X$ is the inclusion of the nonsingular locus of X.
b) w is injective and identifies $\omega_M|_X$ with $\mathcal{O}_X \cdot \Delta^{-1} dx_{c+1} \wedge \ldots \wedge dx_N$.
c) u is injective and the image of $w \circ u$ is $((I_M : I_X) + I_X)/I_X \cdot \Delta^{-1} dx_{c+1} \wedge \ldots \wedge dx_N$.
d) The composition $w \circ u \circ \eta$ is the localization map. Its image is $\mathrm{Jac}(F_1, \ldots, F_c) \cdot \Delta^{-1} dx_{c+1} \wedge \ldots \wedge dx_N$, where $\mathrm{Jac}(F_1, \ldots, F_c)$ denotes the ideal generated in $\mathcal{O}_X$ by the r–minors of the Jacobian matrix of $F_1, \ldots, F_c$.

COROLLARY 9.2. With the above notation, we have the following inclusion

$$\mathrm{Jac}(F_1, \ldots, F_c) \cdot \mathcal{O}_X \subseteq ((I_M \colon I_X) + I_X)/I_X.$$

COROLLARY 9.3. Suppose that X is a normal affine n–dimensional Gorenstein variety. If Z is the first Nash subscheme of X, that is, $I_Z \otimes \omega_X$ is the image of the canonical map $\eta \colon \Omega_X^n \to \omega_X$, then there is an ideal J such that

$$\mathrm{Jac}_X = I_Z \cdot J.$$

PROOF. We choose a closed embedding $X \hookrightarrow A = \mathbf{A}^N$, and let $F_1, \ldots, F_d$ be as above. For every $L = (i_1, \ldots, i_c)$, with $1 \le i_1 < \cdots < i_c \le d$, let I_L denote the ideal generated by $F_{i_1}, \ldots, F_{i_c}$. It follows from Proposition 9.1 that

$$\mathrm{Jac}(F_{i_1}, \ldots, F_{i_c}) \cdot \mathcal{O}_X = I_Z \cdot ((I_L \colon I_X) + I_X)/I_X.$$

If we take $J = \sum_L ((I_L \colon I_X) + I_X)/I_X$, this ideal satisfies the condition in the corollary. $\qquad\square$

PROOF OF PROPOSITION 9.1. Since X is reduced, we may consider its normalization $f \colon \widetilde{X} \to X$. On $\widetilde{X}$ we have a canonical morphism $\widetilde{\eta} \colon \Omega_{\widetilde{X}}^n \to \omega_{\widetilde{X}}$. On the other hand, since f is finite and surjective we have an isomorphism $f_* \omega_{\widetilde{X}} \simeq \mathcal{H}om_{\mathcal{O}_X}(f_* \mathcal{O}_{\widetilde{X}}, \omega_X)$, and the inclusion $\mathcal{O}_X \hookrightarrow f_* \mathcal{O}_{\widetilde{X}}$ induces a morphism $f_* \omega_{\widetilde{X}} \to \omega_X$.

The morphism η is the composition

$$\Omega_X^n \to f_* \Omega_{\widetilde{X}}^n \overset{f_* \widetilde{\eta}}{\to} f_* \omega_{\widetilde{X}} \to \omega_X,$$

where the first arrow is induced by pulling-back differential forms. The construction is compatible with the restriction to an open subset. In particular, the composition

$$\Omega_X^n \to \omega_X \to \Omega_{K/k}^n$$

of η with the morphism going to the stalks at the generic points of the irreducible components of X is the localization morphism corresponding to Ω_X^n.

Note that $\omega_M \simeq \mathcal{E}xt_{\mathcal{O}_A}^c(\mathcal{O}_M, \Omega_A^N)$. Since $F_1, \ldots, F_c$ form a regular sequence, we can compute ω_M using the Koszul complex associated to the F_i's to get

$$\omega_M \simeq \mathcal{H}om_{\mathcal{O}_M}\left(\bigwedge^c (I_M/I_M^2), \Omega_A^N|_M\right).$$

This is a free $\mathcal{O}_M$-module generated by the morphism ϕ that takes $\overline{F_1} \wedge \ldots \wedge \overline{F_c}$ to $dx_1 \wedge \ldots \wedge dx_N|_M$.

Since X is a closed subscheme of M of the same dimension, and since M is Cohen-Macaulay, it follows that $\omega_X \simeq \mathcal{H}om_{\mathcal{O}_M}(\mathcal{O}_X, \omega_M)$. In particular, we have $\omega_X \subseteq \omega_M$. Moreover,

$$\omega_X \otimes \omega_M^{-1} \simeq \mathcal{H}om_{\mathcal{O}_M}(\mathcal{O}_X, \mathcal{O}_M) = (I_M \colon I_X)/I_M.$$

Let u be the composition $\omega_X \hookrightarrow \omega_M \to \omega_M|_X$. Since $M = X$ at the generic point of each irreducible component of X, u is generically an isomorphism. On the other hand, M is Cohen-Macaulay and ω_X is contained in the free $\mathcal{O}_M$-module ω_M, hence ω_X has no embedded associated primes. Therefore u is injective.

Using again the fact that u is an isomorphism at the generic points of the irreducible components of X we get a localization morphism $w \colon \omega_M|_X \to \Omega_{K/k}^n$, and we see that the composition $w \circ u \circ \eta$ is the localization map for Ω_X^n at the generic points.

By construction, w takes the image of ϕ in $\omega_M|_X$ to $\Delta^{-1}dx_{c+1} \wedge \ldots \wedge dx_N$. It follows from our previous discussion that the image of ω_X in $\omega_M|_X$ is $((I_M : I_X) + I_X)/I_X \cdot \omega_M|_X$, from which we get the image of $w \circ u$. The last assertion in d) follows from the fact that if $1 \leq i_1 < \cdots < i_n \leq N$ and if D is the r–minor of the Jacobian of $F_1, \ldots, F_c$ corresponding to the variables different from $x_{i_1}, \ldots, x_{i_n}$, then

$$(w \circ u \circ \eta)(dx_{i_1} \wedge \ldots \wedge dx_{i_n}) = \pm\frac{D}{\Delta}dx_{c+1} \wedge \ldots \wedge dx_N.$$

This completes the proof of the proposition. $\qquad\qquad\qquad\qquad\qquad\square$

Suppose now that X is an affine $\mathbf{Q}$–Gorenstein normal variety. Our goal is to generalize Corollary 9.3 to this setting. Let K_X be a Weil divisor on X such that $\mathcal{O}(K_X) \simeq \omega_X$ and let us fix a positive integer r such that rK_X is Cartier. Note that we have a canonical morphism $p_r : \omega_X^{\otimes r} \to \mathcal{O}(rK_X)$.

We use the notation in Proposition 9.1. Let $\eta_r : (\Omega_X^n)^{\otimes r} \to \mathcal{O}(rK_X)$ be the composition of $\eta^{\otimes r}$ with p_r. Equivalently, if i denotes the inclusion of X_{reg} into X, then η_r is identified with the canonical map $(\Omega_X^n)^{\otimes r} \to i_*((\Omega_{X_{\mathrm{reg}}}^n)^{\otimes r})$. The image of η_r is by definition $I_{Z_r} \otimes \mathcal{O}(rK_X)$, where Z_r is the rth Nash subscheme of X.

Since $\omega_M^{\otimes r}|_X$ is locally-free, the morphism $u^{\otimes r}$ induces

$$u_r = i_*(u^{\otimes r}|_{X_{\mathrm{reg}}}) : \mathcal{O}(rK_X) \to \omega_M^{\otimes r}|_X.$$

This is injective, since this is the case if we restrict to the nonsingular locus of X. If we put $w_r := w^{\otimes r}$, then it follows from Proposition 9.1 that

i) w_r is injective and its image is $\mathcal{O}_X \cdot \Delta^{-r}(dx_{c+1} \wedge \ldots \wedge dx_N)^{\otimes r}$.
ii) The composition $w_r \circ u_r \circ \eta_r$ is the localization map. Moreover, its image is equal to $\mathrm{Jac}(F_1, \ldots, F_c)^r \cdot \Delta^{-r}(dx_{c+1} \wedge \ldots \wedge dx_N)^{\otimes r}$.

We now generalize Corollary 9.3 to the case when X is $\mathbf{Q}$–Gorenstein. Let $\overline{\mathfrak{a}}$ denote the integral closure of an ideal $\mathfrak{a}$. We define the *non-lci subscheme of level r* to be the subscheme of X defined by the ideal $J_r = (\overline{\mathrm{Jac}_X^r} : I_{Z_r})$ (see Remark 9.6 below for a justification of the name).

COROLLARY 9.4. *Let X be a normal $\mathbf{Q}$–Gorenstein n–dimensional variety and let r be a positive integer such that rK_X is Cartier. If Z_r is the rth Nash subscheme of X, and if J_r defines the non-lci subscheme of level r, then the ideals Jac_X^r and $I_{Z_r} \cdot J_r$ have the same integral closure.*

PROOF. It is enough to prove the assertion when X is affine, hence we may assume that we have a closed embedding $X \subset A$ of codimension c, and general elements $F_1, \ldots, F_d$ that generate the ideal of X in A, as above. It is enough to show that for every $L = (i_1, \ldots, i_c)$ with $1 \leq i_1 < \cdots < i_c \leq d$ we can find an ideal $\mathfrak{b}_L$ such that

(9.1) $$I_{Z_r} \cdot \mathfrak{b}_L = \mathrm{Jac}(F_{i_1}, \ldots, F_{i_c})^r.$$

Indeed, in this case if we put $\mathfrak{b} := \sum_L \mathfrak{b}_L$, then Jac_X^r and $I_{Z_r} \cdot \mathfrak{b}$ have the same integral closure. In particular, we have $\mathfrak{b} \subseteq J_r$, and we see that the inclusions

$$I_{Z_r} \cdot \mathfrak{a}_r \subseteq I_{Z_r} \cdot J_r \subseteq \overline{\mathrm{Jac}_X^r}$$

become equalities after passing to integral closure. Note also that $\mathfrak{b}$ and J_r have the same integral closure.

In order to find $\mathfrak{b}_L$, we may assume without any loss of generality that $L = (1, \ldots, c)$. With the above notation, consider the factorization of the localization map $(\Omega_X^n)^{\otimes r} \to (\Omega_{K/k}^n)^{\otimes r}$ as $w_r \circ u_r \circ \eta_r$. If $\mathfrak{b}_L$ is the ideal of $\mathcal{O}_X$ such that the image of u_r is $\mathfrak{b}_L \otimes \omega_M^{\otimes r}|_X$, then (9.1) follows from the discussion preceding the statement of the corollary. $\qquad\square$

REMARK 9.5. Since $I_{Z_{rs}} = I_{Z_r}^s$ for every $s \geq 1$, it follows that $(J_r)^s \subseteq J_{rs}$, and we deduce from the corollary that these two ideals have the same integral closure.

REMARK 9.6. Under the assumptions in Corollary 9.4, the support of the non-lci subscheme of level r is the set of points $x \in X$ such that $\mathcal{O}_{X,x}$ is not locally complete intersection. Indeed, if $\mathcal{O}_{X,x}$ is locally complete intersection, then after replacing X by an open neighborhood of x, we may assume that X is defined in some $\mathbf{A}^N$ by a regular sequence. In this case $I_{Z_1} = \mathrm{Jac}_X$, and we deduce that $J_r = \mathcal{O}_X$. Conversely, suppose that $\mathcal{O}_{X,x}$ is not locally complete intersection, and after restricting to an affine neighborhood of x, assume that we have a closed embedding $X \subset A$ as in our general setting. Note the by assumption, for every complete intersection M in A that contains X, the ideal $((I_M : I_X) + I_X)/I_X$ is contained in the ideal $\mathfrak{m}_x$ defining $x \in X$. On the other hand, following the notation in the proof of Corollary 9.4 we see that given $L = (i_1, \ldots, i_c)$ with $1 \leq i_1 < \cdots < i_c \leq d$, and $I_M = (F_{i_1}, \ldots, F_{i_c})$ we have $\mathfrak{b}_L \subseteq ((I_M : I_X)^r + I_X)/I_X \subseteq \mathfrak{m}_x$. Therefore $\mathfrak{b} \subseteq \mathfrak{m}_x$, and since J_r and $\mathfrak{b}$ have the same support (they even have the same integral closure), we conclude that x lies in the support of J_r.

References

[Amb] F. Ambro, *On minimal log discrepancies*, Math. Res. Lett. **6** (1999), 573–580.

[Bat] V. V. Batyrev, *Stringy Hodge numbers of varieties with Gorenstein canonical singularities*. Integrable systems and algebraic geometry (Kobe/Kyoto, 1997), 1–32, World Sci. Publishing, River Edge, NJ, 1998.

[Bui] A. Buium, *Differential algebra and diophantine geometry*, Actualités Mathématiques **39**, 1998.

[DL] J. Denef and F. Loeser, *Germs of arcs on singular algebraic varieties and motivic integration*, Invent. Math. **135** (1999), 201–232.

[EM] L. Ein and M. Mustaţă, *Inversion of Adjunction for local complete intersection varieties*, Amer. J. Math. **126** (2004), 1355–1365.

[EMY] L. Ein, M. Mustaţă and T. Yasuda, *Log discrepancies, jet schemes and Inversion of Adjunction*, Invent. Math. **153** (2003), 119–135.

[Eis] D. Eisenbud, *Commutative algebra. With a view toward algebraic geometry*, Graduate Texts in Mathematics **150**, Springer-Verlag, New York, 1995.

[Gil] H. Gillet, *Differential algebra–a scheme theory approach.* Differential algebra and related topics (Newark, NJ, 2000), 95–123, World Sci. Publ., River Edge, NJ, 2002.

[Gre] M. Greenberg, *Rational points in henselian discrete valuation rings*, Publ. Math. I.H.E.S. **31** (1966), 59–64.

[Har] R. Hartshorne, *Residues and duality*, Lecture Notes in Mathematics, No. **20**, Springer-Verlag, Berlin-New York 1966.

[IK] S. Ishii and J. Kollár, *The Nash problem on arc spaces of singularities*, Duke Math. J. **120** (2003), 601–620.

[Kaw1] M. Kawakita, *On a comparison of minimal log discrepancies in terms of motivic integration*, math. AG/0608512.

[Kaw2] M. Kawakita, *Inversion of adjunction on log canonicity*, Invent. Math. **167** (2007), 129–133.

[Kln] E. Kolchin, *Differential algebra and algebraic groups*, Academic Press, New York 1973.

[Kol] J. Kollár, *Singularities of pairs.* Algebraic geometry, Santa Cruz 1995, 221–286, volume **62** of Proc. Symp. Pure Math. Amer. Math. Soc. 1997.

[K+] J. Kollár (with 14 coauthors), *Flips and Abundance for Algebraic Threefolds*, vol. **211**, Astérisque, 1992.

[Kon] M. Kontsevich, Lecture at Orsay (December 7, 1995).

[Laz] R. Lazarsfeld, *Positivity in algebraic geometry*. II, Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. **49**, Springer-Verlag, Berlin, 2004.

[Loj] E. Looijenga, *Motivic measures*, in Séminaire Bourbaki, Vol. 1999/2000, Astérisque **276** (2002), 267–297.

[Los] F. Loeser, Seattle lectures on motivic integration, this volume.

[Mus] *Jet schemes of locally complete intersection canonical singularities*, with an appendix by David Eisenbud and Edward Frenkel, Invent. Math. **145** (2001), 397–424.

[NS] J. Nicaise and J. Sebag, *Le* théorème *d'irréductibilité de Kolchin*, C. R. Math. Acad. Sci. Paris **341** (2005), 103–106.

[Seb] J. Sebag, *Intégration motivique sur les schémas formels*, Bull. Soc. Math. France **132** (2004), 1–54.

[Sho] V. V . Shokurov, *Letters of a bi-rationalist. V. Minimal log discrepancies and termination of log flips*, Proc. Steklov Inst. Math. **246** (2004), 315–336.

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